Enhancing Resilience of Strong Structural Controllability in Leader-Follower Networks

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Abstract: This paper explores measures of edge augmentation to enhance resilience of strong structural controllability for control systems modeled as leader-follower networks. Unlike existing methods which typically increase the number of leaders, the proposed approach achieves resilience by strategically adding edges, thus maintaining leader sets with a small cardinality. Using the zero forcing method, conditions are derived to enhance resilience either for specific agents or for the entire network. Numeric simulations validate the approach and show its effectiveness in large and complex networks.

1 INTRODUCTION

Networked Control Systems (NCS) are integral to modern engineering, enabling distributed coordination and scalability in complex systems such as smart grids, traffic systems, swarms and biological networks. In these systems, it is frequently desired to control the network by injecting control actions only for a small subset of nodes. This leads to the notion of leader-follower frameworks, where leaders have a control input, whereas followers do not have one (Porfiri and di Bernardo, 2008; Egerstedt et al., 2012). While this can simplify the control architecture, the system may get more fragile with respect to malfunctions or attacks, if the loss of an agent and/or its connections significantly impairs the system's functionality (Pasqualetti et al., 2020).

Controllability and resilience are thus important properties when investigating in how far such systems can maintain their function if subject to internal or externally triggered changes. Controllability ensures the ability to let the system transition from any initial state to a desired state, while resilience pertains to the system's capacity to withstand (or recover from) disruptions. These properties are crucial, especially in safety-critical applications where failures can have catastrophic consequences.

This paper focuses on the notion of strong structural controllability (SSC) which requires controllability for varying interconnections of states (and thus structures) of the system to be controlled. This is particularly advantageous in NCS, since controllability depends on the network topology and the leader set, rather than only particular weights assigned to constantly existing interconnections. This makes SSC suitable for real-world applications, in which information on edge weights may be uncertain, time-varying, noisy, or even vanishing (Chapman and Mesbahi, 2013; Monshizadeh et al., 2014).

In order to assess SSC of leader-follower networks as well as its resilience against changing network topologies, this paper utilizes the method of zero forcing sets, a graph-theoretic approach that is frequently used in the context of SSC for NCSs (Monshizadeh et al., 2014; Abbas et al., 2020b; Schmidtke et al., 2024). In the context of SSC, the following existing papers have proposed edge augmentation, i.e. the addition of edges to a graph, as a means to modify the system structure for enhanced controllability. In (Chen et al., 2019), minimal edge sets are computed to render non-SSC systems structurally controllable, whereas (Abbas et al., 2020a) and (Mousavi et al., 2021) identify a set of edges that can be added without violating SSC. These approaches leverage edge augmentation as a design tool to either restore or preserve controllability.

A certain body of literature in the given context has examined the relationship between graph-theoretic robustness measures and the minimum number of required leaders: The most often used measure is the Kirchoff index, which formulates the system's resilience against disconnection. (Pasqualetti et al., 2020) and (Abbas et al., 2020b) investigate the trade-off between an increasing connectivity and the minimum number of required leader agents, which is

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also discussed as a central controllability measure in (Egerstedt et al., 2012). The work in (Abbas et al., 2024) identifies edge additions leading to a reduction of the required number of leader agents, while increasing the robustness measure. A design procedure which builds a graph which is SSC with the highest possible Kirchoff index for specified graph parameters is presented in (Patel et al., 2024).

The notion of robustness addressed in the aforementioned work differs from the approach to resilience taken in this paper. Rather than focusing on graph connectivity, resilience of controllability is here specifically considered with respect to preserving SSC in case of structural disruptions such as node or edge failures. Resilient SSC is addressed only in very few contributions: In (Schmidtke et al., 2024), conditions have been specified which allow to check which agents can be removed from the system, while maintaining SSC with the current leader set. The papers (Abbas, 2023) and (Alameda et al., 2024) exploit methods to specify a set of leader agents which guarantees that a certain number of nodes or edges can be removed, while keeping the system SSC. However, these methods require the introduction of far more leader agents – this, on the one hand, leads to leader selection problems which are known to be NP-hard in general (Aazami, 2008). On the other hand, this approach contradicts the goal to have as few leaders as possible. Thus overall, a significant gap in literature is the enhancement of resilience without increasing the number of leader agents. This paper addresses this gap by proposing schemes to augment the graph by additional edges in order to achieve strong structural controllability, including the case that agents leave and/or join the network.

The paper is structured as follows: Section 2 introduces preliminaries on the system class, the property of strong structural controllability, and the zero forcing method. Section 3 formally states the problem of enhancing resilient strong structural controllability through modifying the edges of the graph. Section 4 presents the main results, while Section 5 provides a numeric example to illustrate the approach. Finally, Section 6 concludes the paper, summarizing the findings and their implications.

2 PRELIMINARIES

Networks are in this paper represented by a graph $\tilde{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, in which $\mathcal{V} = \{1, \dots, N\}$ is the set of agents, $\mathcal{E}$ the set of edges, and $\mathcal{W}$ the set of weights assigned to each edge according to $w_{ij} : \mathcal{E} \to \mathbb{R} \setminus 0$. For simplification, it is assumed that $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$, which simplifies the notation. However, $w_{ij} \neq w_{ji}$ is allowed, making the graph directed. It is important to note, that the upcoming results can be straightforwardly extended to cases where the assumption of bi-directional edges is dropped. Edges (i, i) for representing self-loops are permitted. Let all neighbors of a node i be collected in a set $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}, j \neq i\}$.

