On the Asymptotic Stability Analysis of a Certain Type of Discrete-time 3-D Linear Systems

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Abstract: This work is concerned with the analysis of 3-d (3-dimensional) systems. The aim is to establish conditions that guarantee the asymptotic stability of these kinds of systems. To accomplish it, the Lagrange candidate solutions method for partial difference equations is adopted here. We show that the systems are asymptotically stable if the entries of the matrices of their state space descriptions yield a solution in the Lagrange solution sense. Furthermore, the particular cases in which the matrices can be turned into a diagonal matrix by means of the canonical transformation is studied in order to figure out the role of the eigenvalues on the stability conditions.

1 INTRODUCTION

Multidimensional linear systems theory grew out of theoretical and practical research on systems that can be mathematically modeled and described by partial difference and differential equations; and the field, which is an interdisciplinary area that shares common grounds with engineering, mathematics, physics and other sciences, has been developing ever since with each section owing its particular tools and techniques to handle the problems (see for example (Bose, 1982; Tzafestas, 1986; Wall, 1987; Lim, 1990; Leondes, 1995; Jerri, 1996; Zerz, 2000; Du and Xie, 2002; Matsuo and Hasegawa, 2003; Cheng, 2003; Rosenthal and Gilliam, 2003; Wood, 2004; Elaydi, 2005; Russell and Cohn, 2013)).

The investigations of multidimensional linear systems in engineering date back to the 1960s when a certain kind of electrical system network was modeled by as a set of partial differential equations in two independent variables, namely as 2-d linear system (Kasami, 1960; Ansell, 1964). In the 1970s, the state space description (Roesser, 1972; Attasi, 1973; Roesser, 1975; Marchesini, 1978) and the matrix polynomial (Juri, 1978; Bose, 1982) approaches were established and these frameworks prevailed until the 1990's when the energy method matured as a fruitful formalism to the analysis and design of multidimensional control systems (W. S. Lu, 1992). Eventually, this reasoning evolved into linear matrix inequalities that have been applied to 2-d discrete linear multidimensional systems (Izuta, 2007a; Izuta, 2007b; Izuta, 2007c).

More recently, unlike the techniques so far, the authors pursued a solution to 2-d discrete linear control systems on grounds of the Lagrange method for solving partial difference equations from the controller design standpoint (Izuta, 2010a; Izuta, 2010b). Briefly, the key contribution of these studies was to show how to obtain an explicit solution to the system of partial difference equations based on the Lagrange method when a solution to them exists.

Motivated by these works, in this paper we are concerned with the asymptotic stability analysis of discrete-time 3-d linear systems in the scope of Lagrange solutions to partial difference equations. The state space description of the system consists of two matrices on its right hand side. One of them is related to the current states and the other one corresponds to the states with smaller values indices, which in the ordinary 1-d systems theory, these kinds of states are often called systems with state delays.

Taking these into consideration, the aims of this work are: (1) to extend the previous result to 3-d systems; (2) to establish conditions on the entries of the matrix of the state space description in order to guarantee the asymptotic stability; and (3) to shed some light on the link between the eigenvalues and the Lagrange solutions when the matrix composing the system description can be transformed into a diagonal matrix by means of the canonical transformation.

Finally, the paper is organized as follows: in sec-

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tion 2, the discrete-time 3-d linear system and the assumptions are presented; the asymptotic stability conditions are establish in section 3; and some final remarks are enunciated in section 4 .

2 PRELIMINARIES

In this section we provide the definitions and concepts that define the scope as well as the problem to be investigated in the work. To begin with, the system is described in the following definition.

Definition 1. *The 3-d system is described by the system of partial difference equations given by*

$$\begin{bmatrix} x_1(i+1,j,k) \\ x_2(i,j+1,k) \\ x_3(i,j,k+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1(i,j,k) \\ x_2(i,j,k) \\ x_3(i,j,k) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} x_{1d} \\ x_{2d} \\ x_{3d} \end{bmatrix} \tag{1}$$

in which

$$\begin{bmatrix} x_{1d} \\ x_{2d} \\ x_{3d} \end{bmatrix} \doteq \begin{bmatrix} x_1(i-\Delta_1, j-\Delta_2, k-\Delta_3) \\ x_2(i-\Delta_1, j-\Delta_2, k-\Delta_3) \\ x_3(i-\Delta_1, j-\Delta_2, k-\Delta_3) \end{bmatrix}$$

and $i,\ j,\ k,\ \Delta_\star \in \mathbf{Z}$ ($\forall\star$), $x_*(i,j,j)$, $x_*(i,j,j) \in \Re$, $\forall *$, are the states of the system, a_{**} and b_{**}, $\forall * \star$ are real valued entries of the matrices.

