An Industry-focused Advertising Model

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Abstract: In this paper a model is created that may be effectively used to determine the optimal spending trajectory for an advertising campaign. Given a sufficient data set, all parameters present in the model should be easily determinable, or at least accurately approximated, and justifications are given for the form of all parts of the model. Finally, the solution to both the deterministic and stochastic versions of the model are given.

1 INTRODUCTION

The problem of predicting whether an ad campaign will ultimately be successful is an important problem that is difficult to solve. Several models have already been proposed such as the Sethi model (Sethi, 1983) and the older Vidale-Wolfe advertising model (Vidale and Wolfe, 1957). However, all advertising models thus far have been purely theoretical and have had limited applicability due to their assumptions and simplifications. Justifications for each feature of the new model come from a large data set provided by a corporation that is actively engaged in numerous advertising campaigns, however it has not been presented here due to confidentality issues.

Since (Gould, 1970) it has become standard practice to assume that the function relating market share to advertising effort is concave. Indeed, it would be unreasonable to assume differently as that would imply that there would not be a diminished effect from each additional advertisement. However, the exact nature of the relationship between advertising effort and market share is often quite difficult to determine due to the inherent variance in this type of data. Typically there is so much variance that almost any concave function would model the relationship quite well. Lewis and Rao (Lewis and Rao, shed) demonstrate the difficulty in simply proving that a given advertising campaign yielded a positive return on investment, let alone the relationship between advertising spending and the resulting growth in market share. Thus, a quadratic relationship between market share and advertising expenditure is assumed since a quadratic form yields a very simple form for the optimal control. The purpose of the new model is to maximize long-term profit, however future profits must be "discounted" due to the role that re-investment and inflation play. This effect is incorporated into the model by multiplying the profit at time t by $e^{-\delta t}$, where δ is the rate at which the profit is discounted over time. Thus the long-term profit can be described according to the following function:

$$P = \sum_{0}^{\infty} (mx_t - u_t^2)e^{-\delta t}. \tag{1}$$

Where x_t is the market share at time t, m is the revenue per unit of market share, and u_t is the advertising effort at time t.

As in the Sethi model, x_t is normalized by the market share, however unlike the Sethi model the size of the market does not remain constant. Instead, it is assumed that the market size changes according to the predefined function M_t. Modelling market growth and decline is a separate problem and no attempt to do so is undertaken in this paper. The dynamics of the market share are modelled as a discrete version of those found in Equation 5 of (Murray and MacIsaac, 2015). A discrete model is used since firms cannot feasibly control their advertising expenditure in real time but only for periods of time with granularity on the order of months, days, hours, etc. Equation 2 gives the deterministic version of the function describing the dynamics of the state equation where ρ is the effectiveness of advertising, and D is the rate at which market share decays (assumed to be linear for simplicity).

$$x_{t+1} - x_t = \rho u_t \sqrt{M_t - x_t} + r(x_t)(M_t - x_t) - Dx_t, \tag{2}$$

where $0 \leq x_t \leq M_t$. It is known that the function describing the decay of the market share is concave

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however a quadratic relationship ($-\frac{Dx^2}{2}$) once again results in the simplest optimal control. The growth due to ranking, $r(x_t)$ is defined as follows:

$$r(x_t) = \begin{cases} r_k, & \text{if } x_t \geq T_k, \\ & \dots \\ r_1, & \text{if } T_2 > x_t \geq T_1, \\ 0, & \text{otherwise} \end{cases}$$

which captures the effect that ranking systems have on market share growth. Each T_i is the market share threshold to enter into the i^{th} ranking tier. As the market share grows, so too will the ranking and likewise a higher ranking spurs more growth. For more information on the dynamics of the ranking system see (Murray and MacIsaac, 2015). In equation 1 it has been assumed that the rankings are in a state of equilibrium and thus have taken it to only be dependent on the market share. In general, the advertising rate of each company will affect their rankings and thus affect the rankings of other firms as well. For more research on the optimization of advertising revenue in a system with many competing firms see (Fruchter, 1999), (Horsky, 1988) and (Erickson, 1995).

