A Stochastic Version of the Ramsey's Growth Model

Gabriel Zacarías-Espinoza[1], Hugo Cruz-Suárez[2] and Enrique Lemus-Rodríguez[3]
[1]Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, Ave. San Rafael Atlixco 186, Col. Vicentina,09340, México D.F., México
[2]Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, Ave. San Claudio y Río Verde, Col. San Manuel, Ciudad Universitaria, Puebla, Pue., 72570, México
[3]Escuela de Actuaría, Universidad Anáhuac México-Norte, Ave. Universidad Anáhuac 46, Col. Lomas Anáhuac, 52786, Edo. de México, México

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Abstract: In this paper we study a version of Ramsey's discrete time Growth Model where the evolution of Labor through time is stochastic. Taking advantage of recent theoretical results in the field of Markov Decision Processes, a first set of conditions on the model are established that guarantee a long-term stable behavior of the underlying Markov chain.

1 INTRODUCTION

Ramsey's Growth Model has a long and interesting history. In order to give a context to the material of the present paper, we briefly outline it.

The original model presented by Ramsey in (Ramsey, 1928) (formulated in communication with the famous economist Keynes) analyzes optimal global saving in a deterministic continuous time setting, and it is no surprise that it is solved using Calculus of Variations. Since then, several variants have appeared in the Advanced Macroeconomics literature, but, as Prof. Ekeland points out (one of the leading experts in Mathematical Economics): "To the best of my knowledge and understanding, none of the solutions proposed for solving the Ramsey problem is correct with one exception, of course, Ramsey himself, whose own statement was different than the one which is now in current use (Ekeland, 2010)". On the contrary, the discrete time setting allows the straightforward use of Dynamic Programming techniques, and therefore, allows both researchers and practitioners to focus on the analysis of the model itself and its properties. The deterministic case is very clearly stated and analyzed in (Brida et al., 2015), (Le Van and Dana, 2003) and (Sladký, 2012).

Ramsey's seminal work on economic growth has been extended in many ways, but, to the best of our knowledge, the study of a random discrete time version is still in its initial phase. Such study will allow a fruitful interaction between economists and mathematician that will lead to better simulations and consequently, to a better understanding of the effects of the random deviations in the growth of an economy and its impact on the population. And, in this paper a first random model is posed, where the population grows in a stochastic manner.

In this paper a discrete-time stochastic Ramsey growth process is modeling as a discounted Markov Decision Process (MDP) (Hernández-Lerma and Lasserre, 1996) and (Jaśkiewicz and Nowak, 2011). The performance criterion of interest is the total discounted reward. The optimal control problem is to determine a policy that optimizes the performance criterion. The solution of the optimization problem is analyzed through of the Euler Equation (EE) (Cruz-Suárez and Montes-de Oca, 2008) and (Cruz-Suárez et al., 2012). Later, the EE is applied to study the ergodic behavior of the stochastic Ramsey growth process.

2 THE MODEL

Consider an economy in which at each discrete time t, $t = 0, 1, ...,$ there are L_t consumers (population or labor), with consumption c_t per individual, whose growth is governed by the following difference equation:

$$L_{t+1} = L_t \eta_t, \tag{1}$$

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it is assumed that initially the number of consumers, L_0, is known. In this case, $\{\eta_t\}$ is a sequence of independent and identical distributed (i.i.d) random variables. The random variable η_t, $t \geq 0$, represents an exogenous shock that affects the consumer population, for example: epidemics, wars, natural disasters, new technology, etc. Then, in this context, it will be supposed that for each $t \geq 0$: $\eta_t > 0$, almost surely.

Remark 2.1. *In the literature of economic growth models is usual to assume that the number of consumers grow very slowly in time, see, for instance, (Le Van and Dana, 2003) and (Sladký, 2012). Observe that the model presented in this paper is a first step in an effort to weaken that constraint of the model.*

The production function for the economy is given by

$$\begin{aligned} Y_t &= F(K_t, L_t), \\ & K_0 \text{ is known,} \end{aligned}$$

i.e. the production Y_t is a function of capital, K_t, and labor, L_t, where the production function, F, is a homogeneous function of degree one. The output must be split between consumptions $C_t = c_t L_t$ and the gross investment I_t, i.e.

$$C_t + I_t = Y_t. \tag{2}$$

Let $\delta \in (0,1)$ be the depreciation rate of capital. Then the evolution equation for capital is given by:

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{3}$$

Substituting (3) in (2), it is obtained that,

$$C_t - (1-\delta)K_t + K_{t+1} = Y_t. \tag{4}$$

In the usual way, all variables can be normalized into per capital terms, namely, $y_t := Y_t/L_t$ and $x_t := K_t/L_t$. Then (4) can be expressed in the following way:

$$c_t - (1-\delta)x_t + K_{t+1}/L_t = y_t = F(x_t, 1).$$

Now, using (1) in the previous relation, it yields that

$$x_{t+1} = \xi_t(F(x_t,1) + (1-\delta)x_t - c_t),$$

$t = 0,1,2,...$, where $\xi_t := (\eta_t)^{-1}$.