The set of agents is divided into a set of leader agents $\mathcal{V}_l = \{l_1, \dots, l_{m_l}\} \subseteq \mathcal{V}$ and the set of follower agents $\mathcal{V} \setminus \mathcal{V}_l$. The m_l leaders have a control input $u_i(t)$ and the dynamics:

$$\dot{x}_i(t) = w_{ii} \cdot x_i(t) + \sum_{j \in \mathcal{N}_i} w_{ij} \cdot x_j(t) + u_i(t), \qquad (1)$$

while the dynamics of the followers:

$$\dot{x}_i(t) = w_{ii} \cdot x_i(t) + \sum_{j \in \mathcal{N}_i} w_{ij} \cdot x_j(t) \qquad (2)$$

lacks an input. Note that with this definition the leaders are allowed to interact dynamically with follower and leader agents. The initial state of any agent is denoted by $x_i(0) = x_{i,0}$.

The global dynamics of all agents can be derived by introducing the state vector $x(t) := [x_1^T(t), x_2^T(t), \dots, x_N^T(t)]^T \in \mathbb{R}^N$, the input vector $u(t) := [u_1^T(t), u_2^T(t), \dots, u_{m_l}^T]^T \in \mathbb{R}^{m_l}$, the weight matrix W with elements w_{ij}, and the leader selection matrix $B \in [0, 1]^{N \times m_l}$ according to:

$$B = \begin{cases} b_{ij} = 1 & \text{if } i = l_j \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

The global dynamics of the network can then by written as:

$$\dot{x}(t) = Wx(t) + Bu(t), \qquad (4)$$

in which the leader agents can be controlled, whereas the follower agents only respond to the behavior of the leader agents.

2.1 Strong Structural Controllability

As is well known, a system (4) is controllable if the following matrix has full rank (Antsaklis and Michel, 1997):

$$R(W, B) = \begin{bmatrix} B & WB & \dots & W^{n-1}B \end{bmatrix}, \qquad (5)$$

In networked systems, however, edge weights are often either unavailable or time-varying, rendering the direct use of (5) impractical. Then, strong structural controllability (SSC) becomes relevant, as it assesses controllability solely based on the system's structure, independent of the specific edge weights. To define

SSC, the following family of matrices associated with a given graph $\tilde{G}$ is considered:

$$\mathcal{W}(\tilde{G}) = \{W \in \mathbb{R}^{N\times N} : \text{for } i \neq j,\ w_{ij} \neq 0 \Leftrightarrow (j,i) \in \mathcal{E}\}. \quad (6)$$

This matrix family includes all matrices for which the nonzero off-diagonal entries correspond exactly to the edges of $\tilde{G}$. Note that the diagonal elements of every $W \in \mathcal{W}(\tilde{G})$ can be arbitrary. With this notation, SSC can be defined as follows:

Def. 1 (Strong Structural Controllability). *A linear system according to* (4) *is SSC if and only if the pair* (W,B) *is controllable for all* $W \in \mathcal{W}(\tilde{G})$*, where B represents the agents through which inputs are applied.*

Aiming at SSC according to this definition requires that any possible matrix W of the given dimension needs to be considered, while the zero entries W are encoded also by the fact that a corresponding edge is not defined in $\mathcal{E}$. Thus, it is sufficient (without loss of generality) from here on to refer the considerations on SSC to a reduced version $G = (\mathcal{V},\mathcal{E})$ of the graph $\tilde{G}$ without weights.

2.2 SSC and Zero Forcing

The use of *zero forcing* as a method to examine SSC is motivated by its direct correlation to network controllability on graphs, as explained in (Monshizadeh et al., 2014), see also Theorem 1. Zero forcing is defined as follows (AIM Minimum Rank - Special Graphs Work Group, 2008):

Def. 2 (Zero Forcing). *For a graph* $G = (\mathcal{V},\mathcal{E})$*, let* $\mathcal{V}_0^c \subset \mathcal{V}$, $|\mathcal{V}_0^c| > 0$ *be chosen arbitrarily as an initial set of* colored nodes*. Let then* $\mathcal{V}$ *be partitioned into* $\mathcal{V}_0^c$ *and a set* $\mathcal{V}_0^u$ *of* initially uncolored nodes *according to* $\mathcal{V}_0^c \cap \mathcal{V}_0^u = \varnothing$ *and* $\mathcal{V}_0^c \cup \mathcal{V}_0^u = \mathcal{V}$*. If a colored node i then only has one uncolored neighbor j, then j is* forced *to be colored by i (denoted by* $i \rightharpoonup j$*), i.e. j is inserted into* $\mathcal{V}^c$ *and removed from* $\mathcal{V}^u$*. This coloring process continues until no further coloring can be executed, leading to final colored and uncolored sets* $\mathcal{V}_f^c$ *and* $\mathcal{V}_f^u$*.*

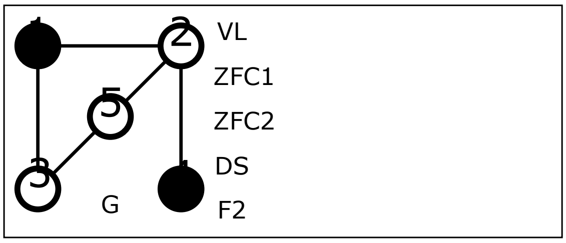

Figure 1: Example for zero forcing with $\mathcal{V}_0^c$ as ZFS with minimal cardinality.

