OFF-LINE ROBUSTIFICATION OF EXPLICIT MPC LAWS  
The Case of Polynomial Model Representation

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Abstract: The paper deals with the predictive control for linear systems subject to constraints, technique which leads to nonlinear (piecewise affine) control laws. The main goal is to reduce the sensitivity of these schemes with respect to the model uncertainties and avoid in the same time a fastidious on-line optimisation which may reduce the range of application. In this idea a two stage predictive strategy is proposed, which synthesizes in a first instant an analytical (continuous and piecewise linear) control law based on the nominal model and secondly robustify the central controller (the controller obtained when no constraint is active). This robustification is then expanded to all the space of the piecewise structure by means of its corresponding noise model.

1 INTRODUCTION

The model predictive control (MPC) laws are optimization based techniques which allow constraints handling from the design stage. The analytical formulation of the optimum and its on-line evaluation avoids a challenging optimization from the point of view of the real-time control environment. Solutions in this direction exist at least for two important classes of problems (linear and quadratic) subject to linear constraints due to the Abadie constraint qualification (Goodwin et al., 2004). It must be said that these are in fact a part of a larger class of multiparametric convex programs (Bemporad et al., 2002b) for which exact or approximate algorithms exist (TØndel et al., 2003; Seron et al., 2003; Olaru and Dumur, 2004; Bemporad and Filippi, 2006).

In the case of robust predictive control laws, the model uncertainties and the disturbances can be taken into account at the design stage. A popular technique in this sense is the use of a min-max criterium (in the case when the extreme combination of disturbances or uncertainties are known) (Kerrigan and Maciejowski, 2004; Bemporad et al., 2002a) which comes finally to the resolution of a single multiparametric linear program. The structure of this ultimate optimization is however quite complex and large prediction horizons cannot be handled due to the exponential growth of disturbances realization to be taken into account. In a slightly different manner, by constructing an estimation mechanism (Goodwin et al., 2004) for the constrained variables, one can obtain alternatively a robust control structure, but the multiparametric optimization remains intricate.

A first study on the robustness improvements for the explicit affine feedback policy constructed upon constrained predictive control strategies was presented in (Olaru and Rodríguez-Ayerbe, 2006). The simplest way to proceed is to consider an observer of the state variables (Goodwin et al., 2004), the dimension of the state space being preserved and the piece-wise structure of controller unchanged. The same observer can be used for all feasible regions and can be viewed as noise characterisation of the model. Nevertheless, the observer does not describe the entire class of stabilizing controllers. The present paper presents an improved result based on the Youla-Kučera parametrization which spans the space of stabilizing controllers. For a two-degree of freedom controller, one has access to all the stabilizing controllers that preserve the same input/output behavior, so the Youla-Kučera parameter offers more degrees of freedom than the use of an observer.

The robustification is made such that the state space dimension of the controller is augmented. The main contribution here is the reconstruction of the noise model induced by the central Youla-Kučera...
parameter, in order to use it to generate the corresponding robust piece-wise controller.

In the following, section 2 briefly recalls the predictive control and the Youla-Kučera parametrization. Section 3 details the explicit formulation of the control laws obtained in the constrained case. Section 4 contains the main contribution: the noise model of the Youla-Kučera parameter and the numerical examples are presented in section 5 and the final conclusions in section 6.

2 PREDICTIVE CONTROL

The Generalized Predictive Control (GPC) strategy, introduced in (Clarke et al., 1987), uses for the prediction a CARIMA plant model:

\[ A(q^{-1})y = B(q^{-1})u_{e-1} + C(q^{-1})\xi / \Delta(q^{-1}) \]  

with \( u, y \) the input and output, \( \xi \) a white noise, \( A \) and \( B \) polynomials in the backward shift operator of degrees \( n_a \) and \( n_b \) respectively, and \( \Delta(q^{-1}) = 1 - q^{-1} \) the difference operator. The \( C \) polynomial is the model argument taking into account the noise influence on the system. In the GPC case the cost function to be minimized over a receding horizon is quadratic:

\[
J = \sum_{j=N_1}^{N_2} \left[ w_{e-j} - \hat{y}_{e-j} \right]^2 + \sum_{j=1}^{N_2} \lambda_j \left[ \Delta u_{e-j-1} \right]^2
\]

(2)

where \( N_1, N_2 \) are the costing horizons, \( N_u \) the control horizon, \( \lambda \) the control weighting factor and \( w \) the set-point.