2 DETERMINISTIC SOLUTION

The current value Hamiltonian from equations 1 and 2 is

$$H = mx_t - u_t^2 + \lambda(\rho u_t \sqrt{M_t - x_t} + r(x_t)(M_t - x_t) - Dx_t), \tag{3}$$

which is almost identical to what is seen in the Sethi model. Indeed, the optimal control takes a similar form to that of the Sethi model:

$$u^*(x) = \frac{\lambda\rho\sqrt{M-x}}{2}. \tag{4}$$

If we attempt to derive λ while taking r to depend on x then the solution becomes intractable. However, by Bellman's Optimality Principle (Bellman, 1957), each sub-path of an optimal path must be optimal. Thus, for each interval where r is constant a solution may be obtained. By solving equation 5, for a particular value of r, we can determine the current value adjoint variable λ_t.

$$\lambda_{t+1} - \lambda_t = \frac{-dH}{dx} = -m + \lambda\left(D + \delta + r + \frac{\rho u^*}{2\sqrt{M-x}}\right). \tag{5}$$

When equation 4 is substituted into equation 5 we get:

$$\lambda_{t+1} - \lambda_t = \frac{-dH}{dx} = -m + \lambda\left(D + \delta + r + \frac{\rho^2\lambda}{4}\right), \tag{6}$$

which is a Ricatti Equation. The optimal path will require $\lambda(0) = \lim_{t\to\infty} \lambda_t$. The optimal long term equilibrium of λ_t can be determined by setting $\lambda_{t+1} - \lambda_t = 0$ and solving for λ_t in equation 6. The solution to which is

$$\lim_{t\to\infty} \lambda_t = \frac{-2(D + r + \delta - A))}{\rho^2} \tag{7}$$

where $A = \sqrt{2m\rho^2 + (r + D + \delta)^2}$. Solving equation 6 with the initial condition $\lambda(0) = \lim_{t\to\infty} \lambda_t$ yields

$$\lambda_t = \frac{-2(D + r + \delta - A)}{\rho^2}. \tag{8}$$

If equation 2 is evaluated with the lambda obtained in equation 8 then we get

$$x_{t+1} - x_t = \frac{\lambda\rho^2(M - x_t)}{2} + r(M - x_t) - Dx_t. \tag{9}$$

The solution of equation 9 with initial condition $x(0) = x_0$ is

$$x_t = (x_0 - \frac{M(\lambda\rho^2 + 2r)}{B})e^{\frac{-Bt}{2}} + \frac{M(\lambda\rho^2 + 2r)}{B}. \tag{10}$$

For a particular value of r where $B = 2D + \lambda\rho^2 + 2r$. The full, piece-wise optimal path of x_t is given by

$$x_t = \begin{cases} x_t|_{r=r_k}, & \text{if } x^{-1}(T_k) \leq t \\ & \dots \\ x_t|_{r=r_1}, & \text{if } x^{-1}(T_1) \leq t < x^{-1}(T_2) \\ x_t|_{r=0}, & \text{otherwise} \end{cases}$$

If equation 8 is substituted into equation 4 then the following optimal feedback control is obtained:

$$u*_t = \frac{-(D + r + \delta - A)\sqrt{M - x_t}}{\rho}. \tag{11}$$

Figure 1 illustrates how the state and control functions change when ranking is introduced. The dramatic decreases in the optimal advertising rate are due to each changes in ranking.

The y-axis for $x(t)$ is market share and the y-axis for $u(t)$ is measured in arbitrary cost units. The market in Figure 1 is being modelled as growing linearly at a very slow rate, but is still visible in the upward bend in the optimal state and control functions.

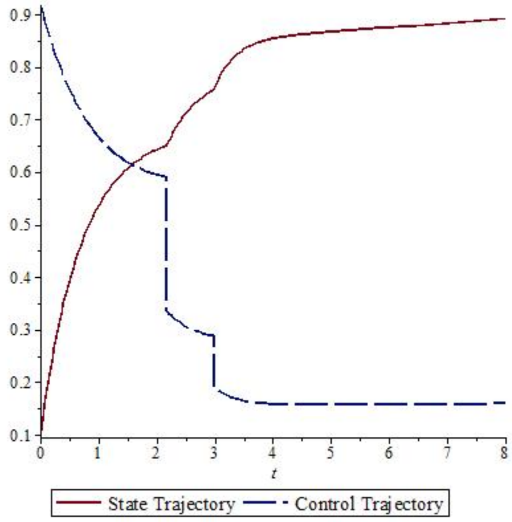

Figure 1: Comparison of $x(t)$ and $u(t)$ in the deterministic case.

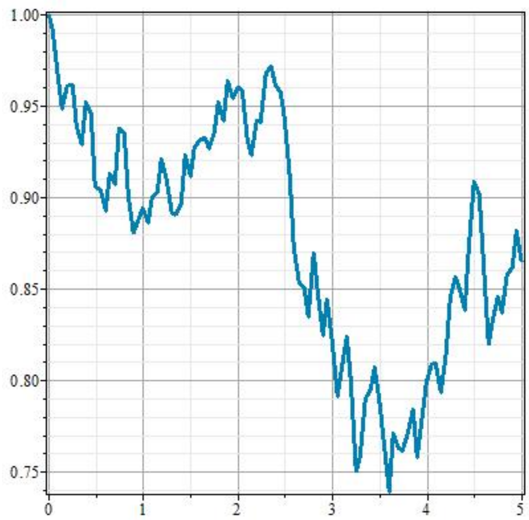

Figure 2: A sample path for the advertising effectiveness over time.

