Proofs and Applications of Cauchy-Goursat Theorem and Cauchy's Integral Formula

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Abstract: This article focuses on some basis formulae and theorems of complex analysis in further mathematics. Complex variable function is a kind of function which contains complex number as its independent variables on the complex plane instead of real number in the real plane. This essay attempts to prove the two theorems found by Cauchy, i.e., Cauchy-Goursat theorem and Cauchy's integral formula primarily and briefly by some approaches which the people who knows the complex analysis firstly can follow and understand. After that, there are several applications in the further mathematical aspects in this paper. This paper may contain some reverse applications about previous conclusions and use the idea of breaking down some harsh problems into many easier questions. This paper has gotten the steps to utilize and further understand the Cauchy-Goursat theorem and Cauchy's integral formula. This article may assist the people who first takes up the further mathematics and complex analysis to realize the Cauchy-Goursat theorem and Cauchy's integral formula and try to construct the confidence of mastering them in the future.

1 INTRODUCTION

Complex analysis is a bit modern part of further mathematics derived from the real analysis in 18th century founded by a well-known mathematician Euler. Many famous mathematics researchers have concentrated on this area, made an effort, made up a new idea and then finally developed a new theorem. These include persons such as Euler, Cauchy, Riemann, Weierstrass and so on (Chen, 2023). Their ideas are broadly used in mechanics, electric engineering, and pure mathematics and other complicated aspect (Cohen, 2007). In addition, it may be linked to the real analysis to produce a new principle or approach in a deeper area (Zhang & Qi, 2018).

In 1825, French mathematician Cauchy pointed out a novel theorem that the result of the integral is independent of the shape of the integral, but has correlation to its origin and endpoint. In 1900, another French mathematician Goursat cut some useless conditions down and give his proof and generalize the theorem to a broader area. So, this theorem is known as "Cauchy-Goursat theorem". Next, on its foundation, there are Cauchy's Integral Formula to solve more realistic questions. There are many applications of other previous theorem such as Cauchy-Riemann equation and ε-δ language to prove the two theorems. After then, it is well-prepared to develop the residue theorem and finally people can get the integral of complex number (Zhou et al, 2022). Additionally, it is also an indispensable basis of the Taylor's series and Laurent expansion. It is an imperative ring of the development history of complex analysis. Although it is complex analysis, it will also generalize to the real analysis and serve some problems in real function (Zhang et al, 2023).

The article will adopt the steps below to assist readers to get to know the main body of two theorems and popularize its influence relatively rapidly. The article below firstly performs the two theorems, explains them briefly and tries to give some evidence that how Cauchy's two findings are valid. Furthermore, the author will introduce the indispensable conditions needed in the theorems to readers relatively in detail. What's more, the paper will focus on proving the theorem briefly by using some methods that the mathematicians have adopted in history and pointing out the main factor of the method, such as breaking down the complicated

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problems and using the old theorem reversely. In the next step there will be some instances to assist reader to realize and master the two conclusions. Finally, the author will conclude the article and make the knowledge have talked about more clearly.

2 METHODS

2.1 Cauchy-Goursat Theorem

To begin with, the author wants to introduce the Cauchy-Goursat Theorem simply. There are three conditions of this principle which is indispensable. Without the three conditions, the formula is not valid so that these conditions must be followed. Firstly, Ω is a simply connected subset of the complex plane. What's more, $f(z)$ is an analytic function of the variable z. In addition, γ is a simply closed curve in Ω. Then if one integrates the function $f(z)$ along the curve γ, people will get the conclusion that

$$\oint_\gamma f(z)dz = 0 \tag{1}$$

To prove this, the definition of the term "simply connected" and "simply closed" are imperative. Simply connected area means that any area in the curve surrounding Ω belongs to Ω. In other words, there isn't any hole occurring in the area Ω. If the curve is closed without cross with itself, people consider it as a simply closed curve.

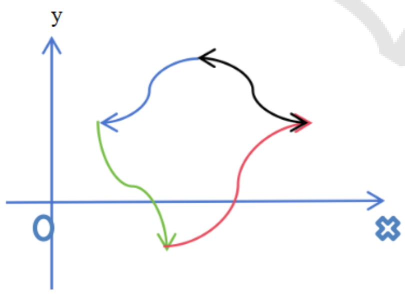

Figure 1: Illustration of the Green's theorem

Then it is enough to gather all conditions to prove the Cauchy-Goursat theorem. The mathematicians usually break any complex function $f(z)$ to the imaginary part v and real part u.

$$f(z) = u(x,y) + v(x,y) \tag{2}$$

Due to $dz = dx + idy$, it is calculated that

$$\oint_\gamma f(z)dz = (u+iv)(dx+idy)$$
$$= \oint_\gamma udx - vdy + i\oint_\gamma vdx + udy \tag{3}$$