Using the model (1) and the solution of some Diophantine equations (Clarke et al., 1987), this control strategy leads to two-degrees of freedom RST controller, implemented through a difference equation (Figure 1):

\[
\begin{align*}
A(q^{-1})v & = B(q^{-1})u_{e-1} + C(q^{-1})\xi / \Delta(q^{-1}) \\
M(q^{-1}) & = \sum_{j=N_1}^{N_2} \alpha_j q^{j}(CE_j - E_j) \\
M & = [\alpha_{N_1} \ldots \alpha_{N_2}] \\
R & = R'C + A\Delta M \\
S & = S'C - q^{-1}BM \\
T & = TC
\end{align*}
\]

(3)

with:

\[
M(q^{-1}) = \sum_{j=N_1}^{N_2} \alpha_j q^{j}(CE_j - E_j)
\]

(4)

\[
M = [\alpha_{N_1} \ldots \alpha_{N_2}]
\]

(5)

\[
R = R'Q_{den} + \Delta AQ_{num}
\]

\[
S = S'Q_{den} - q^{-1}BQ_{num}
\]

\[
T = TQ_{den}
\]

(6)

where \( Q(q^{-1}) \) is a stable transfer function.

The choice of the \( Q \) parameter is a complex problem on its own but it is not the subject of the current paper. The methods presented in (Rodriguez and Dumur, 2005; Rossiter 2003; Ansay et al., 1998; Yoon and Clarke, 1995; Kouvaritakis et al., 1992) can be used for the choice of this parameter.

Comparing (3) and (6) it turns out that the controller for \( C \neq 1 \) is obtained for \( Q = M/C \). As \( M \) depends of \( C \) as shown by (4), the robustification by the \( C \) polynomial has less degrees of freedom than the robustification by Youla-Kučera parameter (Yoon and Clarke, 1995).

3 EXPLICIT CONSTRAINED GPC LAWS

In the case when the GPC law is subject to constraints, the optimization has to be solved with respect to a feasible domain. If the considered constraints are stated on the control action, on the control increment, on the plant outputs or any other signal related by a CARIMA model to the control signal, then one can restate them in a form depending only on the control increment, leading to a set of linear constraints (Ehrlinger et al., 1996):
(7)

\[ \begin{align*}
A_{\mu} k_{\mu} & \leq B_{\mu} p_{t} + b_{in} \\
A_{eq} k_{eq} & = B_{eq} p_{t} + b_{eq}
\end{align*} \]

involving the optimization argument \( k_{\mu}(t) \) and the vector of context parameters:

\[ p_{t} = \begin{bmatrix} y_{past} & \{ y_{past} \}^{T} & \Delta u_{past}^{T} & w^{T} \end{bmatrix}^{T} \]

which regroups a finite sequence of past inputs \( \Delta u \), future setpoints \( w \), past outputs \( y \) and present and future values of the signals under constraints (noted for short \( y_{past} \)).

It is interesting to observe that this set of constraints characterizes in fact a parameterized polyhedron (Olaru and Dumur, 2004) in the optimization argument space. The optimum will lie on a combination of its parameterized vertices and thus one can construct the explicit solution for the multiparametric optimization:

The use of a dual representation of the feasible domain and projection mechanisms (see Olaru and Dumur, 2004 and 2005) can be advantageous in order to express the optimality conditions if there exist unbounded feasible directions.

\begin{align*}
\min_{k_{\mu}(t)} J \\
\text{subject to: } & A_{\mu} k_{\mu}(t) \leq B_{\mu} p_{t} + b_{in} \\
& A_{eq} k_{eq}(t) = B_{eq} p_{t} + b_{eq}
\end{align*} \tag{9}

Subsequently, the predictive control law can be described explicitly upon the solution of (9) as a piecewise affine function of the vector of parameters (Seron et al., 2003; Bemporad et al., 2002; Olaru and Dumur, 2004).

\[ k_{\mu}^{*}(p) = K_{in} p + K_{eq} \text{ for } p \in CR_{\mu} \] \tag{10}

with \( CR_{\mu} \), critical regions in the space of context parameters, The GPC subjet to constraints has a piecewise RST polynomial form:

\[ S_{\mu}(q^{-1}) \Delta u_{t} = -R_{\mu}(q^{-1})y_{t} + T_{\mu}(q)w_{t} + V_{\mu} y_{past} + t_{\mu} \] \tag{11}

for \( \begin{bmatrix} y_{past} & \{ y_{past} \}^{T} & \Delta u_{past}^{T} & w^{T} \end{bmatrix}^{T} \in CR_{\mu} \).