Remark 1. *Ruling out the terms related to $\Delta_\star$, $\forall\star$, the state space notation (1) reduces to one introduced by Roesser (Roesser, 1975).*

Remark 2. *Hereafter for the sake of compact notations and when convenient we reference to the first matrix on the right hand side of (13) as matrix A and the second one as matrix B. Moreover, in some cases we abusively reference to system (1) as $x_{+1} = Ax + Bx_\Delta$.*

Next, we state the definition of asymptotic stability and the kind of solution searched for in this investigation. The statement of asymptotic stability is conceptually the same as in (Jerri, 1996).

Definition 2. *System (1) is asymptotically stable if its solutions satisfy*

$$\begin{array}{l} \lim_{(i+j+k)\to\infty} \mid x_1(i,j,k) \mid \to 0 \\ \lim_{(i+j+k)\to\infty} \mid x_2(i,j,j) \mid \to 0 \\ \lim_{(i+j+k)\to\infty} \mid x_3(i,j,j) \mid \to 0 \end{array} \tag{2}$$

Furthermore, if there exist $0 \neq |\alpha_1|$, $|\beta_1|$, $|\gamma_1|$, $|\alpha_2|$, $|\beta_2|$, $|\gamma_2|$, $|\alpha_3|$, $|\beta_3|$, $|\gamma_3| < 1$ such that the so called Lagrange solution candidates, which are given by

$$\begin{array}{l} x_1(i,j,k) = \alpha_1^i \beta_1^j \gamma_1^k \\ x_2(i,j,k) = \alpha_2^i \beta_2^j \gamma_2^k \\ x_3(i,j,k) = \alpha_3^i \beta_3^j \gamma_3^k \end{array} \tag{3}$$

can be established, then

$$\begin{array}{l} \lim_{(i+j+k)\to\infty}(\mid x_1(i,j,k) \mid + \mid x_2(i,j,k) \mid \\ \quad + \mid x_3(i,j,k) \mid) \to 0 \end{array} \tag{4}$$

holds for $i,\ j \geq 0$ and the symbol $\mid * \mid$ meaning the conventional vector norm of $*$; and consequently, the system is asymptotically stable.

Remark 3. *Basically, (4) means that as the indices increase the sum of absolute values of the states vanish.*

Finally, the aim of this investigation is to establish the conditions that the system has to satisfy in order to be asymptotically stable in the sense of the Lagrange solution method.

3 RESULTS

Here the system is analyzed and asymptotic stability conditions are pursued. First, the existence of a solution is given by the conditions stated in the following theorem.

Theorem 1. *Consider system (1). Then there exists an asymptotically stable solution on the grounds of the Lagrange solutions if either or both of the following conditions hold.*

- *Condition 1:*

$$\begin{array}{l} \alpha_1 = \alpha_2 = \alpha_3 \doteq \alpha \\ \beta_1 = \beta_2 = \beta_3 \doteq \beta \\ \delta_1 = \delta_2 = \delta_3 \doteq \beta \end{array} \tag{5}$$

- *Condition 2:*

$$\begin{array}{l} \alpha_1^{\Delta_1+1}\beta_1^{\Delta_2}\gamma_1^{\Delta_3} - a_{11}\alpha_1^{\Delta_1}\beta_1^{\Delta_2}\gamma_1^{\Delta_3} - b_{11} = 0 \\ \alpha_2^{\Delta_1}\beta_2^{\Delta_2+1}\gamma_2^{\Delta_3} - a_{22}\alpha_2^{\Delta_1}\beta_2^{\Delta_2}\gamma_2^{\Delta_3} - b_{22} = 0 \\ \alpha_3^{\Delta_1}\beta_3^{\Delta_2}\gamma_3^{\Delta_3+1} - a_{33}\alpha_3^{\Delta_1}\beta_3^{\Delta_2}\gamma_3^{\Delta_3} - b_{33} = 0 \end{array} \tag{6}$$