Then, people usually use the Green's formula reversely to attempt to prove the result of the integral is zero. Green's formula used the idea that breaks a complicated problem to many simple problems so one can solve each simple problem then add them up to find the final conclusion. Green divided one integral to four integrals, the upper one (blue), the lower one (red), the right one (black) and the left one (green), like the plot shown in Figure 1. According to the breaking, there are some lines have no contribution to the integral. Then Green got his formula

$$\oint_\gamma f(z)dz = +\iint_\gamma -\frac{\vartheta v}{\vartheta x} - \frac{\vartheta u}{\vartheta y} dx\,dy + i\iint_\gamma \frac{\vartheta u}{\vartheta x} - \frac{\vartheta v}{\vartheta y} dx\,dy \tag{4}$$

By adopting Cauchy-Riemann Function, one finds

$$\frac{\vartheta u}{\vartheta x} = \frac{\vartheta v}{\vartheta y}, \frac{\vartheta v}{\vartheta x} = -\frac{\vartheta u}{\vartheta y} \tag{5}$$

In light of this formula, it is found that $\iint_\gamma -\frac{\vartheta v}{\vartheta x} - \frac{\vartheta u}{\vartheta y} dx\,dy + i\iint_\gamma \frac{\vartheta u}{\vartheta x} - \frac{\vartheta v}{\vartheta y} dx\,dy = 0$. Finally, formula shown in Eq. (1) can be obtained. This finishes the proof of the theorem.

2.2 Cauchy's Integral Formula

By using the idea of Cauchy-Goursat theorem, the Cauchy's integral formula is found below to cope with more specific problems and the theorem is from theories to the realistic.

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-z_0}dz \tag{6}$$

It also uses the idea of breaking the only one integral to 4 integrals to prove that as long as both the outer contour and the inner one are closed, the integrals along the two contours are equal to each other (Egahi & Otache, 2018). The formula below can be got as

$$\oint_\gamma \frac{f(z)}{z-z_0} = \oint_c \frac{f(z)}{z-z_0} \tag{7}$$

Due to the $f(z)$ is analytic in z_0, for given any small number $\varepsilon > 0$, there must exist a number $\delta > 0$ that when $|z - z_0| < \delta$, $|f(z) - f(z_0)| < \varepsilon$, when $|z - z_0|$ belongs to the contour γ, so that $dz = i\varepsilon e^{i\theta}\, d\theta$. Consequently, the formula is arrived

$$\oint_\gamma \frac{f(z)}{z-z_0}dz = +if(z_0)\int_0^{2\pi} 1 d\theta + i\int_0^{2\pi} z_0 + \varepsilon e^{i\theta} - f(z_0)\, d\theta \tag{8}$$

Since the term behind tends to zero, the result of the integral is $2\pi i f(z_0)$.When someone move some of terms to the left-hand side, the conclusion upper will be bringing out.

When using the Cauchy's integral formula, people should consider thatz_0 is the only one strange point in the contour (Mihálka, et al, 2019). In other words, if there are two strange points in the contour, the Cauchy's integral formula can't be used to solve the problem directly and should adopt a further formula to solving that. It is called residue theorem and it stemmed from this theorem.

3 RESULTS AND APPLICATIONS

3.1 Complex Applications

As people all known, the complex variables have many applications in various aspects. It is no exception to the Cauchy-Goursat theorem and Cauchy's integral formula. During the paragraph below, the readers will know the utilization of these theorems. It is not only used in the problem-solving, but also adopted in the novel theorems, such as Laurent's expansion, Taylor's series and the residue theorems.

This part will focus on the application of Cauchy's integral formula in the complex analysis. The author will use two example to distinguish two situation that where the Cauchy's integral formula can be used or there is a translation needed to further theorem adopted.

Example 1. $\int_c \frac{1}{z^2+2} dz$, when C: $|z+i| = \frac{1}{2}$.

It is clear to see that the contour C is a circle whose centre is $-i$, and radius is $\frac{1}{3}$ laying on the complex plane. Its upper boundary of the circle is $\frac{1}{2}$i, while the lower boundary of the circle is$-\frac{3}{2}$i. Thus, there are two strange points situating on the complex plane, which is$\sqrt{2}$i and$-\sqrt{2}$i. Due to $-\sqrt{2}$i is the only one strange point in the contour C, it is suitable for the situation that Cauchy's integral formula is valid.

Hence if Cauchy's integral formula is adopted, the result can be gotten directly. This is also proved indirectly that the Cauchy's integral is valid when there is only one strange point in the contour.

$$\int_c \frac{1}{z^2+2} dz = \int_c \frac{\frac{1}{z-\sqrt{2}i}}{z+i\sqrt{2}} dz = 2\pi i \cdot \frac{1}{-2i\sqrt{2}} = -\pi \quad (9)$$

Example 2. $\int_c \frac{1}{z(z^2+2)} dz$, when C: $|z+i| = \frac{1}{2}$.