The structure of such a piecewise controller is shown in Figure 2. Once the look-up table of local polynomial RST laws is available, an efficient positioning mechanism (based on a search tree) can be constructed such that the on-line evaluation routine can find the optimal control action according to the GPC philosophy (TÖndel et al., 2003).

4 ROBUSTIFICATION

At this stage, it is assumed that the design of initial controller has been performed with \( N_{1}, N_{2}, N_{s}, \lambda \) adjusted to satisfy the required input/output behavior. The resulting piecewise two-degrees of freedom RST controller will be denoted \( R_{\mu}^{*}, S_{\mu}^{*}, T_{\mu}^{*} \), in the following sections.

The observer based robustification corresponds to the consideration of an observer of the state. In the case of GPC, this corresponds to a choice of a \( C \) polynomial. The roots of this polynomial correspond to the poles of the observer. The obtained piecewise controller can be implemented as in Figure 2, see also (Olaru and Rodríguez-Ayerbe, 2006, Camacho and Bordons, 2004 ; Bitmead et al.,1990).

As the use of an observer corresponds to the consideration of a \( C \) polynomial for the noise influence on the CARIMA model (1), the idea is to find the corresponding noise model of the Youla-Kučera parameter. To do this, an extra polynomial is added in the model of the system. A \( D(q^{-1}) \) polynomial will appear as following:

\[ A(q^{-1})y_{t} = B(q^{-1})u_{t-1} + \frac{C(q^{-1})z_{t}}{D(q^{-1})\Delta(q^{-1})} \] \tag{12}
With this new model, Diophantine equations are:
\[
\Delta(q^{-1})A(q^{-1})D(q^{-1})E(q^{-1})F(q^{-1}) + q^{-1}E(q^{-1})F(q^{-1}) = C(q^{-1}) \tag{13}
\]
\[
G(q^{-1})C(q^{-1}) + q^{-1}H(q^{-1}) = B(q^{-1})E(q^{-1})D(q^{-1}) \tag{13}
\]

Finding the relation between the controller obtained for \(C=D=1\) and the one obtained for \(C \neq 1\) and \(D \neq 1\), we obtain something similar to (3). Considering \(\tilde{R}, \tilde{S}, \tilde{T}\) the controller obtained with \(C=D=1\) and \(\tilde{R}, \tilde{S}, \tilde{T}\) whose obtained with \(C \neq 1\) and \(D \neq 1\), the following relations are obtained:
\[
\tilde{R} = R'C + A\tilde{M} \quad \tilde{S} = S'C - q^{-1}B\tilde{M} \quad \tilde{T} = T'C \tag{14}
\]

With, (see Appendix for structural details):
\[
\tilde{M}(q^{-1}) = \sum_{n=1}^{N_2} \alpha_n q^{-(CE_n' - \tilde{E},D)} \tag{15}
\]

So, the \(D\) polynomial corresponding to the considered Youla-Kučera parameter must verify:
\[
Q_{\text{num}}(q^{-1}) = \tilde{M}(q^{-1}) = \sum_{n=1}^{N_2} \alpha_n q^{-(CE_n' - \tilde{E},D)} \tag{16}
\]

Once the corresponding noise model has been obtained, it can be used to regenerate the piecewise affine controller. The same input/output behaviour as for the initial one is assured, in the ideal case of no model errors. A modified close loop behaviour will be observed with respect to disturbance rejection, robustness, etc.

The resolution of (16) is a non linear problem that can be undertaken with standard optimization methods. Nevertheless, it is not always possible to guarantee a real solution. The resolution of (16) and its limitations are raising interesting questions, research being currently conducted on this subject. From a practical point of view, any such limit case can be avoided by retuning the initial predictive control parameters or the robustification specification.

