

GEOMETRIC CONTROL OF A BINOCULAR HEAD

Eduardo Bayro-Corrochano and Julio Zamora-Esquivel

*Department of Electrical Engineering and Computer Science
CINVESTAV, unidad Guadalajara. Jalisco, Mexico*

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Abstract: In this paper the authors use geometric algebra to formulate the differential kinematics of a binocular robotic head and reformulate the interaction matrix in terms of the lines that represent the principal axes of the camera. This matrix relates the velocities of 3D objects and the velocities of their images in the stereo images. Our main objective is the formulation of a kinematic control law in order to close the loop between perception and action, which allows to perform a smooth visual tracking.

1 INTRODUCTION

In this work we formulate the problem of visual tracking and design a control law by velocity feedback that allows us to close the loop between perception and action. Geometric algebra allow us to work with geometric entities like points, lines and planes and helps in the representation of rigid transformations. In this mathematical framework we straightforwardly formulate the direct and differential kinematics of robotic devices like the binocular robot head. On the other hand we show a reformulation of visual Jacobean which relates the velocity of a tridimensional object with the velocity of its projection onto the stereo camera images. Finally we write an expression that relates the joint velocities in the pan-tilt unit and velocities of the points in the camera image.

We start this work presenting a brief description of the geometric entities and conformal transformations them we show the kinematics of a pan-tilt unit and formulate its control law.

In contrast to other authors like (C. Canudas de Wit and Bastin, 1996), (Ruf, 2000) or (Kim Jung-Ha., 1990) we will use multivectors instead of matrices to formulate the control law it reduces the computation and improve the performance of the controller.

2 GEOMETRIC ALGEBRA: AN OUTLINE

Let \mathcal{G}_n denote the geometric algebra of n -dimensions, this is a graded linear space. As well as vector addition and scalar multiplication we have a non-commutative product which is associative and distributive over addition – this is the *geometric* or *Clifford product*.

The inner product of two vectors is the standard *scalar* or *dot* product and produces a scalar. The outer or wedge product of two vectors is a new quantity which we call a *bivector*. We think of a bivector as a oriented area in the plane containing a and b , formed by sweeping a along b .

Thus, $b \wedge a$ will have the opposite orientation making the wedge product anti-commutative. The outer product is immediately generalizable to higher dimensions – for example, $(a \wedge b) \wedge c$, a *trivector*, is interpreted as the oriented volume formed by sweeping the area $a \wedge b$ along vector c . The outer product of k vectors is a k -vector or k -blade, and such a quantity is said to have *grade* k . A *multivector* (linear combination of objects of different type) is *homogeneous* if it contains terms of only a single grade.

2.1 The Geometric Algebra of n-D Space

In this paper we will specify a geometric algebra \mathcal{G}_n of the n dimensional space by $\mathcal{G}_{p,q,r}$, where p , q and r

stand for the number of basis vector which squares to 1, -1 and 0 respectively and fulfill $n = p + q + r$.

We will use e_i to denote the vector basis i . In a Geometric algebra $G_{p,q,r}$, the geometric product of two basis vector is defined as

$$e_i e_j = \begin{cases} 1 & \text{for } i = j \in 1, \dots, p \\ -1 & \text{for } i = j \in p+1, \dots, p+q \\ 0 & \text{for } i = j \in p+q+1, \dots, p+q+r \\ e_i \wedge e_j & \text{for } i \neq j \end{cases}$$

This leads to a basis for the entire algebra:

$$\{1\}, \{e_i\}, \{e_i \wedge e_j\}, \{e_i \wedge e_j \wedge e_k\}, \dots, \{e_1 \wedge e_2 \wedge \dots \wedge e_n\} \quad (1)$$

Any multivector can be expressed in terms of this basis.

3 CONFORMAL GEOMETRY

Geometric algebra $G_{4,1}$ can be used to treat conformal geometry in a very elegant way. To see how this is possible, we follow the same formulation presented in (H. Li, 2001) and show how the Euclidean vector space \mathbb{R}^3 is represented in $\mathbb{R}^{4,1}$. This space has an orthonormal vector basis given by $\{e_i\}$ and $e_{ij} = e_i \wedge e_j$ are bivectorial basis and e_{23}, e_{31} and e_{12} correspond to the Hamilton basis. The unit Euclidean pseudo-scalar $I_e := e_1 \wedge e_2 \wedge e_3$, a pseudo-scalar $I_c := I_e E$ and the bivector $E := e_4 \wedge e_5 = e_4 e_5$ are used for computing the inverse and duals of multivectors.