The contour is same as the last example, but now there are three strange points in the complex plane, which are 0,$\sqrt{2}$i,$-\sqrt{2}$i. There are two strange points locating in the contour C: 0 and $-\sqrt{2}$i. So, the Cauchy's integral formula is not appropriate for this situation. Some people would say it needs residue theorems to perform a further calculation. However, residue is the numerator actually the partial fraction of a complicated fraction. In others words, the complicated fraction should be turned to two simple fractions so that the Cauchy's integral formula could be adopted on it. Two parameters, the capital A and capital B like below are supposed to adopted to complete the partial fraction (Zhu & Luo, 2023). The partition is given by

$$\int_c \frac{1}{z(z^2+2)} dz = \int_c \frac{\frac{1}{z-\sqrt{2}i}}{z(z+\sqrt{2}i)} dz = \int_c \frac{A}{z} dz + \int_c \frac{B}{z+i\sqrt{2}} dz \quad (10)$$

In order to calculate the capital A and capital B easily, on the next step each side could multiply the term $z(z+\sqrt{2}i)$, i.e., $\frac{1}{z-\sqrt{2}i} = A(z+i\sqrt{2}) + Bz$. Thus, one can finally let z equal to 0 and $-\sqrt{2}$i one by one. When z equal to zero, the formula below can be gotten and A will be $\frac{1}{2}$, i.e., $A \cdot i\sqrt{2} = -\frac{1}{\sqrt{2}i}$. This finishes the calculation of A and B.

When z equal to the other critical value, $-\sqrt{2}$i, the formula below can be obtained and B will be -$\frac{1}{2}$ since $\frac{1}{-2\sqrt{2}i} = -\sqrt{2}\mathrm{i} \cdot \mathrm{B}$. Hence the partial fraction is completed and the critical values or residues have been gained. Then the formula could be transferred to a more clearly term:

$$\frac{1}{z-\sqrt{2}i} = \frac{1}{2}(z+i\sqrt{2}) - \frac{1}{2}z \quad (11)$$

Consequently, the formula could be written as the formula below which the Cauchy's integral formula could be adopted and further calculation could complete:

$$\int_c \frac{1}{z(z^2+2)} dz = \int_c \frac{\frac{1}{z-\sqrt{2}i}}{z(z+\sqrt{2}i)} dz = \int_c \frac{1}{2z} dz - \int_c \frac{1}{2(z+\sqrt{2}i)} dz = 0 \quad (12)$$

The second example provides evidence of the universality of the Cauchy's integral formula. It proved that the Cauchy's integral formula could be adopted to calculated a sophisticated integration that two, or of cause more strange points exists in the contour and show the principle that how the residue theorem works (Yang & Zhang, 2006).

3.2 Real Applications

The complex analysis could be spread to real analysis. This may assume that the Cauchy's integral formula could also be spread to real situation.

However, some people may argue that how the extrapolation could be valid. As everybody all known, the complex number contain two parts, the imaginary parts and the real parts. Hence if there is a demand to spread the complex analysis to the real analysis, it must be the binary real function. In real plane, the integral of a simple closed curve is 2π. Similarly, the integral of a simple closed curve in complex plane is 2πi. It is clear that there is a relationship between the results in complex plane and that in real plane.

When breaking the complex function into two parts, it can transfer to the following terms which are divided into several parts:

$$\begin{aligned}\int_c \frac{1}{z-z_0}dz &= \int_c \frac{1}{x+iy-(x_0+iy_0)}dx+idy \\ &= \int_c \frac{1}{x-x_0+i(y-y_0)}dx+idy \\ &= \int_c \frac{x-x_0-i(y-y_0)}{(x-x_0)^2-(y-y_0)^2}dx+idy \\ &= \mathrm{A}+iB\end{aligned}$$

where $A=\int_c \frac{(x-x_0)\,dx+(y-y_0)dy}{(x-x_0)^2-(y-y_0)^2}$ and $B=\int_c \frac{(x-x_0)\,dy-(y-y_0)dx}{(x-x_0)^2-(y-y_0)^2}$. According to Cauchy's integral formula, the results of integration is 2 πi . Consequently, one can get that the real part of the integration is zero, and the imaginary part is 2π.

Just use the steps above, the promotion of a complex theorem to application in real function is finished. Promotion is a necessary process in science research, abundance principle is found by this method that from common to specific, from one hand to another hands. The researcher should not only realize the nature of the principle, but also need to open their mind to link things in distinct areas together and find their correlation (Ji, 2023).

4 CONCLUSIONS

This article aims to assist the beginner to understand the two indispensable theorem in complex analysis, Cauchy-Goursat theorem and Cauchy's integral formula. From their conditions to their proof, the author has adopted appropriate words to introduce the principle in detail. Then the article referred what situations the principle could be used in complex analysis and real analysis. In complex analysis, the author calculated two similar integrals with distinct characteristics to show that how to integrate a formula with only one strange point or two or more strange points, prepared for the proof and application of residue theorems in the next. On the other hand, the theorem is spread from complex analysis to real analysis, it could inspire the ideas of generalization of more new theorems in the future.

However, there are still plenty of drawbacks exists in the article needed to be enhanced and corrected, such as the grammar, the accuracy of the words. Moreover, as it is a mathematics paper, the proof is supposed to be more rigorous and in detail. If the reader thinks anywhere has shortage, don't be shy to contact the author and have a communication. It is my honour to accept contact and correction. In the future, the author will read the authority complex variables mathematics textbook and think carefully, then concentrated on the further complex analysis, prove and apply the Taylor's series, Laurent expansion, and residue theorems.

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