and

$$\begin{array}{l} a_{12}\alpha_2^{\Delta_1}\beta_2^{\Delta_2}\gamma_2^{\Delta_3} + b_{12} = 0 \\ a_{21}\alpha_1^{\Delta_1}\beta_1^{\Delta_2}\gamma_1^{\Delta_3} + b_{21} = 0 \\ a_{31}\alpha_1^{\Delta_1}\beta_1^{\Delta_2}\gamma_1^{\Delta_3} + b_{31} = 0 \end{array} \tag{7}$$

and

$$\begin{array}{l} a_{13}\alpha_3^{\Delta_1}\beta_3^{\Delta_2}\gamma_3^{\Delta_3} + b_{13} = 0 \\ a_{23}\alpha_3^{\Delta_1}\beta_3^{\Delta_2}\gamma_3^{\Delta_3} + b_{23} = 0 \\ a_{32}\alpha_2^{\Delta_1}\beta_2^{\Delta_2}\gamma_2^{\Delta_3} + b_{32} = 0 \end{array} \tag{8}$$

Proof. To begin with, rewrite the equations of system (1) as

$$\begin{aligned} x_1(i+1,j,k) &= a_{11}x_1(i,j,j)+a_{12}x_2(i,j,k)\\ &+a_{13}x_3(i,j,k)\\ &+b_{11}x_1(i-\Delta_1,j-\Delta_2,k-\Delta_3)\\ &+b_{12}x_2(i-\Delta_1,j-\Delta_2,k-\Delta_3)\\ &+b_{13}x_3(i-\Delta_1,j-\Delta_2,k-\Delta_3)\end{aligned}$$

for the first entry of the matrix of the system and

$$\begin{aligned} x_2(i,j+1,k) &= a_{21}x_1(i,j,j)+a_{22}x_2(i,j,k)\\ &+a_{23}x_3(i,j,k)\\ &+b_{21}x_1(i-\Delta_1,j-\Delta_2,k-\Delta_3)\\ &+b_{22}x_2(i-\Delta_1,j-\Delta_2,k-\Delta_3)\\ &+b_{23}x_3(i-\Delta_1,j-\Delta_2,k-\Delta_3)\end{aligned} \tag{9}$$

for the second one. Finally,

$$\begin{aligned} x_3(i,j,k+1) &= a_{31}x_1(i,j,j)+a_{32}x_2(i,j,k)\\ &+a_{33}x_3(i,j,k)\\ &+b_{31}x_1(i-\Delta_1,j-\Delta_2,k-\Delta_3)\\ &+b_{32}x_2(i-\Delta_1,j-\Delta_2,k-\Delta_3)\\ &+b_{33}x_3(i-\Delta_1,j-\Delta_2,k-\Delta_3)\end{aligned}$$

for the last one. Now substitute (3) into it to come up with the equations

$$\begin{aligned} &\alpha_1^{i-\Delta_1}\beta_1^{j-\Delta_2}\gamma_1^{k-\Delta_3}\times(\alpha_1^{\Delta_1+1}\beta_1^{\Delta_2}\gamma_1^{\Delta_3}\\ &-a_{11}\alpha_1^{\Delta_1}\beta_1^{\Delta_2}\gamma_1^{\Delta_3}-b_{11})=\alpha_2^{i-\Delta_1}\beta_2^{j-\Delta_2}\gamma_2^{k-\Delta_3}\\ &\times(a_{12}\alpha_2^{\Delta_1}\beta_2^{\Delta_2}\gamma_2^{\Delta_3}+b_{12})+\alpha_3^{i-\Delta_1}\beta_3^{j-\Delta_2}\gamma_3^{k-\Delta_3}\\ &\times(a_{13}\alpha_3^{\Delta_1}\beta_3^{\Delta_2}\gamma_3^{\Delta_3}+b_{13})\end{aligned} \tag{10}$$

and

$$\begin{aligned} &\alpha_2^{i-\Delta_1}\beta_2^{j-\Delta_2}\gamma_2^{k-\Delta_3}\times(\alpha_2^{\Delta_1}\beta_2^{\Delta_2+1}\gamma_2^{\Delta_3}\\ &-a_{22}\alpha_2^{\Delta_1}\beta_2^{\Delta_2}\gamma_2^{\Delta_3}-b_{22})=\alpha_1^{i-\Delta_1}\beta_1^{j-\Delta_2}\gamma_1^{k-\Delta_3}\\ &\times(a_{21}\alpha_1^{\Delta_1}\beta_1^{\Delta_2}\gamma_1^{\Delta_3}+b_{21})+\alpha_3^{i-\Delta_1}\beta_3^{j-\Delta_2}\gamma_3^{k-\Delta_3}\\ &\times(a_{23}\alpha_3^{\Delta_1}\beta_3^{\Delta_2}\gamma_3^{\Delta_3}+b_{23})\end{aligned} \tag{11}$$