Def. 3 (Derived Set). *If zero forcing has started from* $\mathcal{V}_0^c$ *and terminated with* $\mathcal{V}_f^c$*, the latter set is also called the* derived set *and is denoted by* $\mathcal{D}(G,\mathcal{V}_0^c) := \mathcal{V}_f^c$*.*

Def. 4 (Zero Forcing Set). *If a selected set of initially colored nodes* $\mathcal{V}_0^c$ *leads to* $\mathcal{D}(G,\mathcal{V}_0^c) = \mathcal{V}$ *(i.e. all nodes are colored at the end of zero forcing),* $\mathcal{V}_0^c$ *is called* zero forcing set *(ZFS).*

In addition the following terms are introduced:

Def. 5 (Forcing Chain). *A* forcing chain *(FC)* $\mathcal{C}(G,\mathcal{V}_0^c)$ *contains the list of forces* $i \rightharpoonup j$ *obtained during zero forcing for the graph G, where this list is sorted according to the sequence of forces carried out during the forcing procedure when starting from* $\mathcal{V}_0^c$*.*

Def. 6 (Forcing Agents). *For an agent* $j \in \mathcal{V}$*, the set* $\mathcal{F}_j(G,\mathcal{V}_0^c)$ *contains all* forcing agents *(FA)* $i \in \mathcal{V}$*, which can color j through a force* $i \rightharpoonup j$ *contained in any forcing chain* $\mathcal{C}(G,\mathcal{V}_0^c)$*.*

An example that illustrates these quantities can be found in Fig. 1. As shown there, forcing chains generally lack uniqueness, while the derived set $\mathcal{D}(G,\mathcal{V}_0^c)$ remains unique (AIM Minimum Rank - Special Graphs Work Group, 2008). For any set of FA, the inequality $|\mathcal{F}_j(G,\mathcal{V}_0^c)| \leq |\mathcal{N}_j|$ is always satisfied because only neighbors are capable of coloring an agent. When $\mathcal{V}_0^c$ constitutes a ZFS, it ensures that $|\mathcal{F}_j(G,\mathcal{V}_0^c)| \geq 1$ for all $j \in \mathcal{V}_0^u$, guaranteeing that at least one FC for every uncolored agent exists. Accordingly, $|\mathcal{F}_j(G,\mathcal{V}_0^c,j)| = 0$ for all $j \in \mathcal{V}_0^c$ since these agents are colored initially.
The following theorem from (Monshizadeh et al., 2014) establishes that zero forcing can determine if a leader set $\mathcal{V}_l$ guarantees strong structural controllability for the system defined by the graph G:

Theorem 1 ((Monshizadeh et al., 2014)). *System* (4) *with* $W \in \mathcal{W}$ *and B defined by the leader set* $\mathcal{V}_l$ *is SSC if and only if* $\mathcal{V}_l$ *is a ZFS for G.*

2.3 Resilient SSC

In network controllability, resilience refers to the system's ability to remain controllable despite agent malfunctions or edge failures (Abbas, 2023). In terms of structural controllability, the concept of a ZFS is extended to l-leaky forcing sets, defined as follows:

Def. 7 (l-Leaky Forcing Set). *The leader set* $\mathcal{V}_l$ *is a* $l-$*leaky forcing set (*$l-$*LFS), if* $\forall i \in \mathcal{V}\setminus\mathcal{V}_l :$ $\mathcal{F}_i(G,\mathcal{V}_l) \geq l+1$*.*

Building on the previous definition, (Abbas, 2023) demonstrates that an l-LFS provides resilience against l malfunctions, where a malfunction may involve a node departure or edge removal. (Note that the case

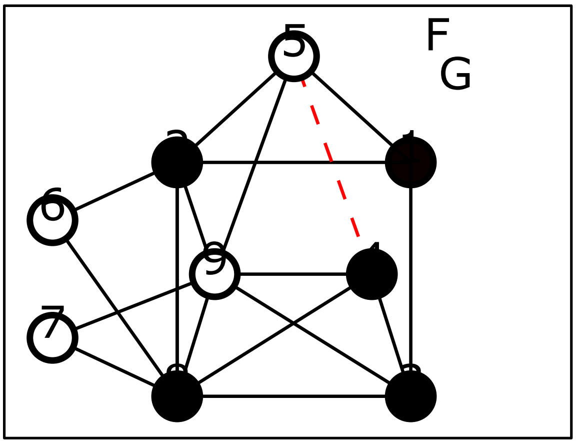

Figure 2: In graph G, node 1 has only node 5 as its only one forcing agent, i.e., the system is no longer SSC if node 1 is removed from the graph. In G', which is obtained by adding the edge indicated by the red dashed line to G, node 5 has an additional forcing agent, and therefore the graph remains SSC, if node 1 is removed.

of entering agents is not considered here, but results on this situation can be found in (Schmidtke et al., 2024).

3 PROBLEM STATEMENT

While in (Abbas, 2023) the l-LFS guarantees resilient SSC by introducing additional leader agents, the present work aims at improving resilience through edge augmentation, i.e., by adding new edges to the graph.

Formally, given a graph $G = (\mathcal{V}, \mathcal{E})$ with a ZFS $\mathcal{V}_l$, the addressed problem is to enhance resilience by adding an edge (i, j) (where $i, j \in \mathcal{V}$ and $(i, j) \notin \mathcal{E}$) such that:

$$|\mathcal{F}_i(G, \mathcal{V}_l)| < |\mathcal{F}_i(G', \mathcal{V}_l)|, \tag{7}$$

for $G' = (\mathcal{V}, \mathcal{E} \cup (i, j))$.

This approach, which can be applied iteratively, follows the principle of the l-LFS methodology by increasing the number of FA for critical nodes. However, this is achieved through the addition of edges rather than by expanding the leader set. The method is particularly well-suited for scenarios in which bottlenecks or vulnerable regions of the network are known in advance. In such cases, edge augmentation can be applied in a targeted manner to strengthen the resilience of specific nodes, thereby improving resilience without the need to introduce additional leader agents.