3 STOCHASTIC SOLUTION

A major departure from previous advertising models is made by incorportating dynamics into the effectiveness of advertising. While the effectiveness, ρ is traditionally treated as a constant it will be assumed that ρ evolves randomly through time according to a white noise process $w(\rho_t)$. Doing so takes into account effects like competing ads, changing ad "creatives" and other phenomenon that may impact how receptive an audience is to a given ad campaign. It should be noted that multiple competing firms would be more accurately modelled by a many-player game theoretic approach as seen in (Prasad and Sethi, 2004). The stochastic model that will be used is similar to Equation 2,

$$x_{t+1} - x_t = \rho_t u_t \sqrt{M_t - x_t} + r(x_t)(M_t - x_t) - Dx_t, \tag{12}$$

however it includes the following dynamics:

$$\rho_{t+1} - \rho_t = w(\rho_t). \tag{13}$$

Where $\rho(0) = \rho_0$ and as in Section 1, $0 \leq x_t \leq M_t \ \forall t$ and $x(0) = x_0$. Again we seek to maximize equation 1 but first we solve for the Value function $V(x_t, \rho_t)$. Note that we need not make restrictions on the parity of ρ_t and indeed, a negative value for advertising effectiveness could be interpreted as a poor public view of the firm and/or its products. Shown in Figure 2 is a sample path for how ρ_t may vary over time.

By (Bensoussan, 1982) the current-value function for this problem satisfies the following HJB equation:

$$\delta V(x_t, \rho_t) = \max_{u_t} \left(\frac{\partial V(x_t, \rho_t)}{\partial x_t} (\rho_t u_t \sqrt{M_t - x_t} - Dx_t + r) + \frac{1}{2} \frac{\partial^2 V(x_t, \rho_t)}{\partial \rho_t^2} w^2(\rho_t) + mx_t - u_t^2 \right). \tag{14}$$

It can be easily verified that the optimal control for this case is:

$$u_t* = \frac{1}{2} \frac{\partial V}{\partial x_t} \rho_t \sqrt{M_t - x_t}. \tag{15}$$

Substituting equation 15 into equation 14 yields

$$\delta V = mx_t + \frac{\partial V}{\partial x_t} \frac{\left(\frac{\partial V}{\partial x_t} \rho_t^2 \sqrt{M_t - x_t} - Dx_t + r \right)}{4} + \frac{1}{2} \frac{\partial^2 V}{\partial \rho_t^2} w^2(\rho_t). \tag{16}$$

The details of the solution to equation 16 are left out for the sake of brevity, however the result is

$$\begin{aligned} V = \frac{1}{\delta} \Bigg(& mx_t + \frac{G_t^2 (M_t - x_t)}{4\rho_t^2} - \frac{G_t D x_t}{\rho_t^2} + \frac{G_t r (M_t - x_t)}{\rho_t^2} \\ & + \frac{\sigma^2}{2} \left(\frac{-2m^2 x_t}{H_t^3} - \frac{6mx_t}{\rho_t^2 H_t} + \frac{6G_t x_t}{\rho_t^4} + \frac{2m^2}{\rho_t \delta H_t^2} \right. \\ & \left. - \frac{5G_t m}{\rho_t^3 \delta H_t} - \frac{G_t m^2}{\rho_t \delta H_t^3} + \frac{3G_t^2}{\rho_t^5 \delta} \right) \Bigg) \end{aligned} \tag{17}$$

where $G_t = 2(\sqrt{(\delta + D + r)^2 + m\rho_t^2} - \delta - D - r)$ and $H_t = \frac{G_t}{2} + \delta + D + r$. Inserting equation 17 into equation 15 yields the optimal feedback control for the stochastic problem as shown in equation 18.

$$u_t* = \frac{1}{2\delta}\Bigg(m - \frac{G_t^2}{4\rho_t^2} - \frac{G_t D}{\rho_t^2} - \frac{G_t r}{\rho_t^2} + \sigma^2\left(\frac{3G_t}{\rho_t^4} - \frac{m^2}{H_t^3} - \frac{3m}{\rho_t^2 H_t}\right)\Bigg)\rho_t\sqrt{M_t - x_t} \tag{18}$$

Figure 3 is a sample path for the stochastic optimal feedback control using the advertising effectiveness path seen in Figure 2.