3.1 The Stereographic Projection

The conformal geometry is related to a stereographic projection in Euclidean space. A stereographic projection is a mapping taking points lying on a hypersphere to points lying on a hyperplane. In this case, the projection plane passes through the equator and the sphere is centered at the origin. To make a projection, a line is drawn from the north pole to each point on the sphere and the intersection of this line with the projection plane constitutes the stereographic projection.

For simplicity, we will illustrate the equivalence between stereographic projections and conformal geometric algebra in \mathbb{R}^1 . We will be working in $\mathbb{R}^{2,1}$ with the basis vectors $\{e_1, e_4, e_5\}$ having the usual properties. The projection plane will be the x -axis and the sphere will be a circle centered at the origin with unitary radius.

Given a scalar x_e representing a point on the x -axis, we wish to find the point x_c lying on the circle that projects to it (see Figure 1). The equation of the line passing through the north pole and x_e is given by $f(x) = -\frac{1}{x_e}x + 1$ and the equation of the circle $x^2 +$

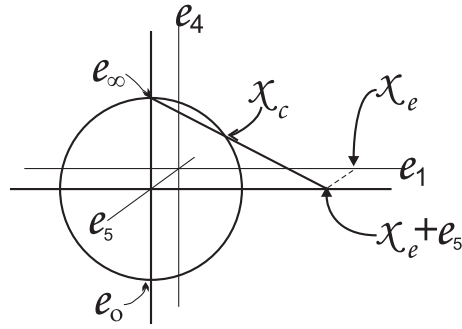


Figure 1: Stereographic projection for 1-D.

$f(x)^2 = 1$. Substituting the equation of the line on the circle, we get the point of intersection x_c which can be represented in homogeneous coordinates as the vector

$$x_c = 2 \frac{x_e}{x_e^2 + 1} e_1 + \frac{x_e^2 - 1}{x_e^2 + 1} e_4 + e_5. \quad (2)$$

From (2) we can infer the coordinates on the circle for the point at infinity as

$$e_\infty = \lim_{x_e \rightarrow \infty} \{x_c\} = e_4 + e_5, \quad (3)$$

$$e_0 = \frac{1}{2} \lim_{x_e \rightarrow 0} \{x_c\} = \frac{1}{2} (-e_4 + e_5), \quad (4)$$

Note that (2) can be rewritten to

$$x_c = x_e + \frac{1}{2} x_e^2 e_\infty + e_0, \quad (5)$$

3.2 Spheres and Planes

The equation of a sphere of radius ρ centered at point $p_e \in \mathbb{R}^n$ can be written as $(x_e - p_e)^2 = \rho^2$. Since $x_c \cdot y_c = -\frac{1}{2}(\mathbf{x}_e - \mathbf{y}_e)^2$ and $x_c \cdot p_c = -\frac{1}{2}\rho^2$, we can rewrite the formula above in terms of homogeneous coordinates as. Since $x_c \cdot e_\infty = -1$ we can factor the expression above to

$$x_c \cdot (p_c - \frac{1}{2}\rho^2 e_\infty) = 0. \quad (6)$$

Which finally yields the simplified equation for the sphere as $s = p_c - \frac{1}{2}\rho^2 e_\infty$. Note from this equation that a point is just a sphere with zero radius. Alternatively, the dual of the sphere is represented as 4-vector $s^* = s I_c$. The advantage of the dual form is that the sphere can be directly computed from four points as

$$s^* = x_{c1} \wedge x_{c2} \wedge x_{c3} \wedge x_{c4}. \quad (7)$$

If we replace one of these points for the point at infinity we get the equation of a plane

$$\pi^* = x_{c1} \wedge x_{c2} \wedge x_{c3} \wedge e_\infty. \quad (8)$$

So that π becomes in the standard form

$$\pi = I_c \pi^* = n + d e_\infty \quad (9)$$

Where n is the normal vector and d represents the Hesse distance.