and finally

$$\begin{aligned} &\alpha_3^{i-\Delta_1}\beta_3^{j-\Delta_2}\gamma_3^{k-\Delta_3}\times(\alpha_3^{\Delta_1}\beta_3^{\Delta_2}\gamma_3^{\Delta_3+1}\\ &-a_{33}\alpha_3^{\Delta_1}\beta_3^{\Delta_2}\gamma_3^{\Delta_3}-b_{33})=\alpha_1^{i-\Delta_1}\beta_1^{j-\Delta_2}\gamma_1^{k-\Delta_3}\\ &\times(a_{31}\alpha_1^{\Delta_1}\beta_1^{\Delta_2}\gamma_1^{\Delta_3}+b_{31})+\alpha_2^{i-\Delta_1}\beta_2^{j-\Delta_2}\gamma_2^{k-\Delta_3}\\ &\times(a_{32}\alpha_2^{\Delta_1}\beta_2^{\Delta_2}\gamma_2^{\Delta_3}+b_{32})\end{aligned} \tag{12}$$

From these equations we have that the left hand terms depend on the indices i, j and k whereas the expressions on the right are independent from them. Moreover since the equalities must hold for all values of the indices, we have that in order to have an asymptotically stable system, the claim of the theorem must hold. $\square$

Theorem 2. *Consider system (1) and let*

$$\begin{aligned} \bar{A}_\star &\doteq \textstyle\sum_{t=1,\cdots,3} a_{\star t}, \quad \star=1,\cdots,3\\ \bar{B}_* &\doteq \textstyle\sum_{s=1,\cdots,3} b_{*s}, \quad *=1,\cdots,3\\ \hat{C}_{\star *} &\doteq \bar{A}_\star\times\bar{B}_*\end{aligned} \tag{13}$$

Then there exists a Lagrange solution on the grounds of the equalities (5) if either of the following conditions are fulfilled.

- *Condition 1: there exists an* α, $|\alpha|<1$ *subject to*

$$\begin{aligned} &m_\alpha<\alpha<M_\alpha\\ &m_\alpha \doteq \max\left\{\frac{\hat{C}_{21}+\hat{C}_{12}-\bar{B}_1}{\bar{B}_2},\frac{\hat{C}_{13}-\hat{C}_{31}-\bar{B}_1}{\bar{B}_3}\right\}\\ &M_\alpha \doteq \max\left\{\frac{\hat{C}_{21}+\hat{C}_{12}+\bar{B}_1}{\bar{B}_2},\frac{\hat{C}_{13}-\hat{C}_{31}+\bar{B}_1}{\bar{B}_3}\right\}\end{aligned} \tag{14}$$

- *Condition 2: there exists a* β, $|\beta|<1$ *subject to*

$$\begin{aligned} &m_\beta<\beta<M_\beta\\ &m_\beta \doteq \max\left\{\frac{\hat{C}_{21}-\hat{C}_{12}-\bar{B}_2}{\bar{B}_1},\frac{\hat{C}_{23}-\hat{C}_{32}-\bar{B}_2}{\bar{B}_3}\right\}\\ &M_\beta \doteq \max\left\{\frac{\hat{C}_{21}-\hat{C}_{12}+\bar{B}_2}{\bar{B}_1},\frac{\hat{C}_{23}-\hat{C}_{32}+\bar{B}_2}{\bar{B}_3}\right\}\end{aligned} \tag{15}$$

- *Condition 3: there exists a* γ, $|\gamma|<1$ *subject to*

$$\begin{aligned} &m_\gamma<\gamma<M_\gamma\\ &m_\gamma \doteq \max\left\{\frac{\hat{C}_{31}-\hat{C}_{13}-\bar{B}_3}{\bar{B}_1},\frac{\hat{C}_{32}-\hat{C}_{23}-\bar{B}_3}{\bar{B}_2}\right\}\\ &M_\gamma \doteq \max\left\{\frac{\hat{C}_{31}-\hat{C}_{13}+\bar{B}_3}{\bar{B}_1},\frac{\hat{C}_{32}-\hat{C}_{23}+\bar{B}_3}{\bar{B}_2}\right\}\end{aligned} \tag{16}$$

