

Kinematic and Dynamic Analysis of Quadruped Legged Robots: A New Formulation Approach

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Abstract: This paper presents a framework for efficient kinematic and dynamic modelling of a quadruped robot using the recursive Newton-Euler method. The robot features 12 actuators—three per leg—and is analysed under both static walking and dynamic trotting gaits. The formulation incorporates assumed ground reaction forces and system over-constraints, enabling the resolution of contact forces through a reduced set of six linear equations. Twelve generalized coordinates are used for static gait analysis, with an additional generalized coordinate introduced for dynamic trotting. Body attitude, velocity, and acceleration are derived from joint-space trajectories, and forward dynamics is computed by inverting the inverse dynamics equations by numerically evaluating the mass matrix and nonlinear torque vectors. By employing a reduced set of generalized coordinates and simplified constraint handling of ground reactions, the proposed framework streamlines rigid-body dynamic simulation for such a high-degree-of-freedom system.

1 INTRODUCTION

Modelling of engineering systems and processes is a critical step toward achieving robust and accurate control. This is equally applicable in the field of robotics, where obtaining a realistic and precise dynamic model is essential for accurate motion control. Achieving high levels of autonomy, accuracy, and precision in robotic systems necessitates a thorough understanding of the underlying kinematic and dynamic principles specific to the robot. These fundamentals are critical for the precise control, modelling, and optimization of robotic behaviour.

Kinematics, which characterizes the mathematical framework governing the motion of robotic systems independent of forces, plays a vital role in the design and analysis phases of robotic system development. In contrast to kinematics, dynamics focuses on the relationship between the forces, torques, and the resulting motion of the respective robotic system. This understanding is fundamental for accurate modelling and control,

enabling the development of control strategies that ensure good performance and stability. Dynamic analysis is essential for designing controllers capable of adapting to varying environmental conditions, thereby enhancing the robot's reliability and robustness. By analysing the forces and torques required to achieve desired motions, one can optimize system design for energy efficiency without sacrificing performance. Similar to kinematics, dynamics also provides insight into the interaction between the robot and its environment, which is critical for executing complex tasks such as terrain exploration and manipulation.

Locomotion in multi-body systems, such as legged robots, presents significant complexity due to the coupled nature of joint interactions and the intermittent ground contacts. Addressing key challenges such as stability control and trajectory planning necessitates a comprehensive understanding of the system's kinematic and dynamic behaviour. Currently, the majority of quadruped dynamic models are formulated using the Euler-Lagrange approach, which systematically derives the equations of motion

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based on energy methods and generalized coordinates (He et al., 2016; Sun et al., 2007). The Euler-Lagrange approach derives the equations of motion by computing the system's kinetic and potential energy, while incorporating closed kinematic chains and high-dimensional constraint conditions. This often necessitates the introduction of additional generalized coordinates and constraint equations, resulting in a set of differential-algebraic equations (DAEs) that are computationally intensive to solve. An alternative that is widely adopted in multibody dynamics is the Newton-Euler formulation, which employs a recursive algorithm to compute the equations of motion and joint torques more efficiently (Bennani & Giri, 1996; Fu & Gao, 2022; Liu et al., 2019; Potts et al., 2011). While the forces and moments can be numerically computed with relative ease (Morais et al., 2021; Shi et al., 2021) analytical solutions using this formulation are often complex and introduce unnecessary constraint forces. In most cases, in addition to the body six variables and their derivatives, 12 joint states, and their derivatives, are required to calculate joint forces and moments.

In this paper, we propose an accurate model that captures the underlying rigid-body dynamics of the quadruped robot while reducing simulation complexity and improving computational efficiency. The model is designed to exploit the kinematic over-constraints present in the quadruped system, reducing the number of generalized coordinates to 12 for static walking and 13 for dynamic walking, thereby simplifying the dynamic equations. Furthermore, the study focuses on formulating a recursive algorithm for computing the input joint torques, enabling efficient dynamic analysis and accurate derivation of the system's equations of motion. Computational efficiency is significantly enhanced by formulating the equations of motion such that only six unknown ground reaction force components need to be solved. This reduction in dimensionality streamlines the overall dynamic computation and improves the accuracy and efficiency of force and moment calculations within the system

2 KINEMATIC MODEL

The quadruped model presented in this study features four identical legs with symmetric kinematic configurations. The coordinate frame assignment for a representative leg is illustrated in Fig. 1.