3.3 Circles and Lines

A circle z can be regarded as the intersection of two spheres s_1 and s_2 as $z = (s_1 \wedge s_2)$. The dual form of the circle can be expressed by three points lying on it

$$z^* = x_{c_1} \wedge x_{c_2} \wedge x_{c_3}. \quad (10)$$

Similar to the case of planes, lines can be defined by circles passing through the point at infinity as:

$$L^* = x_{c_1} \wedge x_{c_2} \wedge e_\infty. \quad (11)$$

The standard form of the line can be expressed by

$$L = l + e_\infty(t \cdot l), \quad (12)$$

the line in the standard form is a bivector, and it has six parameters (Plucker coordinates).

4 RIGID TRANSFORMATIONS

We can express rigid transformations in conformal geometry carrying out reflections between planes.

4.0.1 Reflection

The reflection of conformal geometric entities help us to do any other transformation. The reflection of a point x respect to the plane π is equal x minus twice the direct distance between the point and plane see the image 2, that is $x' = x - 2(\pi \cdot x)\pi^{-1}$ to simplify this expression recalling the property of Clifford product of vectors $2(b \cdot a) = ab + ba$.

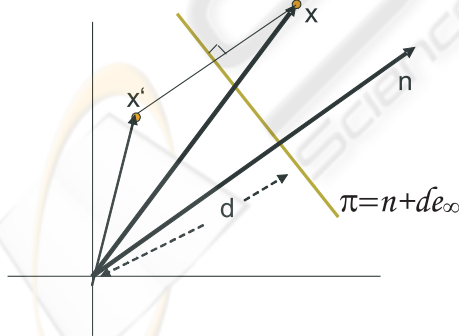


Figure 2: Reflection of a point x respect to the plane π .

For any geometric entity Q , the reflection respect to the plane π is given by

$$Q' = \pi Q \pi^{-1} \quad (13)$$

4.0.2 Translation

The translation of conformal entities can be done by carrying out two reflections in parallel planes π_1 and π_2 see the image (3), that is

$$Q' = \underbrace{(\pi_2 \pi_1)}_{T_a} Q \underbrace{(\pi_1^{-1} \pi_2^{-1})}_{\tilde{T}_a} \quad (14)$$

$$T_a = (n + d e_\infty) n = 1 + \frac{1}{2} a e_\infty = e^{-\frac{a}{2} e_\infty} \quad (15)$$

With $a = 2dn$.

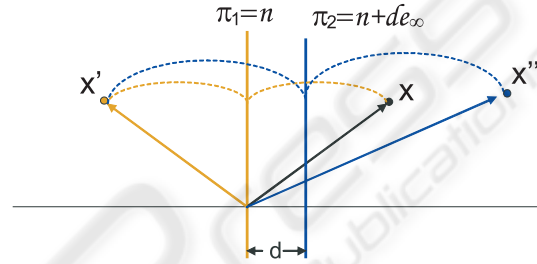


Figure 3: Reflection about parallel planes.

4.0.3 Rotation

The rotation is the product of two reflections between nonparallel planes, (see image (4))

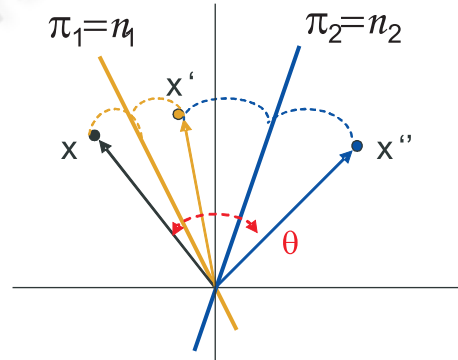


Figure 4: Reflection about nonparallel planes.

$$Q' = \underbrace{(\pi_2 \pi_1)}_{R_\theta} Q \underbrace{(\pi_1^{-1} \pi_2^{-1})}_{\tilde{R}_\theta} \quad (16)$$

Or computing the conformal product of the normals of the planes.

$$R_\theta = n_2 n_1 = \text{Cos}\left(\frac{\theta}{2}\right) - \text{Sin}\left(\frac{\theta}{2}\right) l = e^{-\frac{\theta}{2} l} \quad (17)$$

With $l = n_2 \wedge n_1$, and θ twice the angle between the planes π_2 and π_1 . The screw motion called motor related to an arbitrary axis L is $M = TR\tilde{T}$

$$Q' = \underbrace{(TR\tilde{T})}_{M_\theta} Q \underbrace{((TR\tilde{T}))}_{\tilde{M}_\theta} \quad (18)$$

$$M_\theta = TR\tilde{T} = \text{Cos}\left(\frac{\theta}{2}\right) - \text{Sin}\left(\frac{\theta}{2}\right)L = e^{-\frac{\theta}{2}L} \quad (19)$$

