Application of Feynman's Integral Technique to Representative Integrals and Real Life Scenarios

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Keywords: Definite Integral, Feynman's Integral, Dirichlet Integral.

Abstract: Calculus is a branch of mathematics that studies continuous change and motion, focusing on two fundamental concepts: differential calculus and integral calculus. This paper specifically focuses on integral calculus, which targets the accumulation of quantities and area under curves. Integrals play important roles in many fields, and this is because they possess key characteristics like precision, flexibility and universality. Initially, the great scientists Newton and Leibniz introduced the Newton-Leibniz formula to compute integrals but as time progress, it is insufficient to tackle complicated integrals. Therefore, a technique called Feynman's integral technique will be introduced in this paper. This technique was originated from the Leibniz integral rule and involves differentiation under the integral sign. By applying this technique, complex integrals can be simplified as the integral is being converted to a differential equation. In this paper, how Feynman's integral technique is applied will be demonstrated with detailed examples, which covers a range of different types of integrals, including some classic examples. This paper contributes to extending the idea of integral calculation, facilitates the efficient solution of integral calculations in practical problems and real world application of Feynman's integral technique.

1 INTRODUCTION

Integration, the process of finding the antiderivative of a function, is a fundamental concept in calculus, primarily divided into two types: indefinite and definite integrals. The theory of integration has great importance in mathematical analysis, in fact, in is the one of the twin pillars on which analysis is built. For example, integral is ideal for modelling dynamic systems like fluid flow and heat diffusion as integrals work with continuous functions unlike discrete sums. The theory was originated by the great Newton and Leibniz over three centuries ago, made rigorous by Riemann in the middle of the nineteenth century, and extended by Lebesgue at the beginning of the twentieth century. The fundamental theorem of calculus, the Newton-Leibniz formula is $\int_a^b f(x)\,dx = F(b) - F(a)$, where $f(x) = F'(x)$. From this equation, the following two formulas for indefinite and definite integral respectively can be derived: $\int f(x)\,dx = F(x) + c$ and $\int_a^b f(x)\,dx = \lim_{\|\Delta x\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$. Integration are commonly used to compute areas between curves or under a function, calculate volume of solids and determine the length of arcs and curve. It also has application in various fields, such as physics, economics and even environmental science and medicine.

However, there are many integrals that are very tricky to solve merely using Newton-Leibniz formula due to their complexity. Originally derived from the Leibniz integral rule, Feynman's integral technique is a method that tackles a group of such integrals swiftly (Wang et al, 2020). This group includes oscillatory integrals, logarithmic integrals, Frullani-type integrals, moment generating integrals and conditionally convergent integrals. This technique transforms such challenging integrals into manageable forms by leveraging parameterization and differentiation. Due to its nature of simplifying complex integrals, Feynman's integral trick plays crucial roles in many fields. For example, quantum mechanics in physics, control theory in engineering, analytic number theory in pure mathematics and more. The fundamental of Feynman's integral

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Li, Z.
Application of Feynman's Integral Technique to Representative Integrals and Real Life Scenarios.
DOI: 10.5220/0013815100004708

In *Proceedings of the 2nd International Conference on Innovations in Applied Mathematics, Physics, and Astronomy (IAMPA 2025)*, pages 119-123
ISBN: 978-989-758-774-0

technique is to hinge on transforming a difficult integral into a parameter-dependent function, then leveraging differentiation to simplify it. By introducing an artificial parameter a, the integral becomes a function $I(a)$. Differentiating $I(a)$ with respect to a often simplifies the integrand, allowing the original integral to be recovered via integration and boundary conditions.

This paper is going to cover the following components: what is Feynman's integral technique, how does Feynman's integral technique work with the explanation of its applications in different questions, and the real life applications of Feynman's integral technique in different fields, in this case, precision physics, perturbation theory in quantum chemistry and electromagnetism.