For instance, in scenarios like the one in Fig. 2, resilience is ensured by leveraging the result from (Schmidtke et al., 2024): SSC is preserved, when an agent j leaves G, provided that for all uncolored neighbors $i \in \mathcal{N}_j$ the following holds:

$$|\mathcal{F}_i(G, \mathcal{V}_l) \setminus j| > 0. \tag{8}$$

To date, no published method systematically explores possibilities of modifying the set of FA through leader selection or edge augmentation. Thus, the following section proposes a framework for strategically modifying the network topology to enhance resilience of SSC.

4 IMPROVING RESILIENCE BY EDGE AUGMENTATION

The primary objective of this section is to establish conditions that guarantee (at least locally) enhanced resilience of SSC by augmenting the set $\mathcal{E}$ of the graph G by an additional edge (i, j). To achieve this, the notion of a *forcing graph* is introduced. The latter enables the analysis of the forcing agents of G by examining how the addition of the edge affects the network's resilience. Based on this analysis, a main theorem is formulated stating conditions under which the augmentation leads to improved resilience. Furthermore, this section shows that, under the derived conditions, a ZFS in G retains the SSC properties, ensuring that resilience is either globally or locally enhanced.

A forcing graph is introduced to determine which nodes are colored before a specific uncolored node, under the condition that a designated colored node does not color any other node. The formal definition is as follows:

Def. 8. *Given a graph $G = (\mathcal{V}, \mathcal{E})$, let $i \in \mathcal{V}^u$ be an uncolored node and $j \in \mathcal{V}^c$ a colored node. A forcing graph $\bar{G}_{ij} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ is constructed by introducing two dummy nodes, α and β, thus $\bar{\mathcal{V}} = \{\mathcal{V} \cup \{\alpha; \beta\}\}$, and a modified set of edges is obtained from $\bar{\mathcal{E}} = \{\mathcal{E} \cup \{(\alpha, l) \mid l \in \mathcal{N}_i\} \cup \{(\alpha, j), (\beta, j)\}\}$. In here, α is connected to all neighbors of i and to j, while β is only connected to j.*

An example of a forcing graph is shown on the left hand side of Fig. 3, where the graph from Fig. 2 is expanded by the nodes $i = 5$ and $j = 4$.

Note that the forcing graph is a direct extension of the *extended graph*, introduced in (Schmidtke et al., 2024), which is denoted in this paper by $\bar{G}_{i\{\}}$. In this extended graph, which can be used to determine the FA of node i over all FC (see (Schmidtke et al., 2024)), the node α is only connected to the neighbors of i, and β has no neighbors.

The following result can be obtained for the derived set of the forcing graph:

Lemma 1. *Using a ZFS $\mathcal{V}_{l,1}$ for G, the following condition holds true for the forcing process on the forcing graph $\bar{G}_{ij}$ for every pair (i,j) with $i \in \mathcal{V}^u$, $j \in \mathcal{V}^c$:*

$$\{\alpha, \beta, i\} \notin \mathcal{D}(\bar{G}_{ij}, \mathcal{V}_{l,1}). \tag{9}$$

Proof. Based on the definition of $\bar{G}_{ij}$, it follows that $\mathcal{N}_i \subset \mathcal{N}_\alpha$. Consequently, node i can only be colored after node α has been colored. Given that $\mathcal{N}_\alpha \setminus \mathcal{N}_i = \{j\}$, node j is the only node being able to color α. Furthermore, due to the construction of $\bar{G}_{ij}$, the condition $\mathcal{N}_\beta = \{j\}$ always holds. Neither α nor β are initially colored, and node j cannot color α and β. Thus (9) follows. ■

The forcing graph can be used to draw conclusions about the forcing process for G based on the derived set:

Lemma 2. *Consider $\mathcal{V}_{l,1}$ as a ZFS for G. For all $v \in \mathcal{D}(\bar{G}_{ij}, \mathcal{V}_{l,1})$ with $i \in \mathcal{V}^u$ and $j \in \mathcal{V}^c$, an FC $\mathcal{C}(G, \mathcal{V}_{l,1}) = \{c_1, \ldots, c_{|V|-|V_{l,1}|}\}$ exists with $c_i = u' \rightharpoonup v'$ such that:*

- *$\exists\, c_{i_1}, c_{i_2} \in \mathcal{C}(G, \mathcal{V}_{l,1})$ with:*

$$c_{i_1} = u' \rightharpoonup v, c_{i_2} = \tilde{v} \rightharpoonup i,\ i_1 < i_2 \tag{10}$$

- *$\nexists\, c_{i_3} \in \mathcal{C}(G, \mathcal{V}_{l,1})$ with:*

$$c_{i_3} = j \rightharpoonup \tilde{u} \text{ and } i_3 < i_2. \tag{11}$$

Proof. Lemma 1 states that a forcing graph $\bar{G}_{ij}$ for a ZFS of G prevents the coloring of agent i. Consequently, $v \in \mathcal{V}^u \cap \mathcal{D}(\bar{G}_{ij}, \mathcal{V}_{l,1})$ implies the presence of a FC $\mathcal{C}(G, \mathcal{V}_{l,1})$ in G with the same leader set $\mathcal{V}_{l,1}$, where all nodes v are colored prior to the coloring of node i, according to (10). Moreover in $\bar{G}_{ij}$, node j consistently has two uncolored neighbors that remain uncolored (as established by Lemma 1), ensuring that all nodes v can be colored before node i without node j coloring any of its neighbors. This directly validates condition (11). ■

