Complexity and Approximability of Hyperplane Covering Problems

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Abstract. The well known N.Megiddo complexity result for Point Cover Problem on the plane is extended onto d-dimensional space (for any fixed $d > 1$). It is proved that Min-dPC problem is L-reducible to Min-$(d+1)$PC problem, therefore for any fixed $d > 1$ there is no PTAS for Min-dPC problem, unless $P = NP$.

1 Introduction

Settings of geometric covering problem and related problems are usual in various operations research domains [1-3]: optimal facility location theory, cluster analysis, pattern recognition, etc. Mathematically, family of such problems can be partition into two classes.

The first one contains special cases and modifications of well-known abstract Set Cover problem. The main general feature shared by these problems is the *finiteness* of the initial family of subsets, for which it is required to find a subfamily (or just prove its existence) covering some target set and satisfying given optimality conditions. There are many papers studying problems from this class (see survey at [4]). The classical papers [5-7] seem to be the most important among them. First two papers contain intractability proof of Set Cover problem and two main design patterns for constructing approximation algorithms for this problem. The last paper proves the optimality of these patterns, unless $P = NP$.

The second class consists of problems without the mentioned above finiteness constraint. Usually, the initial family of subsets is given here implicitly in terms of some geometric property characterizing its elements. For instance, for a given set it is required to find a minimal cardinality cover by straight lines, circles of a given radii, etc.

2 Point Cover (2PC) Problem

In the paper, a series of hyperplane covering problems for given finite sets in finite-dimensional vector spaces of fixed dimension $d > 1$ is considered. The first element of

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this family (for $d = 2$), also known as Point Covering on the plane (2PC) problem was studied by N.Megiddo and A.Tamir [8] who proved its intractability in the strong sense.

We extend this result on to the case of appropriate fixed dimensionality $d > 1$ and prove that all these problems are Max-SNP-hard and consequently have no PTAS, unless $P = NP$.

Problem 1. ‘Point covering by lines on the plane’ (2PC). A finite subset $P = \{p_1, \dots, p_n\} \subset \mathbb{Z}^2$ and natural number B are given. Is there exists a finite family C of straight lines covering P such that $|C| \leq B$?

Obviously, in the particular case when the set P is in *the general position*, i.e. each triple of its points does not belong to the same straight line, the 2PC problem has a trivial solution (’Yes’ whether $B \geq \lceil |P|/2 \rceil$ and ’No’ otherwise), which can be found in a polynomial time. But in the general case this problem is intractable.

Theorem 1 ([8]). *The 2PC problem is NP-complete in the strong sense.*

Note that Theorem 1 applies that 2PC problem could not be solved by only polynomial time algorithms, but also pseudo-polynomial time.

3 Hyperplanes Covering Problems

Let us consider the more general problem settings.

Problem 2. ‘Hyperplane covering in d-dimensional space’ (dPC). For a fixed $d > 1$, a finite subset $P = \{p_1, \dots, p_n\} \subset \mathbb{Z}^d$ and natural number B are given. Is there exists a cover C of P by hyperplanes such that $|C| \leq B$?

Problem 3. ‘Minimal hyperplane covering in d-dimensional space’ (Min-dPC). Let a finite subset $P = \{p_1, \dots, p_n\} \subset \mathbb{Z}^d$ be given. It is required to find a minimum cardinality partition $J_1, \dots, J_L$ of a set $\mathbb{N}_n = \{1, \dots, n\}$ such that for each $i \in \mathbb{N}_L$ there is a hyperplane H_i and

$$\{p_j \in P \,:\, j \in J_i\} \subset H_i.$$

We extend the result of Theorem 1 onto the case of d-dimensional space for any fixed $d > 1$. We start with construction of polynomial-time reduction of $(d-1)$PC to dPC problem. Let an instance of $(d-1)$PC be given by subset $P = \{p_1, \dots, p_n\} \subset \mathbb{N}_M^{d-1}$ and $B \in \mathbb{N}$. We use a natural isomorphic embedding of $(d-1)$-dimensional into d-dimensional vector space:

$$x \in \mathbb{R}^{d-1} \mapsto [x, 0] \in \mathbb{R}^d.$$

Map any point $p_i \in P$ into couple of points in $\mathbb{Z}^d$ by the formula

$$\bar{p}_{2i-1} = [p_i, -w_i],\ \bar{p}_{2i} = [p_i, w_i],$$

where

$$w_i = (K+2)^{i-1} \text{ and } K = \left\lceil (d-1)^{\frac{d-1}{2}} (M-1)^{d-1} \right\rceil.$$

Such a way, we construct the subset $\bar{P} \subset \mathbb{Z}^d$ and the setting $(\bar{P}, B)$ of the dPC problem.