Proof. Since (5) holds, we can write equations (10) through (12) as

$$\begin{aligned} (\alpha-\bar{A}_1)\alpha^{\Delta_1}\beta^{\Delta_2}\gamma^{\Delta_3} &= \bar{B}_1\\ (\beta-\bar{A}_2)\alpha^{\Delta_1}\beta^{\Delta_2}\gamma^{\Delta_3} &= \bar{B}_2\\ (\gamma-\bar{A}_3)\alpha^{\Delta_1}\beta^{\Delta_2}\gamma^{\Delta_3} &= \bar{B}_3\end{aligned} \tag{17}$$

from which we get the equations

$$\begin{aligned} \alpha\bar{B}_2-\beta\bar{B}_1 &= \hat{C}_{12}-\hat{C}_{21}\\ \alpha\bar{B}_3-\gamma\bar{B}_1 &= \hat{C}_{13}-\hat{C}_{31}\\ \beta\bar{B}_3-\gamma\bar{B}_2 &= \hat{C}_{23}-\hat{C}_{32}\end{aligned} \tag{18}$$

from which the solutions α, β and γ can not be uniquely established since an equivalent matrix representation of (18) given by

$$\begin{aligned} &\begin{bmatrix} \bar{B}_2 & -\bar{B}_1 & 0\\ \bar{B}_3 & 0 & -\bar{B}_1\\ 0 & \bar{B}_3 & -\bar{B}_2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}\\ &\qquad =\begin{bmatrix}\hat{C}_{12}-\hat{C}_{21}\\ \hat{C}_{13}-\hat{C}_{31}\\ \hat{C}_{23}-\hat{C}_{32}\end{bmatrix}\end{aligned} \tag{19}$$

has a singular matrix on its left hand side. Thus working on the first equation of (18), we have that

$$\beta=\frac{(\alpha-\bar{A}_1)\bar{B}_2+\hat{C}_{21}}{\bar{B}_1} \tag{20}$$

and since we must have $|\beta|<1$ to have an asymptotically stable solution, the right side of (20) yields

$$\frac{-\bar{B}_1+\hat{C}_{21}+\hat{C}_{12}}{\bar{B}_2}<\alpha<\frac{\bar{B}_1+\hat{C}_{21}+\hat{C}_{12}}{\bar{B}_2} \tag{21}$$

Furthermore, plugging (20) into the third equation in (18) leads to

$$\gamma = \frac{(\alpha - \bar{A}_1)\bar{B}_3 + \hat{C}_{31}}{\bar{B}_1} \tag{22}$$

which is part of an asymptotically stable solution if $|\gamma| < 1$, which is equivalent to the condition

$$\frac{-\bar{B}_1 - \hat{C}_{31} + \hat{C}_{13}}{\bar{B}_3} < \alpha < \frac{\bar{B}_1 + \hat{C}_{31} + \hat{C}_{13}}{\bar{B}_3} \tag{23}$$

It is straightforward that (21) and (23) reduce to (15).

In order to show (16), note that the first two equations of (18) provide

$$\alpha = \frac{(\beta - \bar{A}_2)\bar{B}_1 + \bar{A}_1\bar{B}_2}{\bar{B}_2} \tag{24}$$

and

$$\gamma = \frac{(\beta - \bar{A}_2)\bar{B}_3 + \bar{A}_3\bar{B}_2}{\bar{B}_2} \tag{25}$$

which under the conditions $|\alpha| < 1$ and $|\gamma| < 1$ produce the desired result.

Likewise, to establish (17), consider the second and first equations of (18), which render

$$\alpha = \frac{(\gamma - \bar{A}_3)\bar{B}_1 + \bar{A}_1\bar{B}_3}{\bar{B}_3} \tag{26}$$

and

$$\beta = \frac{(\gamma - \bar{A}_3)\bar{B}_2 + \hat{C}_{23}}{\bar{B}_3} \tag{27}$$

and the claim is fulfilled by taking into account $|\alpha| < 1$ and $|\beta| < 1$. □

Remark 4. *Theorem 3 implies that we will have an asymptotically stable system if the matrices of the state space description provide the Lagrange solutions according to one of the conditions established in the proof.*

Next we examine the conditions under which it is possible to establish a Lagrange solution to system (1)