Frame 0 is defined as the reference frame for the leg and maintains a constant orientation relative to the body frame. Its axes are aligned such that the x , y , and

z axes of Frame 0 correspond to the y , x , and z axes of the body frame, respectively. Frame 1 is assigned to the actuator responsible for the Hip Abduction/Adduction (HAA) joint. Frame 2 is associated with the Hip Flexion/Extension (HFE) joint, while Frame 3 corresponds to the Knee Flexion/Extension (KFE) joint. Frame 4 is located at the end of the leg, representing the end-effector. The coordinate frames are assigned according to the Denavit-Hartenberg (D-H) convention. The Z axes of Frames 1 through 3 are aligned with the axes of rotation of their respective joints, ensuring consistency with the joint actuation directions. Due to the orthogonality between the first and second joint axes, the Z axes of Frames 1 and 2 are orthogonal, whereas the Z axes of Frames 2 and 3 are parallel. The X axis in each frame is defined to be orthogonal to both its own Z axis and the Z axis of the subsequent frame. Additionally, the X axes of Frames 2 and 3 are aligned with the axes of their corresponding links. Joint angles are defined as the relative angles between the X axes of two consecutive frames. The system's home configuration, illustrated in Fig. 1.a, corresponds to all joint angles being set to 0° , resulting in the alignment of all X axes across the kinematic chain as shown in the figure. From this configuration, a sequential set of joint rotations is applied to reach the final standing posture depicted in Fig. 1.d. In Fig. 1.b, Links 2 and 3 are simultaneously rotated by 90° , with respect to Frame 0. Fig. 1.c shows Link 2 rotated by 45° , relative to Frame 1. Finally, in Fig. 1.d, Link 3 is rotated by 45° , relative to Frame 2, completing the transition to the standing configuration.

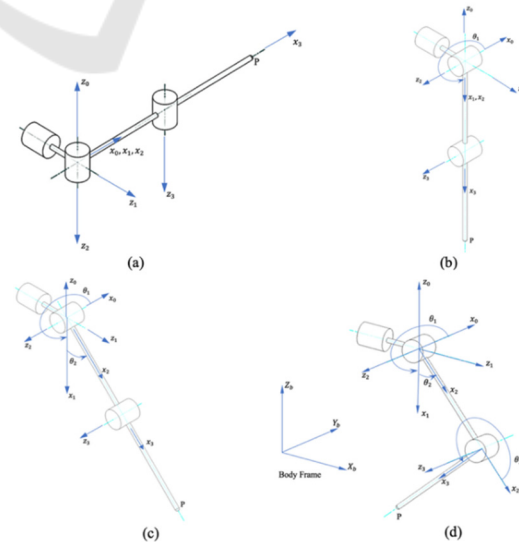


Figure 1: Coordinate frame assignment for a single leg of the quadruped robot.

In this work, identical frame assignments are applied to all four legs of the quadruped system to ensure consistency and simplify the kinematic formulation. The coordinate frames are defined using transformation matrices derived according to the D-H convention. The corresponding D-H parameters for a single leg, as represented in the standing configuration shown in Figure 4.2d, are summarized in Table 1.

Table 1: Implemented D-H table for the legged system.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	90	0	0	θ_1
2	-90	0	0	θ_2
3	0	l_2	0	θ_3
4	0	l_3	0	0

3 INVERSE DYNAMICS FORMULATION STRATEGY

A primary objective of this research is to develop a systematic strategy for conducting dynamic analysis and formulating the equations of motion required for simulating the rigid-body dynamics of the quadruped system. The resulting methodology, which will be elaborated upon in the subsequent section, on the dynamic analysis, establishes the foundation for accurate and computationally efficient simulation model. Prior to presenting the full dynamic formulation, this section outlines the overall strategy, which involves the development of a recursive algorithm for computing the input joint torques.

The input joint torques are computed from joint reaction moment vectors that are determined starting from assumingly known ground reaction forces. The system is analysed recursively, starting from the foot and progressing to Frame 1 at the HFE joint, solving for forces and moments on each link. Fig. 2.a shows the forces and moments exerted on the torso at all four HFE joints, while Fig. 2.b illustrates the reaction force and moment on a single leg at the HFE joint and the ground.