Define $h(x) := F(x,1) + (1-\delta)x$, $x \in X := [0,\infty)$, h henceforth to be identified as the production function. Then, the transition law of the system is given by

$$\begin{aligned} x_{t+1} &= \xi_t(h(x_t) - c_t), && (5) \\ x_0 = x & \quad \text{known,} && (6) \end{aligned}$$

where $c_t \in [0, h(x_t)]$ and $\{\xi_t\}$ is a sequence of i.i.d. random variables with a density function Δ.

Observation 2.2. *Observe that, if $x_t = 0$, for some $t \in \{0,1,2,..\}$ then $x_k = 0$ for each $k \geq t$. This fact is a consequence of relation (5), and in this case zero is considered an absorbtion state.*

A plan or consumption sequence is a sequence $\pi = \{\pi_n\}_{n=0}^{\infty}$ of stochastic kernel π_n on the control set given the history

$$h_n = (x_1, c_1, \cdots, x_{n-1}, c_{n-1}, x_n),$$

for each $n = 0, 1, \cdots$. The set of all plans will be denoted by Π.

Given an initial capital $x_0 = x \in X$ and a plan $\pi \in \Pi$, the performance index used to evaluate the quality of the plan π is determined by

$$v(\pi, x) = \mathbb{E}_x^{\pi}\left[\sum_{n=0}^{\infty} \alpha^n U(c_n)\right], \tag{7}$$

where $U : [0,\infty) \to R$ is a measurable function known as utility function and $\alpha \in (0,1)$ is a discount factor.

The goal of the controller is to maximize utility of consumption on all plans $\pi \in \Pi$, that is:

$$V(x) := \sup_{\pi\in\Pi} v(\pi, x),$$

$x \in X$.

Throughout of this paper the model will be called a Stochastic version of the Ramsey Growth (SRG) model.

The following assumptions it will be considered in the rest of the document.

Assumption 2.3. *The production function h, satisfies:*

a) $h \in C^2((0,\infty))$,

b) *h is a concave function on X,*

c) $h' > 0$ *and* $h(0) = 0$.

d) *Let $h'(0) := \lim_{x \downarrow 0} h'(x)$. Suppose that $h'(0) > 1$ and*

$$\alpha h'(0) > E[\xi^{-1}]. \tag{8}$$

Assumption 2.4. *The utility function U satisfies:*

a) $U \in C^2((0,\infty), \mathbb{R})$, *with* $U' > 0$ *and* $U'' < 0$,

c) $U'(0) = \infty$ *and* $U'(\infty) = 0$,

d) *There exists a function ϑ on S such that $E[\vartheta(\xi)] < \infty$, and*

$$\left|U'(h(s(h(x)-c)))h'(s(h(x)-c))s\Delta(s)\right| \leq \vartheta(s), \tag{9}$$

$s \in S$, $c \in (0, h(x))$.

Observation 2.5. *Observe that in Assumption 2.3 is not considered the Inada condition in zero. In the literature, it is known to have the rather unrealistic implication that each unit of capital must be capable of producing an arbitrarily large amount of output with a sufficient amount of labor (Kamihigashi, 2006).*

3 DYNAMIC PROGRAMMING APPROACH

In this section it will be presented an analysis of the optimization problem introduced in the previous section. Dynamic Programming approach have been used to study different type of problems and in various context. In particular have been applied to Markov Decision Processes (MDPs).