Corollary 1. *If $v \notin \mathcal{D}(\bar{G}_{ij}, \mathcal{V}_{l,1})$ with $\mathcal{V}_{l,1}$ as a ZFS for G, then $\nexists\, \mathcal{C}(G, \mathcal{V}_{l,1})$ such that conditions* (10) *and* (11) *are satisfied.*

It is shown next how the forcing graph can be used in order to find an edge (i,j) to specifically add node j to the set of FA of another node i:

Theorem 2. *Consider $i \in \mathcal{V}^u$, $j \in \mathcal{V}^c$, $i \notin \mathcal{N}_j$ and $\mathcal{V}_{l,1}$ as a ZFS for G. Adding the undirected edge (i,j) to G, resulting in $G' = (\mathcal{V}, \mathcal{E}')$ with $\mathcal{E}' = \{\mathcal{E} \cup (i,j)\}$ leads to $j \in \mathcal{F}_i(G', \mathcal{V}_{l,1})$ if and only if the following condition holds for all nodes $l \in \mathcal{N}_j$:*

$$l \in \mathcal{D}(\bar{G}_{ij}, \mathcal{V}_{l,1}). \tag{12}$$

Proof. Based on Lemma 2, condition (12) satisfied $\forall l \in \mathcal{N}_j$ implies that a FC $\mathcal{C}(G, \mathcal{V}_{l,1})$ exists for which (10) and (11) holds. This implies that all neighbors of j can be colored before i is colored, without requiring j to force any of its neighbors. Thus, the force $j \rightharpoonup i$ is possible in G', leading to $j \in \mathcal{F}_i(G', \mathcal{V}_{l,1})$.
To demonstrate the necessity of condition (12), consider the case in which the edge (i,j) is added to G, while

$$\exists\, l' \in \mathcal{N}_j : l' \notin \mathcal{D}(\bar{G}_{ij}, \mathcal{V}_{l,1}) \tag{13}$$

applies. According to Corollary 1, $\nexists\, \mathcal{C}(G, \mathcal{V}_{l,1})$ such that l' is colored while (10) and (11) are true, i.e. either $j \rightharpoonup l'$ or $i_1 > i_2$ with $c_{i_1} : \tilde{v} \rightharpoonup l'$, $c_{i_2} = u' \rightharpoonup i$. In the first case, edge (i,j) prevents $j \rightharpoonup l'$ because i is an additional uncolored neighbor, making it impossible for j to color i or l', consequently $j \notin \mathcal{F}_i(G, \mathcal{V}_{l,1})$. In the second case, $j \notin \mathcal{F}_i(G, \mathcal{V}_{l,1})$ follows directly from the fact that i is colored before l', implying that up to the point of the coloring of i, node j has more than one uncolored neighbors, preventing $j \rightharpoonup i$. Thus (12) has to apply $\forall l \in \mathcal{N}_j$ such that $j \in \mathcal{F}_i(G', \mathcal{V}_{l,1})$ after adding (i,j) to graph G, thereby establishing the necessity of (12). ■

Fig. 3 shows an example in which condition (12) holds $\forall l \in \mathcal{N}_4$, such that adding edge $(5,4)$ to the graph results in $4 \in \mathcal{F}_5(G', \mathcal{V}_l)$ (as already illustrated in Fig. 2). Note that the computational complexity of verifying condition (12) for a specific pair of nodes is equivalent to that of computing $\mathcal{D}(\bar{G}_{ij}, \mathcal{V}_{l,1})$, for which the effort is of order $O(N + |E|)$ (see (Brimkov et al., 2019)).

In order to show that adding the edge (i,j) by Theorem 2 increases the cardinality of the set of FA of i, as required in (7), the following result is derived first:

Lemma 3. *For $l \in \mathcal{F}_i(G, \mathcal{V}_{l,1})$, let an edge (i,j) be added to G under the conditions of Theorem 2, leading to G'. Then $l \in \mathcal{F}_i(G', \mathcal{V}_{l,1})$ holds.*

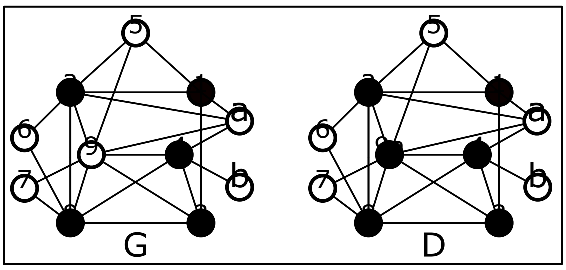

Figure 3: Left: The forcing graph corresponding to graph G from Fig 2 with $i = 5$ and $j = 4$. Right: The derived set of the forcing graph with $\mathcal{V}_l$ being a ZFS for G.

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Proof. Since l is a forcing agent of i in graph G, l and all its neighbors except of i are colored before i is colored. This can be expressed by the forcing graph (cf. (Schmidtke et al., 2024) Theorem 3):

$$l \in \mathcal{D}(\bar{G}_{i\{\}}, \mathcal{V}_{l,1}) \ \wedge \ \mathcal{N}_l \setminus i \subset \mathcal{D}(\bar{G}_{i\{\}}, \mathcal{V}_{l,1}). \quad (14)$$

Theorem 2 ensures that all neighbors of j can be colored by other nodes than j, before i is colored. Therefore, no coloring in the derived set of the forcing graph depends on j, implying:

$$\mathcal{D}(\bar{G}_{ij}, \mathcal{V}_{l,1}) = \mathcal{D}(\bar{G}_{i\{\}}, \mathcal{V}_{l,1}). \quad (15)$$