5 EXAMPLE

Consider the position control of an induction motor, with 1.0724 ms as sampling period
\[
H(q^{-1}) = \frac{\theta(q^{-1})}{\tau_{\text{ref}}(q^{-1})} = \frac{10^{-4}(0.821q^{-1} + 0.8206q^{-2})}{(1 - q^{-1})(1 - 0.998q^{-1})} \tag{17}
\]

Constraints in control amplitude are considered:
\[\tau_{\text{ref}} \in [\tau_{\text{max}} - \tau_{\text{min}}] \text{ and } \tau_{\text{max}} = 1.8. \]

An initial GPC controller is designed with \(C = D = 1\) with the following tuning parameters: \(N_1 = 1, \quad N_2 = 16, \quad \lambda = 0.0001\) and \(N_s = 2\). The position of the motor is obtained through an encoder of 14400 points per rotation, and the highs dynamics of the system (current loop, inverter dynamic, mechanic dynamics in high frequency) have been not identified.

This initial controller is obtained with (9). A piecewise linear controller with 9 regions is obtained. The central region corresponds to the case where no constraint is active. This controller will be noted \(R_0, S_0, T_0\).

To robustify off-line this piecewise controller, the idea is to robustify the central one \((R_0, S_0, T_0)\) and expand this robustification to other regions. In this way a Youla-Kučera parameter has been obtained by method described in (Rodriguez and Dumur, 2005). The following parameter is considered.
\[
Q = -4196.2 + 10499.99q^{-1} + 8902.17q^{-2} + 2541.93q^{-3} \\
1 - 3.565q^{-1} + 4.858q^{-2} - 2.973q^{-3} + 0.7q^{-4} \tag{18}
\]

With (6), we obtain the controller \(R_{\text{off}}, S_{\text{off}}, T_{\text{off}}\). Solving (16) with \(C = Q_{\text{off}}\), the following \(D\) polynomial is obtained:
\[
D = 1 - 0.873q^{-1} + 0.472q^{-2} - 0.018q^{-3} + 0.426q^{-4} \tag{19}
\]

This value has been obtained by available optimization methods (classical Matlab routines in occurrence) as long as (16) represents a set of non linear equations difficult to solve analytically.

With this \(D\) polynomial, the optimization problem (9) can be solved but this time with matrices obtained from (13) for \(C = Q_{\text{off}}\) and \(D\) as in (19). The solution of this new optimization problem leads to a new piecewise controller with 9 regions, as the initial one. The central controller of this piecewise controller correspond to \(R_{\text{off}}, S_{\text{off}}, T_{\text{off}}\).

Figures 3 and 4 show the obtained simulations results for a filtered step reference considering a second order neglected dynamic in high frequency of the following characteristics: \(\omega_n = 1000 \text{rad/s}\) and \(\xi = 0.3\).

In these figures we can observe that the obtained behaviour is stable in the case of robustified controller and instable in the case of initial controller. So, the robustified controller has better
behaviour towards uncertainties in high frequency and the continuity between regions is guaranteed.

Figure 3: Position reference, position error, control signal and active region for the initial controller and uncertain model.

Figure 4: Position reference, position error, control signal and active region for the robustified controller and uncertain model.

6 CONCLUSIONS

The paper investigated the robustification methods for the control laws obtained in a constrained predictive control framework. The idea is to design in a first instance a piecewise polynomial controller which satisfy the basic demands in terms of tracking performances. In a second stage, the same predictive control structure (prediction horizon, weightings, etc.) is robustified using the model arguments accounting for the noise influence. The idea is similar to that of using a fixed observer, but exploring all the class of stabilizing controllers of the unconstrained system. This increases the number of degrees of freedom.

The robustification of initial unconstrained controller is made using the Youla-Kučera parametrization, and then this robustification is expanded to all the piecewise structure of the controller. For this, the noise model corresponding to the Youla-Kučera parameter is found, and use to regenerate the robust piecewise controller by preserving the same input/output behavior but being more robust.

The limitations of the method are in the existence of the corresponding noise model of the Youla-Kučera parameter. This is transparent in the resolution of a non linear equation system. The robustification being done off-line, any infeasibility can be handled by retuning the GPC parameters.