The HFE joint reaction force and moment vectors, $\mathbf{f}_{1ij}$ and $\mathbf{n}_{1ij}$ are decomposed into two components: one resulting from the leg's inertial forces and moments, and the other from the ground reaction forces, as shown below in (1) and (2), for a single leg.

$${}^b\mathbf{f}_1 = {}^b\bar{\mathbf{f}}_1 + {}^b\hat{\mathbf{f}}_1 \quad (1)$$

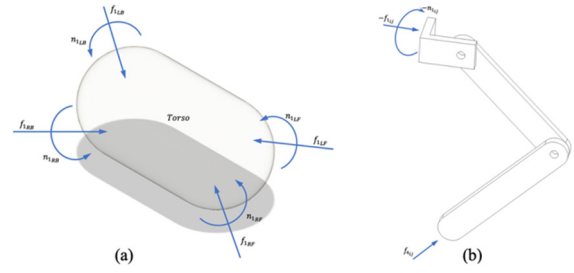


Figure 2: Schematic illustration of forces and moments acting on the torso and a representative leg.

$${}^b\mathbf{n}_1 = {}^b\bar{\mathbf{n}}_1 + {}^b\hat{\mathbf{n}}_1 \quad (2)$$

where, ${}^b\mathbf{f}_1$ and ${}^b\mathbf{n}_1$ represents the reaction force and moments exerted by link 1 on the body, excluding the contribution from the ground reaction force, and ${}^b\bar{\mathbf{f}}_1 = {}^bR {}^4\mathbf{f}_4$ and ${}^b\bar{\mathbf{n}}_1 = {}^bP_4 \times {}^bR {}^4\mathbf{f}_4$ represents the reaction forces and moments exerted by the legs on the body due to ground reaction force.

All force and moment vectors here are expressed in body-fixed reference frame.

In the initial stage, ${}^b\bar{\mathbf{f}}_1$ and ${}^b\bar{\mathbf{n}}_1$ are computed under the assumption that all legs are suspended in air, implying that $\mathbf{f}_{4ij}$ and $\mathbf{n}_{4ij}$ to be 0. The algorithm then recursively propagates calculating forces and moments through each joint, progressing toward the HFE joint for each leg. In the second stage, the still unknown ground reaction force and moment components are determined by solving the dynamic force and moment balance equations as shown in (3) and (4).

$$\sum {}^bR_{ij} {}^4\mathbf{f}_{4,ij} = m_b {}^b\mathbf{a}_b - {}^b\mathbf{G}_b - \sum {}^b\bar{\mathbf{f}}_{1,ij} \quad (3)$$

$$\sum {}^bP_{4,ij} \times {}^bR_{ij} {}^4\mathbf{f}_{4,ij} = {}^bI_b {}^b\boldsymbol{\alpha}_b - {}^bI_b {}^b\boldsymbol{\omega}_b \times {}^b\boldsymbol{\omega}_b - \sum {}^b\bar{\mathbf{n}}_{1,ij} \quad (4)$$

Equations (3) and (4) are formulated with known right-hand side terms, while the unknowns are grouped into left-hand side expressions involving ${}^4\mathbf{f}_{4,ij}$ for the four legs. In-depth analyses of the solution are presented in the subsequent sections.

The system's inverse dynamic analysis is carried out using the recursive Newton-Euler algorithm (Craig, 2005). Joint velocities and accelerations are computed from the body frame to the foot frame using an outward iteration process. Using these velocity and acceleration vectors, the inertial force vector $\mathbf{F}_i$ and inertial moment vector $\mathbf{N}_i$ acting on the i th link are computed via the Newton-Euler equations. The reaction force $\mathbf{f}_i$ and reaction torque $\mathbf{n}_i$ at the i th joint are computed recursively using the reaction force and torque vectors from the subsequent joint $i+1$, based

on the previously calculated force and moment vectors.

The presented formulations are defined with respect to a single leg and must therefore be applied independently to all four legs. The recursive computation initiates at the last link (Link 3), where the ground reaction force is either evaluated or assumed to be zero in the case of a non-contact (swing) phase. A systematic approach for computing the ground reaction forces is developed and will be detailed in the subsequent section.