It is evident, that any hyperplane cover of P induces the equivalent cover (with the same number of hyperplanes) of $\bar{P}$ in $\mathbb{R}^d$. The converse statement should be proved.

Denote by π_0 the hyperplane $\{[x, 0] \ : \ x \in \mathbb{R}^{d-1}\}$. Let $\mathrm{Pr}_{\pi_0} Q$ be an orthogonal projection of the subset $Q \subset \mathbb{R}^d$ onto π_0.

Lemma 1. *Let subsets $Q \subset P$ and $\bar{Q} \subset \bar{P}$ be related by $Q = \mathrm{Pr}_{\pi_0} \bar{Q}$ and the following inequalities be valid*

$$|\bar{Q}| \geq d + 1,$$
$$\dim \mathrm{aff} \bar{Q} \leq d - 1.$$

Then $\dim \mathrm{aff} Q \leq d - 2$.

Lemma 2. *Let $\bar{\Pi} = \{\bar{\pi}_1, \ldots, \bar{\pi}_t\}$ be a hyperplane cover of subset $\bar{P}$. The subset P also has a hyperplane cover Π such that $|\Pi| \leq t$.*

Lemma 3. *The described above reduction $(d-1)$PC to dPC can be done in polynomial time of $Length((d-1)PC)$.*

On the basis of these lemmas we can prove the following

Theorem 2. *For an arbitrary fixed $d > 1$, the dPC problem is NP-complete (and the Min-dPC problem is NP-hard) in the strong sense.*

Now we show that the supposed above $(d-1)$PC to dPC reduction can be reformulated as L-reduction [9] from Min-$(d-1)$PC to Min-dPC problem.

Definition 1. *Let sets $\Im$ and S, set-valued map $F : \Im \to 2^S$ and some target function $c : \bigcup_{I \in \Im} F(I) \to \mathbb{R}_+$ be given. The quadruple $A = (\Im, S, F, c)$, where each $I \in \Im$ is mapped to optimization problem*

$$\min\{c(s) : \ s \in F(I)\},$$

is called a combinatorial minimization problem.

W.o.l.g., any $I \in \Im$ is called an instance of the problem A and its optimum value is denoted by $OPT(I)$.

Definition 2. *Consider problems A and B of combinatorial minimization. It is called, that there is an L-reduction from A into B, if there are two LSPACE-computable functions R and S and positive constants α and β such that the following conditions are valid:*

1. *for each instance I of the problem A, $R(I)$ is an instance of B and*

$$OPT(R(I)) \leq \alpha OPT(I);$$

2. *for each feasible solution z of $R(I)$, $S(z)$ is a feasible solution of I such that*

$$c_A(S(z)) - OPT(I) \leq \beta(c_B(z) - OPT(R(I))),$$

where c_A, c_B are target functions of A and B correspondingly.

Now we are ready to formulate a recurrent L-reduction of problems in question.

Theorem 3. *For each fixed $d > 2$, there is an L-reduction of Min-$(d-1)$PC to Min-dPC problem.*

Taking into account the following known result

Theorem 4 ([11]). *Min-2PC problem is Max-SNP-hard.*

one can formulate the last

Theorem 5. *For each fixed $d > 1$, the Min-dPC problem is Max-SNP-hard.*

Consequently, Min-dPC problem has no polynomial-time approximation schema (PTAS) for each fixed $d > 1$, unless $P = NP$.

4 Conclusions

We show that Hyperplane covering problem remains intractable and poorly approximatable even in fixed dimension spaces (for any $d > 1$). This result extends the well known Point Cover intractability result obtained by N. Megiddo and A. Tamir. Obviously, Min-dPC problem can be trivially approximated in polynomial time within $O(n/d)$ approximation guarantee. But the question on the existence of polynomial time algorithms with lower (e.g. fixed) approximation guarantee is still open.

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