REFERENCES


**APPENDIX**

By solving the first Diophantine equation of (13) for \( C=R=1 \), \( D \neq 1 \), a relationship between the \( R' \) polynomial obtained for \( C=R=1 \) and the \( \tilde{R} \) obtained for \( C \neq 1, D \neq 1 \) is obtained:

\[
\begin{align*}
\left[ \Delta(q^{-1})A(q^{-1})E'_i(q^{-1}) + q^{-1}F'_i(q^{-1}) \right] &= 1 \\
\left[ \Delta(q^{-1})A(q^{-1})D(q^{-1})E'_i(q^{-1}) + q^{-1}F'_i(q^{-1}) \right] &= C(q^{-1}) \\
\Rightarrow \tilde{F}'_i &= CF'_i + \Delta Aq^i (CE'_i - D\tilde{E}_i) \\
R \text{ is obtained as: } R &= \sum_{i=1}^{N_1} \alpha_i F'_i \text{ (see Yoon and Clarke 1995).} \\
\end{align*}
\]

For \( \tilde{R} : 
\begin{align*}
\tilde{R} &= \sum_{i=1}^{N_1} \alpha_i \tilde{F}'_i = \sum_{i=1}^{N_1} \alpha_i \left( CF'_i + \Delta Aq^i (CE'_i - D\tilde{E}_i) \right) \\
&= CR' + \Delta \sum_{i=1}^{N_1} \alpha_i q^i \left( CE'_i - D\tilde{E}_i \right) \\
&= CR' + \Delta M
\end{align*}
\]

With the same development, the corresponding expression for \( \tilde{S} \) of (14) is obtained. Solving (16) comes to the identification of a \( D \) polynomial corresponding to \( Q_{\text{con}}(q^{-1}) \). It must be noted that the first \( i \) coefficients of \( E'_i \) are the same than those of \( E_i \) polynomials. (\( E_i \) is a polynomial of degree \( i-1 \).) With this, the solution of \( D \) for the numerical example has been obtained solving the following non linear equations:

\[
\begin{align*}
D &= 1 + d_1 q^{-1} + d_2 q^{-2} + d_3 q^{-3} + d_4 q^{-4} \\
C &= 1 + c_1 q^{-1} + c_2 q^{-2} + c_3 q^{-3} + c_4 q^{-4} \\
E' &= 1 + e'_1 q^{-1} + \ldots + e'_{15} q^{-15} \\
\tilde{E} &= 1 + \tilde{e}_1 q^{-1} + \ldots + \tilde{e}_{15} q^{-15} \\
M &= m_0 + m_1 q^{-1} + m_2 q^{-2} + \ldots + m_{15} q^{-15}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
c_1 & c_2 & c_3 & 1 & e'_1 & \ldots & e'_{15} \\
c_2 & c_1 & 0 & 0 & 1 & \ldots & e'_{14} \\
c_3 & c_2 & c_1 & 0 & 0 & 0 & \ldots & e'_{13} \\
c_4 & c_3 & c_2 & c_1 & 1 & \ldots & e'_{12} \\
0 & c_4 & c_3 & c_2 & c_1 & 0 & \ldots & e'_{11} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots & e'_{1} \\
1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\vdots \\
\alpha_{15}
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
d_1 & d_2 & d_3 & d_4 & \tilde{e}_1 & \ldots & \tilde{e}_{15} \\
d_2 & d_1 & 0 & 0 & 1 & \ldots & \tilde{e}_{14} \\
d_3 & d_2 & d_1 & 0 & 0 & 0 & \ldots & \tilde{e}_{13} \\
d_4 & d_3 & d_2 & d_1 & 1 & \ldots & \tilde{e}_{12} \\
0 & d_4 & d_3 & d_2 & d_1 & 0 & \ldots & \tilde{e}_{11} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & d_4 & d_3 & d_2 & d_1 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\vdots \\
\alpha_{15}
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\end{align*}
\]

With \( x = (d_1 d_2 d_3 d_4 \tilde{e}_1 \ldots \tilde{e}_{15}) \), the following problem has been solved:

\[
F(x) = 0
\]

With \( F \) defined by the matrix relations (20) and (21), and using a standard optimization routine.