4 INVERSE DYNAMIC SOLUTION

The inverse dynamics analysis, which involves computing joint torques based on the specified motion of the robot, including the robot's body, is conducted with respect to a specific set of ground reaction forces. Depending on the gait pattern, multiple leg support configurations are possible. The nominal configuration corresponds to the quadruped standing on all four legs, permitting body motions such as roll, pitch, yaw, and vertical displacement. In contrast, during a static walking gait, three legs remain in contact with the ground at any given time, while one leg is in the swing phase. In dynamic gaits such as trotting or pacing, two diagonally or laterally opposed legs are in contact with the ground at any given time, while the remaining two are in the swing phase. Additionally, there exist phases, such as during jumping or bounding, where all four legs are airborne. To compute the joint torques, required for rigid-body dynamics, it is essential to determine the corresponding ground reaction forces. The ground reaction force computation for each of these scenarios is discussed in the following subsections.

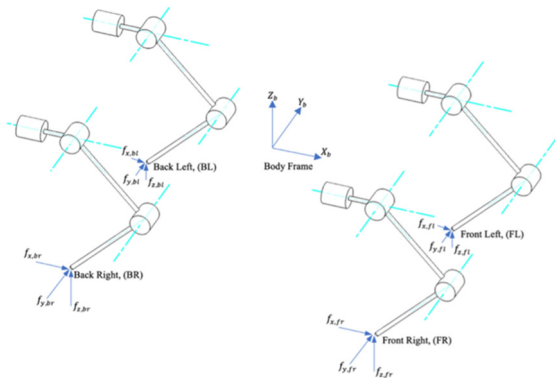


Figure 3: Ground reaction force on the legs of quadruped system.

4.1 Ground Support on Four Legs

When all four legs are in contact with the ground, ground reaction forces act on each leg, as depicted in Fig. 3. This configuration introduces twelve unknown force components, represented as $[f_{i,j}]^T$, where i refers to the left or right leg and j indicates front or rear leg. However, the dynamic equilibrium of the robot's body yields only six equations, three for linear forces and three for moments, corresponding to (3) and (4). As the system contains six more unknowns than available equations, it becomes statically indeterminate and hence requires six additional constraint equations to fully resolve the ground reaction forces.

To address this issue, a set of simplifying assumptions is introduced to provide the necessary constraint equations. The vertical ground reaction forces (Z-direction) for all four legs are treated as independent, contributing four separate unknowns. In contrast, the ground reaction forces in the X-direction are assumed to be equal across all four legs, reducing the associated unknowns to a single variable. Likewise, the Y-direction ground reaction forces are assumed identical for all legs, introducing only one additional unknown. These assumptions reduce the total number of unknowns to six, allowing the system to be fully determined from the available dynamic equilibrium equations.

4.2 Ground Support on Three Legs

During static walking, the robot maintains ground contact with three legs, resulting in nine unknown ground reaction force components, still exceeding the number of available equilibrium equations. To address this, the same assumptions applied in the full support case are maintained for the Z and X directions: three independent vertical (Z-axis) reaction forces and one common reaction force along the X-axis, reducing the number of unknowns in these directions to four. However, the reaction force components along the Y-direction are viewed as follows. When one leg is lifted, it is assumed that the leg on the same side experiences a distinct reaction force component from the ground along the Y than the two legs on the opposite side, which are assumed to receive equal lateral reaction force components from the ground. This introduces two additional unknowns along the Y-axis. With these assumptions, the total number of unknowns is reduced to six, matching the number of dynamic equilibrium equations and allowing the reaction forces to be determined. The static condition in these scenarios

justifies the simplifying assumptions used. In dynamic walking, however, all ground reaction force components are treated independently as outlined next.

4.3 Ground Support on Two Legs and During Airborne

In dynamic gaits such as trot and pace, the robot maintains ground contact with only two legs at any given time. Under these conditions, the reaction force problem becomes more tractable, as the system involves only six unknown ground reaction force components, three per contact point, which can be determined directly from the equations of motion without necessitating additional assumptions or constraints.

Conversely, during a pronk gait, the robot enters an aerial phase wherein all limbs are off the ground, resulting in zero ground contact. Consequently, the ground reaction forces at the feet are identically zero. This absence of external contact forces simplifies the system dynamics, allowing for direct computation of the internal reaction forces and moments exerted by the legs on the robot's body based solely on its inertial properties and joint torques.

5 FORWARD DYNAMICS

With the forward dynamic formulation, the objective is to determine the acceleration components corresponding to the generalized coordinates assuming the knowledge of the input torque vector. The focus of this study is on two different gait patterns, namely the static walking gait and the dynamic trotting gait.