Theorem 3. *There exists a Lagrange solution on the grounds of the equalities (6) if*

$$\frac{b_{21}}{a_{21}} = \frac{b_{31}}{a_{31}}, \quad \frac{b_{12}}{a_{12}} = \frac{b_{32}}{a_{32}}, \quad \frac{b_{13}}{a_{13}} = \frac{b_{23}}{a_{23}} \tag{28}$$

along with $|\alpha_1|$, $|\beta_2|$ *and* $|\gamma_3|$, *which are represented respectively by*

$$\begin{array}{lcll} |\, a_{11} - \frac{a_{21}b_{11}}{b_{21}} \,| & = & |\, a_{11} - \frac{a_{31}b_{11}}{b_{31}} \,| & < 1 \\ |\, a_{22} - \frac{a_{12}b_{22}}{b_{12}} \,| & = & |\, a_{22} - \frac{a_{32}b_{22}}{b_{32}} \,| & < 1 \\ |\, a_{33} - \frac{a_{13}b_{33}}{b_{13}} \,| & = & |\, a_{33} - \frac{a_{23}b_{33}}{b_{23}} \,| & < 1 \end{array} \tag{29}$$

hold.

Proof. Since equations (6) through (8) translate into

$$\begin{array}{l} (\alpha_1 - a_{11})\alpha_1^{\Delta_1}\beta_1^{\Delta_2}\gamma_1^{\Delta_3} = b_{11} \\ (\beta_2 - a_{22})\alpha_2^{\Delta_1}\beta_2^{\Delta_2}\gamma_2^{\Delta_3} = b_{22} \\ (\gamma_3 - a_{11})\alpha_3^{\Delta_1}\beta_3^{\Delta_2}\gamma_3^{\Delta_3} = b_{33} \\ \alpha_1^{\Delta_1}\beta_1^{\Delta_2}\gamma_1^{\Delta_3} = \frac{-b_{21}}{a_{21}} = \frac{-b_{31}}{a_{31}} \\ \alpha_2^{\Delta_1}\beta_2^{\Delta_2}\gamma_2^{\Delta_3} = \frac{-b_{12}}{a_{12}} = \frac{-b_{32}}{a_{32}} \\ \alpha_3^{\Delta_1}\beta_3^{\Delta_2}\gamma_3^{\Delta_3} = \frac{-b_{13}}{a_{13}} = \frac{-b_{23}}{a_{23}} \end{array} \tag{30}$$

and the claims follow straightforwardly by recalling the conditions that the Lagrange solution must satisfy to guarantee the asymptotic stability of the system. □

Remark 5. *The conditions (28) in theorem 3 say that (6) is met only if the system is described by some very particular kinds of matrices. In general, systems satisfying these conditions are not common.*

In what follows, the framework developed so far is used to get some insights into the role of the eigenvalues of the matrices in the system stability.

3.1 Analysis of Diagonalizable Systems

In the sequel we examine systems which have diagonalizable matrices in its state space system description, and pursue the conditions on the eigenvalues to have asymptotically stable systems.

Theorem 4. *Consider system (1) and let it be composed by a diagonalizable matrix B by means of the canonical transformation (Gantmatcher, 1959; Suda, 1978). And let matrix $\tilde{A}$ be the matrix obtained from the canonical transformation of B applied on A. Also, assume that the Lagrange solution candidate is composed by $\rho_\star$, $\sigma_\star$ and $\phi_\star$, for $\star = 1, \cdots, 3$. Then the system is asymptotically stable in the sense of the Lagrange solution method if the eigenvalues of matrix B defined by λ_1, λ_2 and λ_3 satisfy either of the conditions.*

- *Condition 1:*

$$\begin{array}{l} -1 - \tilde{A}_* < \lambda_* < 1 - \tilde{A}_* \\ * = 1, \cdots, 3 \\ \tilde{A}_* = \text{sum of row entries of } \tilde{A} \end{array} \tag{31}$$

  for $\rho_1 = \rho_2 = \rho_3 \doteq \rho$, $\sigma_1 = \sigma_2 = \sigma_3 \doteq \sigma$ and $\phi_1 = \phi_2 = \phi_3 \doteq \phi$