4.1 Kinematic Chains

The direct kinematics for serial robot arms is a succession of motors as you can see in (Bayro-Corrochano and Kahler, 2000) and it is valid for points, lines, planes, circles and spheres.

$$Q' = \prod_{i=1}^n M_i Q \prod_{i=1}^n \tilde{M}_{n-i+1} \quad (20)$$

5 DIRECT KINEMATICS OF A PAN-TILT UNIT

We implement algorithm for the velocity control of a pan-tilt unit (PTU Fig. 5) assuming three degree of freedom. We consider the stereo depth as one virtual D.O.F. thus the PTU has a similar kinematic behavior as a robot with three D.O.F.

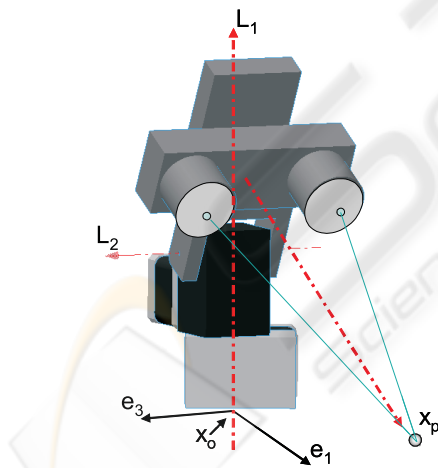


Figure 5: Pan tilt unit.

In order to carry out a velocity control, we need first to compute the direct kinematics, this is very easy to do, as we know the axis lines:

$$L_1 = -e_{31} \quad (21)$$

$$L_2 = e_{12} + d_1 e_1 e_\infty \quad (22)$$

$$L_3 = e_1 e_\infty \quad (23)$$

Since $M_i = e^{-\frac{1}{2}q_i L_i}$ and $\tilde{M}_i = e^{\frac{1}{2}q_i L_i}$, The position of end effectors is computed as

$$x_p(q) = x'_p = M_1 M_2 M_3 x_p \tilde{M}_3 \tilde{M}_2 \tilde{M}_1, \quad (24)$$

The state variable representation of the system is as follows

$$\begin{cases} \dot{x}'_p = x' \cdot (L'_1 \quad L'_2 \quad L'_3) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ y = x'_p \end{cases} \quad (25)$$

where the position of end effector at home position x_p is the conformal mapping of $x_{pe} = d_3 e_1 + (d_1 + d_2) e_2$ (see eq. 5), the line L'_i is the current position of L_i and u_i is the velocity of the i -junction of the system. As L_3 is an axis at infinity M_3 is a translator, that is, the virtual component is a prismatic junction.

5.1 Exact Linearization via Feedback

Now the following state feedback control law is chosen in order to get a new linear an controllable system.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = (x'_p \cdot L'_1 \quad x'_p \cdot L'_2 \quad x'_p \cdot L'_3)^{-1} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad (26)$$

Where $V = (v_1, v_2, v_3)^T$ is the new input to the linear system, then we rewrite the equations of the system

$$\begin{cases} \dot{x}'_p = V \\ y = x'_p \end{cases} \quad (27)$$

5.2 Asymptotical Output Tracking

The problem of follow a constant reference x_t is solved computing the error between end effectors position x'_p and the target position x_t as $e_r = (x'_p \wedge x_t) \cdot e_\infty$, the control law is then given by.

$$V = -ke \quad (28)$$

This error is small if the control system is doing it's job, it is mapped to an error in the joint space using the inverse Jacobian.

$$U = J^{-1}V \quad (29)$$

Doing the Jacobian $J = x'_p \cdot (L'_1 \quad L'_2 \quad L'_3)$

$$j_1 = x'_p \cdot (L_1) \quad (30)$$

$$j_2 = x'_p \cdot (M_1 L_2 \tilde{M}_1) \quad (31)$$

$$j_3 = x'_p \cdot (M_1 M_2 L_3 \tilde{M}_2 \tilde{M}_1) \quad (32)$$

Once that we have the Jacobian is easy to compute the dq_i using the crammer's rule.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = (j_1 \wedge j_2 \wedge j_3)^{-1} \cdot \begin{pmatrix} V \wedge j_2 \wedge j_3 \\ j_1 \wedge V \wedge j_3 \\ j_1 \wedge j_2 \wedge V \end{pmatrix} \quad (33)$$

This is possible because $j_1 \wedge j_2 \wedge j_3 = \det(J)I_e$. Finally we have dq_i which will tend to reduce these errors.