SRG can be identified as a MDP. In this case, the space of states is $X := [0,\infty)$, the admissible action space is $A(x) := [0,h(x)], x \in X$, in consequence, the action space is $A := \bigcup_{x\in X} A(x) = [0,\infty)$. The transition law is given by the stochastic kernel, defined as

$$\begin{aligned} Q(B|x_t = x, a_t = c) &= \Pr(x_{t+1} \in B|x_t = x, a_t = c) \\ &= \int_B w(x,y,c)dy, \end{aligned}$$

with $B \in \mathcal{B}(X)$, ($\mathcal{B}(X)$ denotes the Borel sigma algebra of X), where the function $w : [0,\infty)^3 \to [0,\infty)$ is defined as:

$$w(x,y,c) = \Delta\left(\frac{y}{h(x)-c}\right)\frac{1}{h(x)-c}, \qquad (10)$$

for $x,y \in X$, $c \in [0,h(x))$ and Δ is the density function of the sequence $\{\xi_t\}$. Define $\mathbb{K} := \{(x,c)|x \in X, c \in A(x)\}$. Finally, the reward-per-stage function is identified as the utility function, $U : X \to [0,\infty)$, defined in the previous section. Then the model is referred as the quintuplet: $\mathcal{M} := (X, A, \{A(x) : x \in X\}, Q, U)$.

As it was mentioned above, a plan is a sequence $\pi = \{\pi_n\}_{n=0}^{\infty}$ of stochastic kernel defined on A given the history of the process. Furthermore, it is assumed that $\pi_n(C(x_n)|h_n) = 1$, $n = 0,1,\cdots$, this assumption guarantee that in each decision epoch, it is possible to choose an admissible action. A particular class in Π is the class of *stationary plans*,

$$\mathbb{F} := \{f : X \to A|f(x) \in [0,h(x)], \text{ for all } x \in X\}.$$

In this case, a stationary plan $\pi = (f, f, ...)$ is denoted by f.

Under Assumption 2.3 and Assumption 2.4, for each $x \in X$, it follows that:

(a) The *optimal value function* V satisfies the *following equation (optimality equation)*

$$V(x) = \sup_{c\in A(x)}\left\{U(c) + \alpha\int_0^{\infty} V(y)w(x,y,c)dy\right\}. \qquad (11)$$

(b) There exists and *optimal stationary policy* $f \in \mathbb{F}$ such that

$$V(x) = U(f(x)) + \alpha\int_0^{\infty} V(y)w(x,y,f(x))dy.$$

(c) For every $x \in X$, $v_n(x) \to V(x)$ when $n \to \infty$, where v_n is defined by

$$v_n(x) = \sup_{c\in A(x)}\left\{U(c) + \alpha\int v_{n-1}(y)w(x,y,c)dy\right\},$$

with $v_0(x) = 0$.

Remark 3.1. *The functions, v_n, $n \geq 0$, defined on (c) are known as value iteration functions, (Hernández-Lerma and Lasserre, 1996).*

4 MAIN RESULTS ABOUT SRG

4.1 Euler Equation

In this subsection it will be presented a functional equation, which characterize the optimal value function. In the literature of MDP's, this functional equation is known as Euler Equation (EE), (Cruz-Suárez et al., 2012). The validity of EE is guaranteed due to properties of differentiability of the optimal value function and the optimal policy, (Cruz-Suárez and Montes-de Oca, 2008). Then, it just is necessary to verified that the optimal policy is interior, according to Theorem 3.3 in (Cruz-Suárez et al., 2011), this fact is verified in the following result.

Lemma 4.1. *The optimal plan f satisfies that $f(x) \in (0,h(x))$, for each $x > 0$.*

Proof. Let $x > 0$ fixed, if the optimal policy is $f(\cdot) \equiv 0$, then

$$V(x) = v(0,x) = \frac{U(0)}{1-\alpha},$$

where v is defined in (7).

Since U and h are strictly increasing (see Assumption 2.2 and 2.3), it is obtained that

$$V(x) = \frac{U(0)}{1-\alpha} < U(h(x)) + \frac{\alpha}{1-\alpha}U(0),$$

but this is a contradiction, given that

$$v(h,x) = U(h(x)) + \frac{\alpha}{1-\alpha}U(0).$$

On the other hand, if $h \in \mathbb{F}$ is the optimal policy, then

$$\begin{aligned} V(x) &= v(h,x) \\ &= U(h(x)) + \frac{\alpha}{1-\alpha}U(0). \end{aligned}$$

Let $g : [0,h(x)] \to \mathbb{R}$ be a function defined as

$$g(c) := U(c) + \alpha E[U(h(\xi(h(x)-c)))] + \frac{\alpha^2}{1-\alpha}U(0).$$

Observe that g is continuous and strictly concave function. Then, there exists an unique $\bar{c} \in [0, h(x)]$, which maximizes to g. If $\bar{c} \neq h(x)$, then

$$V(x) \geq g(\bar{c}) > g(h(x)) = V(x),$$

which is impossible. Therefore $\bar{c} = h(x)$.