In the case of a static walking gait, three legs maintain ground contact at any instant of time. This statically stable support configuration allows the estimation of the robot body's pose through the known joint angles of the supporting legs. Under the assumption of static equilibrium and using forward kinematic relations, the system is described by twelve generalized coordinates, corresponding to the three joint angles $(\theta_{1,i}, \theta_{2,i}, \theta_{3,i})$ for each of the four legs, resulting in a total of twelve joint coordinates. On the other hand, during the dynamic trotting gait, only two diagonally opposed legs are in contact with the ground at any given moment, which results in a reducing the support polygon to a line. The associated body rotation cannot be determined solely from the displacement of the attachment points of the two supporting legs. To address this limitation, the roll

angle $\emptyset$ of the quadruped system is introduced as an additional generalized coordinate, increasing the total number of generalized coordinates to thirteen.

The accelerations of the generalized coordinates, twelve in the case of the static walking gait and thirteen in the case of the dynamic trotting gait, are obtained through the forward dynamics formulation. The resulting accelerations are numerically integrated twice to obtain the joint velocities and displacements. The body's orientation and angular rates are subsequently determined through forward kinematics, as described in the previous section. To evaluate the dynamic model, the inverse dynamics formulation based on the Newton-Euler recursive algorithm is used to numerically compute the system's inertia matrix, as well as the gravity, Coriolis, and centrifugal torque vectors (Harib & Srinivasan, 2003; Walker & Orin, 1982).

Once the inertia matrix, Coriolis and centrifugal torque vector, and gravity torque vector have been computed, they are combined with the input torque vector, generated by the control law, to solve for the joint acceleration vector as shown in (5). This acceleration vector is then numerically integrated twice to obtain the joint velocity and displacement vectors.

$$\ddot{q} = M(q)^{-1}(\tau - C(q, \dot{q}) - G(q)) \quad (5)$$

A comprehensive analysis of the system dynamics, control implementation, and experimental validation are presented in (Sureshkumar, 2025).

6 SIMULATION RESULTS

A simulation was conducted in Simulink to validate the proposed dynamic formulation. This section presents a brief overview of the open-loop control results, wherein the acceleration, velocity, and displacement vectors obtained from the forward dynamics are compared against the corresponding vectors derived from the desired trajectory, computed using inverse kinematics.

Fig. 4, illustrates the path planning profile for the dynamic trotting gait, depicting the positional trajectories of both the robot's body and the end-effectors (feet) over the course of the gait cycle. These trajectories serve as essential inputs for the inverse kinematics computation, which subsequently yields the desired joint angle trajectories required to achieve the planned motion. For the dynamic trotting gait, as depicted in Figure 5.7, the back-right (BR) and front-left (FL) legs move in unison during one step phase, followed by the back-left (BL) and front-right (FR)

legs during the subsequent phase.

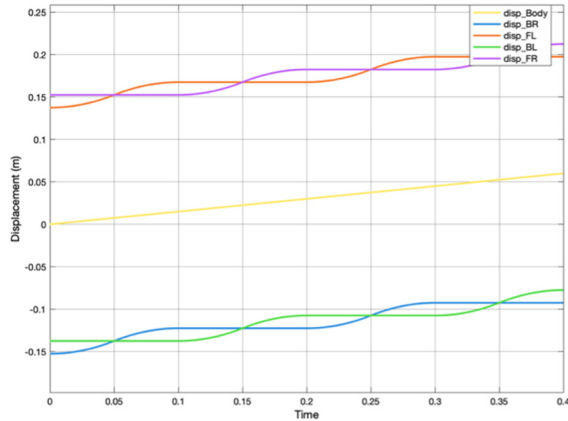


Figure 4: Gait profile for dynamic trotting.

Initially, the inverse kinematics is used to compute joint positions, velocities, and accelerations, which serve as inputs to the inverse dynamics model for calculating the required joint torques. As discussed in the previous section, the inertia matrix, Coriolis and centrifugal torque vector, and gravitational torque vector are determined numerically. These, along with the input torque vector, are then used in the forward dynamics formulation to solve for joint accelerations as described in (5). The computed joint accelerations are subsequently integrated twice over time to obtain the corresponding joint velocities and displacements. The resulting joint kinematic trajectories are then compared against those derived from the inverse kinematics to validate the proposed methodology. The comparison results in the form of acceleration, velocity and displacement of the HFE joints of the back-right leg are shown in Fig. 5 – 7.

Figure 5-7 illustrates high agreement of the simulation of the joint trajectories obtained from the forward dynamic simulation with those computed via the inverse kinematics based on the desired trajectory. The observed discrepancies are minimal, with a maximum percentage error of only 0.0006%.