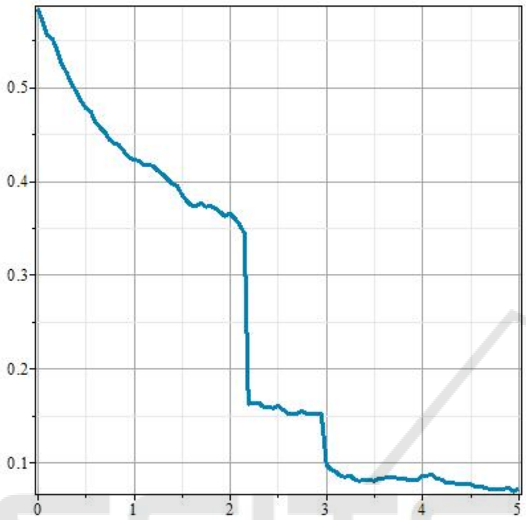

Figure 3: A sample path for optimal advertising spending over time.

4 CONCLUSION

Both the deterministic and stochastic problems have been solved and although the form of the optimal control in the stochastic case is quite complex, it is still straight-forward to implement computationally, allowing advertisers to automate their spending on as granular a level as is feasible. Shown below is a comparison of the controls for the stochastic versions of the Sethi model and the model presented in this paper.

Although the discrete nature of the model does not impact the form of the optimal control, it has been formulated in this way to be more immediately applicable to problems in industry. As Figure 4 illustrates, the ranking plays a large part in the difference between the new and old forms of the optimal control.

A more subtle difference comes from the non-constant size of the market. In Figures 1, 3 and 4 the market was assumed to slightly grow at a constant rate. This is most noticeable in the tail of the functions shown in Figure 1, however Figure 5 exaggerates the growth to demostrate the impact that a non-constant market has on the optimal control. In well-established industries the non-constant market will not likely play a very big part in determining the advertising budget. However, if the market is either quickly exanding or quickly diminishing then this must be taken into account when determining the advertising strategy and the models presented in this paper dictate such a strategy.

The models presented in this paper could be extended to allow for interactions between advertisers, although the budgets and spending of competing advertisers is not typically known and thus such an ex-

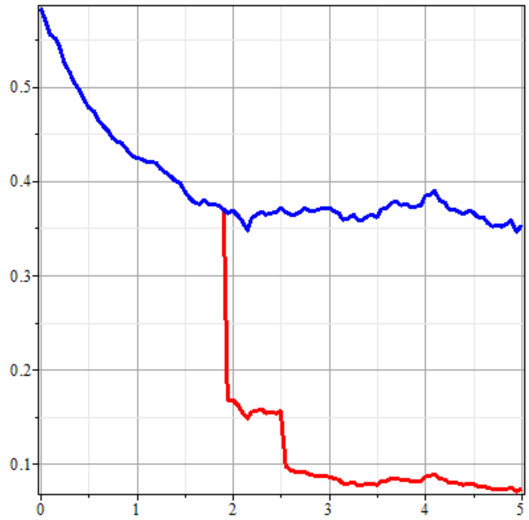

Figure 4: Sample paths for optimal advertising spending over time under the Sethi model (upper path) and new model (lower path).

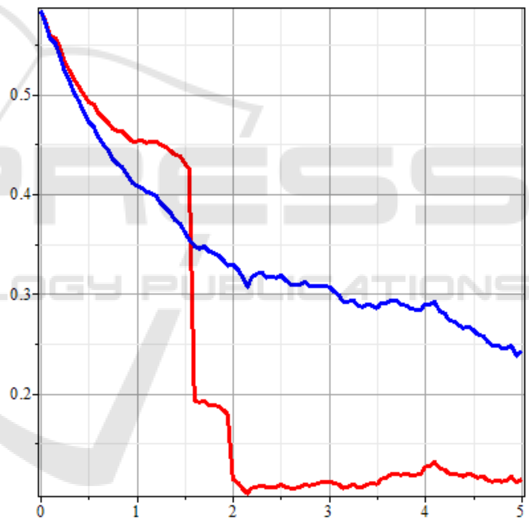

Figure 5: Sample paths for optimal advertising spending over time under the Sethi model and new model where the new model assumes a large linear growth in the market over time.

tension would serve as a largely theoretical exercise. However, Chintagunta and Vilcassim (Chintagunta and Vilcassim, 1992) show that if each player only has knowledge of the past strategies of their opponent then the resulting optimal control formulation more closely aligns with what is empirically observed. Furthermore, the interaction between advertising and the market could be explored. If the market is defined as the set of people who would be willing to purchase the product then for nascent industries the market would be quite small, but could be expanded through advertising efforts.

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