5.3 Visual Jacobian

A point in the image is given by $s = (x, y)^T$ whereas a 3-D point is represented as X . The relationship between s and \dot{X} is called visual Jacobian.

Taking a camera in general position his projection matrix is represented by the planes π_1 , π_2 y π_3 more details in (Hartley and Zisserman, 2000).

$$P = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}, \quad (34)$$

The point X is projected in the image in the point

$$s = \begin{pmatrix} \pi_1 \cdot X \\ \pi_2 \cdot X \\ \pi_3 \cdot X \end{pmatrix} \quad (35)$$

To simplify the explanation the x variable is introduced and his time derivative \dot{x} defined as

$$x = \begin{pmatrix} \pi_1 \cdot X \\ \pi_2 \cdot X \\ \pi_3 \cdot X \end{pmatrix} \quad \dot{x} = \begin{pmatrix} \pi_1 \cdot \dot{X} \\ \pi_2 \cdot \dot{X} \\ \pi_3 \cdot \dot{X} \end{pmatrix} \quad (36)$$

Now s is given by $s_1 = \frac{x_1}{x_3}$ and his derivative

$$\dot{s}_1 = \dot{x}_1 \frac{1}{x_3} + x_1 \left(\frac{-\dot{x}_3}{x_3^2} \right) \quad (37)$$

$$\dot{s}_1 = \frac{x_3 \dot{x}_1 - x_1 \dot{x}_3}{x_3^2} \quad (38)$$

By substitution of x and \dot{x} in the equation 38 is obtained

$$\dot{s}_1 = \kappa [(\pi_3 \cdot X) \pi_1 - (\pi_1 \cdot X) \pi_3] \cdot \dot{X} \quad (39)$$

$$\dot{s}_1 = \kappa [X \cdot (\pi_3 \wedge \pi_1)] \cdot \dot{X} \quad (40)$$

where $\kappa = \frac{1}{x_3^2}$. Doing the same steps for s_2 it possible to write the equation

$$\dot{s} = \kappa X \cdot \begin{pmatrix} \pi_1 \wedge \pi_3 \\ \pi_2 \wedge \pi_3 \end{pmatrix} \cdot \dot{X} \quad (41)$$

Geometrically $\pi_1 \wedge \pi_3$ represents a line of intersection of the planes π_1 and π_3 . Denoting by L_x and L_y the lines of this intersection as

$$L_x = \pi_1 \wedge \pi_3 \quad (42)$$

$$L_y = \pi_2 \wedge \pi_3 \quad (43)$$

It is possible to rewrite 41 as

$$\dot{s} = \kappa X \cdot \begin{pmatrix} L_x \\ L_y \end{pmatrix} \cdot \dot{X} \quad (44)$$

In order to close the loop between the perception and action, the relationship between velocities in the points of the image and the velocities in the joints of the pan-tilt unit is computed.

Taking the equation of differential kinematics 25 and visual Jacobian 44 it is possible to write a new expression

$$\dot{s} = \kappa \begin{pmatrix} (X' \cdot L'_x) \cdot (X' \cdot L'_1) & (X' \cdot L'_x) \cdot (X' \cdot L'_2) \\ (X' \cdot L'_y) \cdot (X' \cdot L'_1) & (X' \cdot L'_y) \cdot (X' \cdot L'_2) \end{pmatrix} \dot{q} \quad (45)$$

We can write a similar expression using the differential kinematics of the Barrett Hand. The equation 45 is very useful to design a control law to track an object or to grasp it.

5.4 Exact Linearization via Feedback

Now the following state feedback control law is chosen in order to get a new linear an controllable system.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (X' \cdot L'_x) \cdot (X' \cdot L'_1) & (X' \cdot L'_x) \cdot (X' \cdot L'_2) \\ (X' \cdot L'_y) \cdot (X' \cdot L'_1) & (X' \cdot L'_y) \cdot (X' \cdot L'_2) \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Where $V = (v_1, v_2)^T$ is the new input to the linear system, then we rewrite the equations of the system

$$\begin{cases} \dot{s}'_p = V \\ y = s'_p \end{cases} \quad (46)$$

5.5 Experimental Results

In this experiment the binocular head should smoothly track a target. The figure (6) show the 3D coordinates of the focus of attention. The figure (7) show examples of the image sequence. We can see that the curves of the 3D object trajectory are very rough, however the control rule manages to keep the trajectory of the pan-tilt unit smooth.