- *Condition 2: assuming the same reasoning of condition 2 in theorem 1, the matrix $\tilde{A}$ must also be a diagonal matrix and the Lagrange solutions can be established such that the eigenvalues of B can be written as*

$$\begin{array}{l} \lambda_1 = \rho_1^{\Delta_1}\sigma_1^{\Delta_2}\phi_1^{\Delta_3}(\rho_1 - \tilde{a}_{11}) \\ \lambda_2 = \rho_2^{\Delta_1}\sigma_2^{\Delta_2}\phi_2^{\Delta_3}(\sigma_2 - \tilde{a}_{22}) \\ \lambda_3 = \rho_3^{\Delta_1}\sigma_3^{\Delta_2}\phi_3^{\Delta_3}(\phi_3 - \tilde{a}_{33}) \end{array} \tag{32}$$

Proof. Since matrix B is diagonalizable by means of canonical transformation, there exists a matrix T such that $T \times B \times T^{-1}$, so that (1) writes

$$Tx_{+1} = TAT^{-1}(Tx) + TBT^{-1}(Tx_{\Delta}) \quad (33)$$

where TBT^{-1} turns into the diagonal matrix D with eigenvalues λ_1, λ_2 and λ_3 of B as entries. Thus defining vector z as $z \doteq Tz$, (33) is described by

$$z_{+1} = \tilde{A}z + Dz_{\Delta} \quad (34)$$

which is a loose representation of the equations

$$\begin{aligned} z_1(i+1,j,k) &= \tilde{a}_{11}z_1(i,j,j) + \tilde{a}_{12}z_2(i,j,k) \\ &+\tilde{a}_{13}z_3(i,j,k) \\ &+\lambda_1 z_1(i-\Delta_1, j-\Delta_2, k-\Delta_3) \\ z_2(i,j+1,k) &= \tilde{a}_{21}z_1(i,j,j) + \tilde{a}_{22}x_2(i,j,k) \\ &+\tilde{a}_{23}z_3(i,j,k) \\ &+\lambda_2 z_2(i-\Delta_1, j-\Delta_2, k-\Delta_3) \\ z_3(i,j,k+1) &= \tilde{a}_{31}z_1(i,j,j) + \tilde{a}_{32}z_2(i,j,k) \\ &+\tilde{a}_{33}z_3(i,j,k) \\ &+\lambda_3 z_3(i-\Delta_1, j-\Delta_2, k-\Delta_3) \end{aligned} \quad (35)$$

and letting the Lagrange solution candidates as

$$\begin{aligned} z_{\star} &= \rho_{\star}^{i}\sigma_{\star}^{j}\phi_{\star}^{k} \\ \star &= 1, \cdots, 3 \end{aligned} \quad (36)$$

and proceeding as in theorem 1, one concludes that similar conditions as (5) - (8) as long as the corresponding terms are suitably changed.

Now with $\rho_1 = \rho_2 = \rho_3 \doteq \rho$, $\sigma_1 = \sigma_2 = \sigma_3 \doteq \sigma$ and $\phi_1 = \phi_2 = \phi_3 \doteq \phi$, the equations to be satisfied are

$$\begin{aligned} \rho^{\Delta_1}\sigma^{\Delta_2}\phi^{\Delta_3}[\rho - (\tilde{A}_1 + \lambda_1)] = 0 \\ \rho^{\Delta_1}\sigma^{\Delta_2}\phi^{\Delta_3}[\sigma - (\tilde{A}_2 + \lambda_2)] = 0 \\ \rho^{\Delta_1}\sigma^{\Delta_2}\phi^{\Delta_3}[\phi - (\tilde{A}_3 + \lambda_3)] = 0 \end{aligned} \quad (37)$$

from which (31) follows.

On the other hand, focusing on the equivalent condition to (6), one realizes that to fulfill it we must have

$$\begin{aligned} \rho_1^{\Delta_1}\sigma_1^{\Delta_2}\phi_1^{\Delta_3}[\rho_1 - \tilde{a}_{11}] - \lambda_1 = 0 \\ \rho_2^{\Delta_1}\sigma_2^{\Delta_2}\phi_2^{\Delta_3}[\sigma_2 - \tilde{a}_{22}] - \lambda_2 = 0 \\ \rho_3^{\Delta_1}\sigma_3^{\Delta_2}\phi_3^{\Delta_3}[\phi_3 - \tilde{a}_{11}] - \lambda_3 = 0 \\ \tilde{a}_{12} = \tilde{a}_{13} = \tilde{a}_{21} = \tilde{a}_{23} = \tilde{a}_{31} = \tilde{a}_{32} = 0 \end{aligned} \quad (38)$$

from which one reaches to (32). □

Remark 6. *Theorem 4 says that one can figure out whether the system is asymptotically stable by just checking out values of the eigenvalues of the transformed system since condition 2, which imposes that both matrices be diagonal, is quite unlikely to be satisfied in general.*

Finally, we investigate the systems with diagonalizable matrix A. The results are packed up in the following theorem.