From this equality, it directly follows that:

$$l \in \mathcal{D}(\bar{G}_{ij}, \mathcal{V}_{l,1}) \ \wedge \ \mathcal{N}_l \setminus i \subset \mathcal{D}(\bar{G}_{ij}, \mathcal{V}_{l,1}). \quad (16)$$

This implies that adding the edge (i, j) does not change the coloring of all FA $l \in \mathcal{F}_i(G, \mathcal{V}_{l,1})$ in G', since the coloring of each of these nodes does not depend on the coloring of j, as shown by (15). Thus, each FA can still be colored before i is colored. This eventually means that:

$$l \in \mathcal{F}_i(G, \mathcal{V}_{l,1}) \implies l \in \mathcal{F}_i(G', \mathcal{V}_{l,1}), \quad (17)$$

holds if G' results from adding (i, j) to G under the condition of Theorem 2 holds. ■

Lemma 3 shows that all FA in G remain FA in G', if the edge is added under the conditions of Theorem 2. Thus, as a consequence of Lemma 3 together with Theorem 2, the following result can be stated which directly validates (7):

Corollary 2. *If the graph G is augmented by adding the edge (i, j) and if Theorem 2 holds, then*

$$|\mathcal{F}_i(G, \mathcal{V}_{l,1})| < |\mathcal{F}_i(G', \mathcal{V}_{l,1})|, \quad (18)$$

holds true.

While augmenting G with an edge (i, j) under the conditions of Theorem 2 guarantees an increase in the number of FA for node i, Corollary 2 does not provide insights into the set of FA of other nodes. To address this shortcoming, the following proposition demonstrates that a ZFS of G remains valid for the augmented graph G' under the same conditions:

Proposition 1. *If $V_{l,1}$ is a ZFS of G and if an edge (i, j) is added to G accrding to Theorem 2, $V_{l,1}$ remains a ZFS for the resulting graph.*

Proof. The graph G' follows from G, by connecting an uncolored node i with a colored node j, while $l \in \mathcal{D}(G_{ij}, \mathcal{V}_{l,1})$ holds $\forall l \in \mathcal{N}_j$ with $\mathcal{V}_{l,1}$ being a ZFS for G. This condition directly implies:

$$l \in \mathcal{D}(G_{\{\}j}, \mathcal{V}_{l,1}) \ \forall \ l \in \mathcal{N}_j, \quad (19)$$

where $G_{\{\}j}$ is the forcing graph with α and β only connected to node j. This means that all neighbors of j can be colored by other nodes in $G_{\{\}j}$, ensuring that the addition of the edge (i, j) does not disrupt the zero forcing process initiated by $\mathcal{V}_{l,1}$. Therefore, $\mathcal{V}_{l,1}$ remains a ZFS of G'. ■

Since $\mathcal{V}_{l,1}$ remains a ZFS in G', the following condition holds true $\forall i' \in \mathcal{V}^u \setminus i$:

$$\mathcal{F}_{i'}(G', \mathcal{V}_{l,1}) \geq 1. \quad (20)$$

Comparing the set $\mathcal{F}_{i'}(G', \mathcal{V}_{l,1})$ of each agent $i' \in \mathcal{V}^u \setminus i$ to $\mathcal{F}_i(G', \mathcal{V}_{l,1})$, the following two cases can occur:

$$|\mathcal{F}_{i'}(G', \mathcal{V}_{l,1})| \geq |\mathcal{F}_{i'}(G, \mathcal{V}_{l,1})| \ \ \forall i' \in \mathcal{V}^u \setminus i \text{ or} \quad (21)$$

$$\exists i' \in \mathcal{V}^u \setminus i : |\mathcal{F}_{i'}(G', \mathcal{V}_{l,1})| < |\mathcal{F}_{i'}(G, \mathcal{V}_{l,1})|. \quad (22)$$

In case (21), the number of FA of all uncolored nodes stays the same or has increased. This is the desired case, as the resilience of the whole networked system has increased. In the second case (22), the number of FA has decreased for at least one uncolored node in the graph. This implies that, while the resilience has increased for at least node i, it has decreased for at least one other node. Thus, in this case, only a local increase in resilience is achieved. This can be advantageous, for instance, when a forcing agent is added to a node with a small FA set, even if another node with a large FA set loses one of its FAs in the process.

Fig. 2 illustrates the case (21). Interestingly, in this case, the augmentation by the edge $(5, 4)$ makes the leader set $\mathcal{V}_l = \{1; 2; 3; 4; 8; 9\}$ a $1-$LFS of graph G', meaning that the system stays SSC despite any removal of an edge or a node.

Investigating how to ensure the desired case (21) will be a focus of future work. The following section investigates how frequently the conditions introduced hold in random graphs, allowing for the addition of edges while enhancing resilient strong structural controllability.

5 NUMERICAL EXAMPLE

This section investigates the effectiveness of the proposed approach through numerical experiments for random graphs. The study proceeds in two parts: firstly, the proposed approach is evaluated with respect to graph size, network density, and leader set size, and secondly, the approach is compared to an existing method from literature.

5.1 Analysis Across Network Parameters

The following three questions are addressed through numerical experiments:

- How frequently can an edge be found that enhances resilience, either locally or globally, depending on the size and density of the network?
- Which share of edges contributes to resilience of the whole network?
- How does the size of the leader set affect the occurrence of edges which enhance resilience?