Now, Assumptions 2.2 and 2.3 imply that if $c \in (0, h(x))$,

$$g'(c) = U'(c) - \alpha E[U'(h(\xi(h(x)-c)))h'(\xi(h(x)-c))\xi],$$

it follows that

$$\lim_{c \to h(x)} g'(c) = -\infty.$$

Therefore, there exists $\tilde{c} \in (0, h(x))$ such that $g'(\tilde{c}) < 0$. This implies that g is decreasing in $[\tilde{c}, h(x)]$ which $h(x)$ can not be the maximizer, i.e. it is a contradiction.

□

Theorem 4.2. *Under Assumption 2.2 and 2.3, it follows that:*

a) $V \in C^2((0,\infty), \mathbb{R})$ and the optimal plan $f \in C^1((0,\infty))$.

b) The value iteration functions satisfies

$$\frac{v_n'(x)}{h'(x)} = \alpha E\left\{ v_{n-1}'\left[\xi\left(h(x) - U'^{-1}\left(\frac{v_n'(x)}{h'(x)}\right)\right)\right]\xi\right\},$$

for each $x > 0$, where $U'^{-1}(\cdot)$ is the inverse of $U'(\cdot)$.

c) The optimal plan f satisfies the following Euler equation:

$$U'(f(x)) = \alpha E[U'(c^*(x))h'(\xi(h(x)-f(x)))\xi],$$

for each $x > 0$, where $c^*(x) := f(\xi(h(x) - f(x)))$.

Proof. The proof of this result is a consequence of Lemma 5.2 and Lemma 5.6 in (Cruz-Suárez et al., 2011). □

Remark 4.3. *Observe that if $f \in \mathbb{F}$ satisfies (4.2) and*

$$\lim_{n\to\infty} \alpha^n E_x^f \left[h'(x_n)U'(f(x_n))x_n\right] = 0,$$

then f is an optimal plan.

4.2 Stability of the SRG

It is known (Hernández-Lerma and Lasserre, 1996) that if $f \in \mathbb{F}$ is the optimal plan then the optimal process $\{x_n\}$ is a Markov process, where

$$x_{n+1} = \xi_n(h(x_n) - f(x_n)),$$

$n = 0, 1, 2, \ldots$, $x_0 = x \in X = [0, \infty)$. In addition,

$$\begin{aligned} Q(B|x, f(x)) &= \int_B w(x, y, f(x))dy \\ &= E[I_B(\xi(h(x) - f(x)))]. \end{aligned}$$

Furthermore, in the literature of economic growth it was studied optimal process stability using Inada conditions. However, (Nishimura and Stachurski, 2005) and (Stachurski, 2009) make use of the Euler equation. In this spirit we present this subsection.

Define Γ for $x \in (0, \infty)$ as

$$\Gamma(x) := [U'(f(x))h'(x)]^{1/2} + x^p + 1, \qquad (12)$$

where $p > 1$. Let us consider to Γ as a weight function.

Let $\mathbb{B}_\Gamma(X)$ be the space of measurable and Γ-bounded function on X with norm $\|\cdot\|_\Gamma$ defined as

$$\|g\|_\Gamma := \sup_{x\in X} \frac{|g(x)|}{\Gamma(x)},$$

for a measurable function g on X. Let φ be a signed measure defined on $\mathcal{B}(X)$. Then, for each $g \in \mathbb{B}_L(X)$, φg denotes

$$\varphi g := \int g(y)\varphi(dy).$$

Observe that for the transition Kernel Q, Qg has the form

$$Qg = \int g(y)Q(dy|x, f(x)),$$

and the k-th transition kernel is

$$Q^k(B|x, f(x)) = \int Q(B|y, f(y))Q^{k-1}(dy|x, f(x)),$$

$B \in \mathcal{B}(X)$, for $k \geq 1$ with $Q^0 := \delta_x$, where δ_x is Dirac's measure on $x \in X$.