Such results indicate strong validation for the accuracy and effectiveness of the proposed methodology in modelling the system dynamics. The close agreement between the forward dynamics outputs and the reference trajectories derived from inverse kinematics demonstrates the reliability of the implemented numerical framework. Additionally, the consistency across results confirms the correctness of the developed code and simulation environment, ensuring its proper functionality for dynamic analysis. This validation reinforces the robustness of

the computational framework and supports its applicability to real-world quadruped robotic systems for dynamic modelling and control.

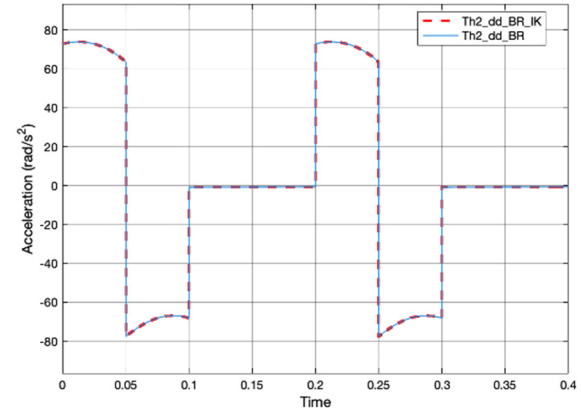


Figure 5: Comparison of the HFE joint acceleration for the back-right (BR) leg.

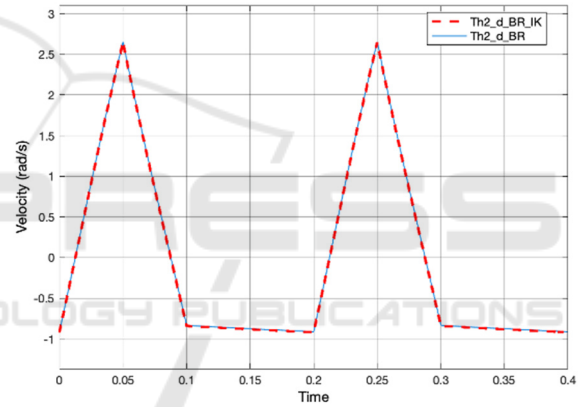


Figure 6: Comparison of the HFE joint velocity for the back-right (BR) leg.

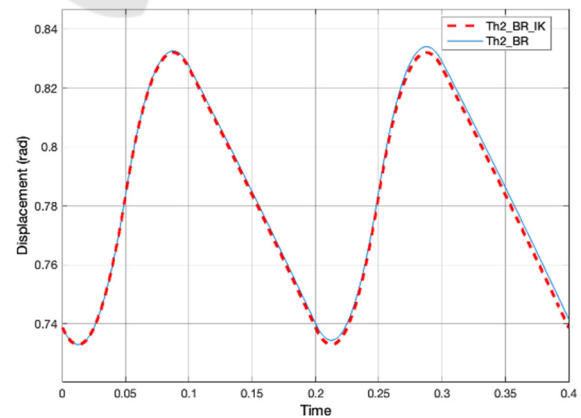


Figure 7: Comparison of the HFE joint displacement for the back-right (BR) leg.

7 CONCLUSIONS

This study presents the development of a novel framework for formulating the system dynamics of a quadruped robot through the systematic selection of kinematic and dynamic constraints. The dynamic model is constructed using the recursive Newton-Euler formulation to ensure computational efficiency. The forward dynamics is implemented by deriving the joint acceleration vector from the system rigid body dynamic equation. A numerical approach is used to compute the inertia matrix, Coriolis and centrifugal torque vector, and gravitational torque vector. These dynamic terms are then utilized to solve for the joint acceleration vector which could be then integrated twice in the simulation to obtain the joint velocity and displacement vectors.

A simulation environment was established for a quadruped system comprising twelve actuators, with each leg driven by three independently actuated joints, enabling full mobility and control authority. Motion trajectories were generated based on a predefined dynamic trotting gait. The accuracy of the dynamic formulation and the developed simulation model was validated by simulating the system under representative trajectories corresponding to dynamic trotting gaits.

A comprehensive analysis of the dynamic formulation, with a particular focus on the forward dynamics could include closed-loop control simulations employing a feedforward and feedback control strategies, followed by experimental validation to assess the accuracy and effectiveness of the proposed approach.

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