In the experiment the coordinate system is in the center of the camera. Then the principal planes of the camera are given by

$$\pi_1 = f_x e_1 + x_o e_3 \quad (47)$$

$$\pi_2 = f_y e_2 + y_o e_3 \quad (48)$$

$$\pi_3 = e_3 \quad (49)$$

whre f_x , f_y , x_o y y_o are the camera's parameters. Using this planes we compute the lines L_x y L_y , by the way the axis of the pan-tilt are known.

$$L_1 = e_{23} + d_1 e_2 \quad (50)$$

$$L_2 = e_{12} + d_2 e_2 \quad (51)$$

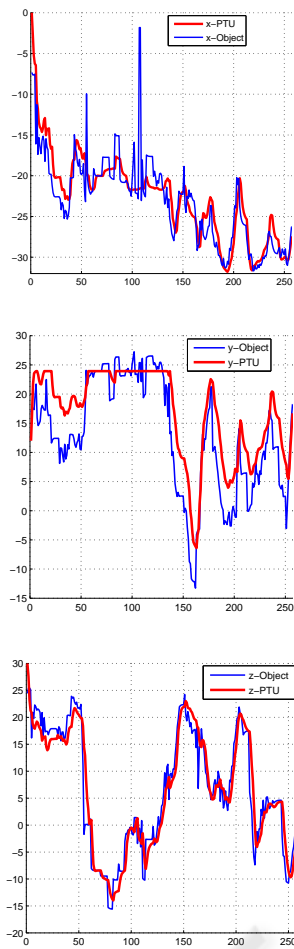


Figure 6: x , y and z coordinate of the focus of attention.

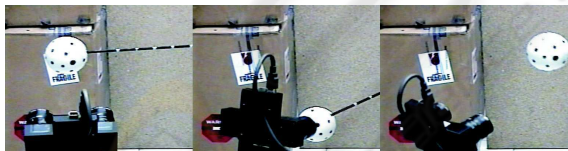


Figure 7: Sequence of tracking.

Note that the tilt axis is called L_1 and the pan axis is L_2 , because the coordinate system is in the camera. also L'_2 is a function of the tilt angle $L'_2 = M_1 L_2 \tilde{M}_1$ with $M_1 = \cos(\theta_{tilt}) + \text{sen}(\theta_{tilt})L_1$. In this experiment a point over the board was selected and using the KLT algorithm was tracked the displacement in the image is transformed to velocities of the pan-tilt's joint using the visual-mechanical Jacobean Eq. 45.

As result in the image (8) we can see a sequence of pictures captured by the robot. In these images the position of the board do not change while the background is in continuous change.

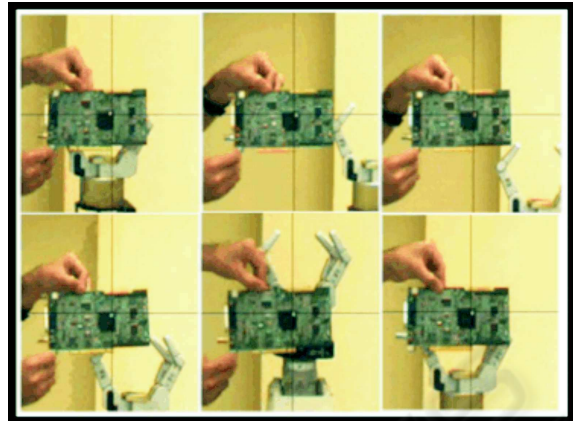


Figure 8: Sequence of tracking.

6 CONCLUSION

The authors show that is possible and easy to write a control law using the lines of the camera's axes as bivectors (Plücker coordinates) in the conformal geometry instead of the interaction matrix. This formulation combines the information of the camera's parameters with the axes of the pan tilt unit in order to create a matrix of the visual-mechanical Jacobian useful to write a velocity control law. The experiments confirm the effectiveness of our approach.

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