Let μ be a probability measure on $\mathcal{B}(X)$. The measure μ is invariant with respect to the Markov chain $\{x_n\}$, if $\mu Q = \mu$, where

$$\mu Q(B) := \int Q(B|y)\mu(dy), B \in \mathcal{B}(X).$$

Define for each measure φ on $\mathcal{B}(X)$

$$\|\varphi\|_\Gamma := \sup_{\|g\|_\Gamma \leq 1} |\varphi g|.$$

Definition 4.4. *Given a set $C \in \mathcal{B}(X)$. C is a small set with respect to the Markov chain $\{x_n\}$, if there exist a finite measure μ on $\mathcal{B}(X)$ and $n \in \mathbb{N}$, such that for each $x \in C$*

$$Q^n(B|x) \geq \mu(B),$$

for each $B \in \mathcal{B}(X)$.

Define for $A \in B(X)$ the measure

$$\Xi(A) := \int_B \Delta(s)ds.$$

Lemma 4.5. *The optimal process $\{x_n\}$ of SRG is Ξ-irreducible and strongly aperiodic.*

Proof. Let $B \in \mathcal{B}(X)$ such that $\Xi(B) > 0$ and $x > 0$. Then, we know that $h(x) - f(x) > 0$ because f is an interior point in the corresponding interval. Moreover,

$$\Pr(x_1 \in B) = \int_X I_B((h(x) - f(x))s)\Delta(s)ds,$$

where I_B denotes the indicator function of the set B. As Δ is positive, it follows that $\Pr(x_1 \in B) > 0$. Consequently, the optimal process Ξ is irreducible.
On the other hand, let $a, b \in (0,\infty)$, $a < b$ and consider $C := [a,b]$. Once again, due to the fact that the optimal policy takes values in the interior of the corresponding interval, it turns out that $h - f$ is an increasing function. In fact, as the optimal value function V is concave, we conclude that V' is decreasing and, by the envelope formula (Cruz-Suárez and Montes-de Oca, 2008) it follows that:

$$V'(x) = U'(h(x) - f(x))h'(x).$$

If $h - f$ is decreasing, for $x, y \in X$, $x < y$:

$$h(y) - f(y) \leq h(x) - f(x)$$

and, as both U' and h' are decreasing and positive, we get

$$U'(h(x) - f(x))h'(x) \leq U'(h(y) - f(y))h'(y)$$

that is, $V'(x) \leq V'(y)$, contradiction. Then, $h - f$ is increasing. Consequently, for each $x \in C$ we have

$$0 < h(a) - f(b) \leq h(x) - f(x) \leq h(b) - f(b).$$

Then, due to Δ is positive function,

$$m := \inf_{(x,s)\in C\times C} \Delta\left(\frac{s}{h(x) - f(x)}\right)\frac{1}{h(x) - f(x)} > 0.$$

for μ a measure on $B \in \mathcal{B}(X)$ defined by

$$\mu(B) := m\int_B I_C(x)dx,$$

we have

$$Q(B|x) \geq \mu(B).$$

The set C is small, see Definition 4.4. Clearly $\mu(C) > 0$: the process is strongly aperiodic. ☐

Analogously to (Nishimura and Stachurski, 2005) and (Stachurski, 2009), it is possible to show that the optimal process is ergodically stable.

Lemma 4.6. *Γ (see (12)) is a Lyapunov function.*

Proof. Consider $a \in \mathbb{R}$ and

$$N_a := \{x \in (0, +\infty) | \Gamma(x) \leq a\}.$$

Suppose that $a \leq 1$, as $\Gamma(x) > 0$, for each $x > 0$, then $N_a = \emptyset$ and hence its closure is compact. On the other hand, if $a > 1$ and $\{x_n\}$ is a sequence in N_a such that $x_n \to x$, due to the continuity of Γ it follows that $x \in N_a$ and consequently, N_a is a closed set. Due to Inada's condition on the model and the definition of Γ it is immediate that N_a is bounded and hence, compact, and trivially, with compact closure.
As a is arbitrary we conclude that Γ is a Lyapunov function. ☐

Lemma 4.7. *The optimal process $\{x_n\}$ converges Γ-ergodically to a unique invariant probability measure μ, that is, there exists non-negative constants R and ρ, $\rho < 1$, such that for each $k = 0, 1, \ldots$*

$$\left\|Q^k - \mu\right\|_\Gamma \leq R\rho^n. \tag{13}$$

The main idea to prove the previous lemma is applying the Euler equation to show that Γ is a Lyapunov function. Then, there exist constants λ and b such that $\lambda \in (0,1)$ and for each $x \in (0,\infty)$:

$$E[\Gamma(\xi(h(x) - f(x)))] \leq \lambda\Gamma(x) + b.$$

Furthermore, there exists a measure φ on $\mathcal{B}(X)$ such that the optimal process is φ-irreducible and aperiodic strongly. Finally, the result follows of Theorem 16.1.2 in (Meyn and Tweedie, 2009).