To investigate these questions, Erdös–Rényi graphs $G(N, p)$ are used, in which N represents the set of nodes and p denotes the probability of an edge existing between two nodes (Erdös and Rényi, 1959). A higher value of p results in a denser network, as it increases the expected number of edges in the graph. The analysis focuses on the addition of edges to *critical nodes*, defined as nodes with a forcing agent set of cardinality one, i.e., $\mathcal{K} : \{i \in \mathcal{K} \mid \mathcal{F}_i(G, \mathcal{V}_l) = 1\}$. These nodes represent structural bottlenecks in the sense that SSC depends on a single neighbor to ensure controllability. The removal of this neighbor results in a loss of SSC. To mitigate this vulnerability, the possibility of adding FA by adding edges is examined, which would enhance resilience without introducing further leader agents.

The set $\hat{\mathcal{E}}$ consists of edges satisfying Theorem 2, with $i \in \mathcal{K}$ and $j \in \mathcal{V}_l$. The subset of edges that enhance resilience for the whole network is denoted by $\mathcal{E}^\star$, meaning that the condition (21) holds when any edge in $\mathcal{E}^\star$ is added to G. Furthermore, the ratio $|\mathcal{E}^\star|/|\hat{\mathcal{E}}|$ reflects the share of added edges for which condition (21) holds, relative to all edges that can be added according to Theorem 2. For all of the upcoming tests, 10 instances of graphs are randomly generated and the average is shown below (considering only connected graphs $G(N, p)$). In all tests $\mathcal{V}_{l,min}$ denotes the leader set with minimal cardinality determined by exhaustive search.

First, the dependency of the sizes of $\mathcal{K}$, $\hat{\mathcal{E}}$, and $\mathcal{E}^\star$ on the size $|\mathcal{V}_l|$ of the leader set is examined for a fixed number of agents $N = 20$ and an edge probability of $p = 0.2$. To increase the size of each leader set, an additional leader is randomly chosen from among the uncolored nodes. The results are presented in Table 1.

For the leader set $\mathcal{V}_{l,\min}$, only very few edges in $\hat{\mathcal{E}}$ can be identified, and none of them belongs to $\mathcal{E}^\star$. This occurs because, in the smallest possible leader set configuration, there is typically not enough redundancy to color the neighbors of a leader if the leader itself is prevented from coloring, as required by Theorem 2. As a result, edges in $\hat{\mathcal{E}}$ are rarely identified in this setting. Increasing the cardinality of the leader set leads to a reduction in $|\mathcal{K}|$, while the size of $\hat{\mathcal{E}}$ increases. Notably, the portion of edges in $\hat{\mathcal{E}}$ that also belongs to $\mathcal{E}^\star$ appears to grow with the size of the leader set. Adding just one agent to $\mathcal{V}_{l,\min}$ already results in a significant increase in both $|\hat{\mathcal{E}}|$ and $|\mathcal{E}^\star|$. For this reason, this configuration is used in the following simulations.

Table 1: Evaluation of the effect of the size of the leader set size on the number edges which can be added (according to Theorem 2) to critical nodes $\mathcal{K}$ in random graphs with $N = 20$ and $p = 0.2$.

	$\lvert\mathcal{K}\rvert$	$\lvert\hat{\mathcal{E}}\rvert$	$\lvert\mathcal{E}^\star\rvert$	$\lvert\mathcal{E}^\star\rvert \cdot \lvert\hat{\mathcal{E}}\rvert^{-1}$
$\lvert\mathcal{V}_l\rvert = \lvert\mathcal{V}_{l,min}\rvert$	7.6	0.5	0	0
$\lvert\mathcal{V}_l\rvert = \lvert\mathcal{V}_{l,min}\rvert + 1$	5.8	5.8	2.6	0.45
$\lvert\mathcal{V}_l\rvert = \lvert\mathcal{V}_{l,min}\rvert + 2$	3.6	6.6	3.1	0.47
$\lvert\mathcal{V}_l\rvert = \lvert\mathcal{V}_{l,min}\rvert + 3$	2.1	5.2	3	0.58

Next, the influence of the network density on the applicability of Theorem 2 is investigated. To this end, the edge probability p is varied, where smaller values result in sparser networks and larger values correspond to denser topologies. The results for a fixed number of $N = 20$ nodes are summarized in Table 2.

Table 2: Investigation of the effect of the network density on the number of edges which can be added according to Theorem 2, considering critical nodes $\mathcal{K}$ in random graphs with $N = 20$ and $|\mathcal{V}_l| = |\mathcal{V}_{l,min}| + 1$.

p	$\lvert\mathcal{V}_l\rvert$	$\lvert\mathcal{K}\rvert$	$\lvert\hat{\mathcal{E}}\rvert$	$\lvert\mathcal{E}^\star\rvert$	$\lvert\mathcal{E}^\star\rvert \cdot \lvert\hat{\mathcal{E}}\rvert^{-1}$
0.15	6.4	8.6	14.8	7.5	0.51
0.2	7.1	5.8	5.8	2.6	0.45
0.25	8.6	5.8	5.5	4.1	0.75
0.3	9.5	5.1	4.3	3.3	0.77
0.35	10.5	3.4	4.4	1.8	0.41

With increasing network density, a larger leader set is required to ensure SSC, while the number of critical nodes $\mathcal{K}$ decreases – an effect already observed in Table 1. In contrast, sparser networks allow for a greater number of edges in $\hat{\mathcal{E}}$, as redundancy in forcing is easier to establish. The share of edges in $\mathcal{E}^\star$ also belonging to $\hat{\mathcal{E}}$ ranges between 41% and 77%, with the maximum observed at $p = 0.3$ and the minimum at $p = 0.35$. Investigating the cause of this variation is subject of future work.

Lastly, the influence of the number of nodes N on the applicability of Theorem 2 is examined. To this end, N is varied while keeping the edge probability p and $|\mathcal{V}_l|$ constant. The results are summarized in Table 3.