Theorem 5. *Let system (1) have diagonalizable matrix B by means of the canonical transformation, and matrix $\tilde{B}$ be the result of the canonical transformation of A applied on B. Then based on the Lagrange solution candidates $\rho_\star$, $\sigma_\star$ and $\phi_\star$, for $\star = 1, \cdots, 3$, the eigenvalues of matrix A given by λ_1, λ_2 and λ_3 satisfy either of the conditions in order to assure the asymptotic stability.*

- *Condition 1: $\rho_1 = \rho_2 = \rho_3 = \rho$, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ and $\phi_1 = \phi_2 = \phi_3 = \phi$. Furthermore, all the sets ranging from maximum to minimum values as defined by*

$$\begin{aligned} \max\{\lambda_1 - (\lambda_2+1)\tfrac{\tilde{B}_1}{\tilde{B}_2}, \lambda_1 - (\lambda_3+1)\tfrac{\tilde{B}_1}{\tilde{B}_3}\} \\ \min\{\lambda_1 - (\lambda_2-1)\tfrac{\tilde{B}_1}{\tilde{B}_2}, \lambda_1 - (\lambda_3-1)\tfrac{\tilde{B}_1}{\tilde{B}_3}\} \end{aligned} \quad (39)$$

and

$$\begin{aligned} \max\{\lambda_2 - (\lambda_1+1)\tfrac{\tilde{B}_2}{\tilde{B}_1}, \lambda_2 - (\lambda_3+1)\tfrac{\tilde{B}_2}{\tilde{B}_3}\} \\ \min\{\lambda_2 - (\lambda_1-1)\tfrac{\tilde{B}_2}{\tilde{B}_1}, \lambda_2 - (\lambda_3-1)\tfrac{\tilde{B}_2}{\tilde{B}_3}\} \end{aligned} \quad (40)$$

and

$$\begin{aligned} \max\{\lambda_3 - (\lambda_1+1)\tfrac{\tilde{B}_3}{\tilde{B}_1}, \lambda_3 - (\lambda_2+1)\tfrac{\tilde{B}_3}{\tilde{B}_2}\} \\ \min\{\lambda_3 - (\lambda_1-1)\tfrac{\tilde{B}_3}{\tilde{B}_1}, \lambda_3 - (\lambda_2-1)\tfrac{\tilde{B}_3}{\tilde{B}_2}\} \end{aligned} \quad (41)$$

contain at least one non-null element whose absolute value is less than unity.

- *Condition 2: proceeding similarly to condition 2 in the previous theorem 4, matrix $\tilde{A}$ must be diagonal and there must exist Lagrange solutions such that the eigenvalues of matrix B are to satisfy*

$$\begin{aligned} \lambda_1 &= \rho_1 - \frac{\tilde{a}_{11}}{\rho_1^{\Delta_1}\sigma_1^{\Delta_2}\phi_1^{\Delta_3}} \\ \lambda_2 &= \sigma_2 - \frac{\tilde{a}_{22}}{\rho_2^{\Delta_1}\sigma_2^{\Delta_2}\phi_2^{\Delta_3}} \\ \lambda_3 &= \phi_3 - \frac{\tilde{a}_{33}}{\rho_3^{\Delta_1}\sigma_3^{\Delta_2}\phi_3^{\Delta_3}} \end{aligned} \quad (42)$$

Proof. The steps to show the claims are close to the ones describe in the previous theorems. □

Remark 7. *Theorem 5 requires the computation of the maximums and minimums in order to establish the range of testing. In this sense, whenever possible, the conditions of theorem 4 are a bit easier to check.*

4 FINAL REMARKS

In this paper we studied the asymptotic stability of discrete time 3-d linear systems. Conditions to assure the stability were established and collected in 5 theorems. Theorem 1 gives the existence condition of the solutions. In practice, to check whether the system is asymptotically stable, one basically has to test either theorem 2 or theorem 4 or theorem 5. Theorem 2 allows us to decide it by making the computations directly on the entries of the matrices, whereas the others require the application of the canonical transformation of the matrices. The striking point of the use of canonical transformation is that it yield some information on the eigenvalues of the matrices, which provides deeper insights into the relationships between the matrix structure and asymptotic stability.

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