Theorem 4.8. *The RSL optimal process converges in L^1 to a random variable with probability measure μ given in Lemma 4.7.*

Proof. Let $x_0 = x \in X$, $\{x_n\}$ be the optimal process. It is known that there exists constants λ and b with $\lambda \in (0,1)$ such that $E[x_1^p|x] \leq \lambda x^p + b$. Hence, for $n = 0, 1, \ldots,$

$$E[x_{n+1}^p|x_n] \leq \lambda x_n^p + b. \tag{14}$$

Iterating and applying standard conditional expectation properties in (14) for $n = 0, 1, \ldots,$ and as $\lambda \in (0,1)$, it follows that for each $n \in \mathbb{N}$:

$$E[x_{n+1}^p] \leq x^p + \frac{b}{1-\lambda} < \infty,$$

hence

$$\sup_n E[x_n^p] < \infty.$$

Moreover, as $\{x_n\}$ is almost surely positive it follows that the optimal process is uniformly integrable (see (Peligrad and Gut, 1999), Theorem 4.2, p. 215). Furthermore, by Lemma 4.7 it is known that the sequence $\{x_n\}$ converges in distribution to the invariant probability measure μ.

Finally, by Theorem 5.9, p. 224 in (Peligrad and Gut, 1999), the result follows. $\square$

5 EXAMPLES: COBB-DOUGLAS UTILITY

Consider the following utility function:

$$U(c) = \frac{b}{\gamma} c^{\gamma},$$

for $c > 0$, where $b > 0$ and $\gamma = 1/3$. The transition law is determined by

$$x_{t+1} = \xi_t \left(x_t - a_t\right),$$

$a_t \in [0, x_t]$, $t = 0, 1, 2, \ldots$, $x_0 = x \in (0, \infty)$. Observe that in this case the production function $h(x) = x$, $x \in (0, \infty)$. Suppose that $\{\xi_t\}$ is a sequence of i.i.d. random variables independent of x_0. Let ξ a generic element of $\{\xi_t\}$ and consider that ξ with log-normal distribution with mean $3/2$ and variance 1. Then:

$$\mu_\gamma := E[\xi^\gamma] = e^{5/9}$$

it is easy to see that $0 < \alpha\mu_\gamma < 1$, where the discount factor $\alpha < e^{5/9}$. Moreover

$$E\left[\xi^{-1}\right] = e^{-1}$$

and Assumption 2.2-d) holds.

Remark 5.1. *Assumption 2.2-d) holds for a log-normal distribution if and only if $\sigma^2 < 2\mu$ where μ and σ^2 are mean and variance, respectively.*

Define $\delta := \left(\alpha\mu_\gamma\right)^{1/(\gamma-1)}$. It is shown in (Cruz-Suárez et al., 2011) that

$$f(x) := \left(\frac{\delta - 1}{\delta}\right) x, \tag{15}$$

$x \in X$, is the optimal plan.

Consider the process corresponding to the optimal plan:

$$x_{t+1} = \xi_t \left(x_t - f(x_t)\right)$$

$t = 0, 1, 2, \ldots, x_0 = x \in X$; easy calculations show that

$$x_{t+1} = \frac{\xi_t x_t}{\delta}$$

iterating this last equation we get

$$x_{t+1} = \frac{x}{\delta^t} \prod_{i=0}^{t-1} \xi_i.$$

Taking the expectation and using the independence of $\xi_0, \xi_1, \ldots, \xi_{t-1}$, it yields that

$$E\left[x_{t+1}\right] = \frac{x}{\mu}\left(\frac{\mu}{\delta}\right)^t,$$

where $\mu := E\left[\xi\right] = e^2$.

Finally, if $\mu < \delta$ then $E[X_n]$ increasing indefinitely whit respect the time; if $\mu > \delta$ then $E[X_n]$ decreasing to zero; if $\mu = \delta$ then $E[X_n] = x/\mu$.

6 CONCLUSIONS

The study of the Ramsey Growth Model in a discrete time and stochastic setting opens an interesting research field, where not only stochastic labor, stochastic depreciation or other variants may be studied. For instance, we believe that a multidimensional case (for instance, where labor is disaggregated into two sub-populations, regarding their different saving capabilities) may be studied combining his techniques with the Euler Equation approach.

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