Table 3: Effect of the graph size on the number of edges which can be added according to Theorem 2 for critical nodes $\mathcal{K}$ in random graphs with $p = 0.2$ and $|\mathcal{V}_l| = |\mathcal{V}_{l,min}| + 1$.

N	$\lvert\mathcal{V}_l\rvert$	$\lvert\mathcal{K}\rvert$	$\lvert\hat{\mathcal{E}}\rvert$	$\lvert\mathcal{E}^\star\rvert$	$\lvert\mathcal{E}^\star\rvert \cdot \lvert\hat{\mathcal{E}}\rvert^{-1}$
10	4.3	3	3.7	2.7	0.73
15	5.5	5.8	8.4	4.6	0.55
20	5.8	5.8	5.8	2.6	0.45
25	10.4	6.2	5.8	2.7	0.47
30	13.3	6.5	7.4	4.2	0.57

As the graph size increases, more leader agents are required to ensure SSC, and the number of critical nodes $\mathcal{K}$ also grows. However, the size of $\hat{\mathcal{E}}$ appears to be largely independent of N. The same holds for the ratio $|\mathcal{E}^\star|/|\hat{\mathcal{E}}|$, with the highest value observed for $N = 10$, followed by $N = 30$, and the lowest for $N = 20$.

In summary, increasing the cardinality of the leader set slightly beyond $\mathcal{V}_{l,\min}$ has the most significant impact. The number of edges in $\hat{\mathcal{E}}$ increases with sparser and larger graphs. In contrast, the portion of edges in $\mathcal{E}^\star$ relative to $\hat{\mathcal{E}}$ (i.e., those improving resilience at the network level) appears largely independent of the graph size and density, ranging between 41% and 77%.

5.2 Comparison of Edge Augmentation and an LFS Approach

The proposed method is now compared to the use of a 1-LFS (cf. Section 2.3), i.e., a set of leaders that guarantees SSC is maintained in the case a single agent leaves the network. The evaluation is performed on the Erdös–Rényi graph G with $N = 30$ and $p = 0.12$ shown in Fig. 4. A critical node is assumed to be known: with the zero-forcing set $\mathcal{V}_l = \{1,2,3,4,6,10,20,23,29,30\}$ (of size $|\mathcal{V}_{l,\min}| + 1$), node 11 is critical since $\mathcal{F}_{11}(G,\mathcal{V}_l) = \{12\}$. Thus, if node 12 leaves the graph, node 11 cannot be colored and the network loses the SSC property.

To evaluate condition (12), the derived set on the forcing graph $\bar{G}_{ij}$ is computed for $i = 12$ and $j \in \mathcal{V}_l$. For $j = 1$, Theorem 2 holds, implying that adding the edge $(1,11)$ to the graph expands the set of forcing agents of node 11 to $\mathcal{F}_{11}(G',\mathcal{V}_l) = \{1,12\}$. Consequently, even if node 12 leaves, the system remains SSC since the coloring of no other agent depends solely on node 12. The evaluation of Theorem 2 requires 60 ms in MATLAB R2021b on a machine with 32 GB RAM and an Intel Core i7-14700 processor.

In contrast, the 1-LFS $\mathcal{V}_{l,\text{1-LFS}} = \{1,2,4,5,7,10,11,13,14,20,23,30\}$, obtained by using the heuristic from (Abbas, 2023), is computed in 4.12 s and introduces two additional leaders in comparison to $\mathcal{V}_l$. Since the procedure is heuristic, minimality of the set size is not guaranteed. Using the integer-programming-based method from (Alameda et al., 2024) to compute a 1-LFS of minimal cardinality does not produce a solution within one hour of computation, which illustrates the hardness of the problem.

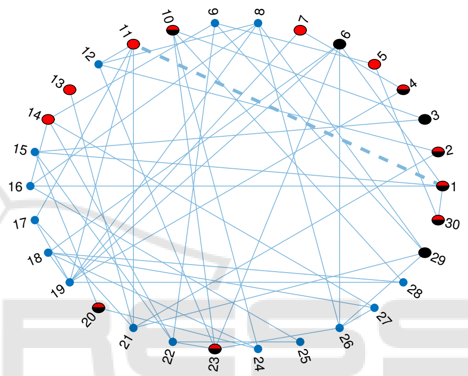

Figure 4: Erdös–Rényi graph $G(30,0.12)$ used for evaluating resilience. Node 11 is a critical node when considering the zero-forcing set $\mathcal{V}_l = \{1,2,3,4,6,10,20,23,29,30\}$ (indicated by the black nodes). Adding the dashed edge $(1,11)$ expands the set of forcing agents of node 11 by including node 1 to enhance the resilience of the network. The red nodes indicate a $1-$LFS for the network.

6 CONCLUSIONS

Conditions were specified for identifying edges that can be added to a network in order to enhance resilient strong structural controllability, either locally or for the entire network. The proposed method is based on the notion of forcing graphs, which extend the original graph by connecting dummy agents in a specific manner. Utilizing the concept of zero-forcing, a condition was established that must hold for the neighbors of a leader agent in the forcing graph. This condition ensures that adding an edge from the leader agent to a follower agent increases the follower's resilience.

Simulation studies demonstrate that such edges can frequently be identified when the size of the leader set exceeds the minimally required size to achieve strong structural controllability. Furthermore, the sparser and larger the graph is, the more edges can be found that enhance network resilience. Notably, the portion of added edges that improve resilience globally, rather than just locally, remains independent of the graph's sparsity and size.

Future research should explore how the proposed method can be extended to directly identify edges which enhance resilient SSC for the entire network.

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