2 FEYNMAN'S INTEGRATION METHOD

Feynman's integration method, also known as "differentiation under the integral sign", is introduced by renowned physicist Richard Feynman to tackle complex integration questions in an unconventional way. The main idea of this integration method is to introduce an auxiliary parameter into the integral, making it a function of the new parameter (Liu et al, 2024). This will simplify the integration process by a large extent as the original integral can often be found by substituting a specific value of the new function due to the introduction of the parameter and thus the final result of the original integral can be computed by differentiating the new integral with respect to the parameter, and integrating the result, then substituting in the value that will transform the new integral to the original integral.

The general formula for Feynman's integral technique is

$$\frac{dI}{da} = \int_c^d \frac{\partial}{\partial a} f(x,a)dx \quad (1)$$

where d and c are fixed limits. Consider this classic example that is significantly simplified by Feynman's integration method to solve. This involves the application of Feynman's trick in improper integral (Nahin, 2015). The typical example is

$$\mathrm{I} = \int_0^\infty \frac{\sin x}{x} dx \quad (2)$$

To evaluate this integral conventionally, $\frac{\sin x}{x} = Im\left(\frac{e^{ix}}{x}\right)$ is first to be taken. Then the function can be written as

$$\mathrm{I} = \mathrm{Im}\left(\int_0^\infty \frac{e^{ix}}{x} dx\right) \quad (3)$$

Due to the singularity at $x = 0$, the contour integration approach should be used. Let the function $f(z) = \frac{e^{iz}}{z}$, and integrate it over a keyhole contour in the complex plane. The contour consists of a small semicircle of radius ϵ around the origin, which avoids singularity at $z = 0$, a large semicircle of radius R in the upper-half plane and two straight lines along the real axis from $\epsilon \to R$ and from $-R \to \epsilon$. Since this function is analytic both on the contour and inside as no poles are enclosed, Cauchy-residue theorem can be applied and the integral of f(z) over this enclosed contour is

$$\oint_C f(z)dz = 0 \quad (4)$$

Now, the author shall consider the three contributions. For the large semicircle, $|z| = R$, as $R \to \infty$, the integral over the large semicircle vanishes as $\left|e^{iz}\right| = e^{-Im(z)}$ decays exponentially in the upper half plane. For the small semicircle, $|z| = \epsilon$, as $\epsilon \to 0$, the integral over the small semicircle contributes $-\pi i$, which is half the residue at $z = 0$. Then for the straight lines, they combine to give $\int_\epsilon^R \frac{e^{ix}}{x} dx + \int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx$. By substituting x=-x into the integral, it becomes $\int_\epsilon^R \frac{e^{ix}-e^{-ix}}{x} dx = 2i \int_\epsilon^R \frac{\sin x}{x} dx$. Taking the limit $R \to \infty$ and $\epsilon \to 0$, the integral will become $2i \int_0^\infty \frac{\sin x}{x} dx$, and the equation $2i \int_0^\infty \frac{\sin x}{x} dx - \pi i = 0$ can be formed. The result of the original integral $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

This process however, can be simplified by merely introducing a new parameter, a , to the function such that it becomes

$$I(a) = \int_0^\infty \frac{sinx}{x} e^{-ax} dx \quad (5)$$

The next step is to differentiate I(a) with respect to a:

$$\frac{dI}{da} = -\int_0^\infty \sin x \, e^{-ax} dx \quad (6)$$

Then by integration by parts, the following can be obtained: $\frac{dI}{da} = -\frac{1}{1+a^2}$. To recover $I(a)$, integrate $\frac{dI}{da}$ with respect to a to get

$$I(a) = -\int \frac{1}{1+a^2} da = -\tan^{-1} a + c \quad (7)$$

where c is a constant.

In order to obtain c, take $a \to \infty$, this causes $\tan^{-1} a \to \frac{\pi}{2}$, and $-e^{-ax}$ to strongly oppose the integrand, which results in $I(a) \to 0$. Therefore, $-\frac{\pi}{2} + c = 0$ and $c = \frac{\pi}{2}$, then $I(a) = \frac{\pi}{2} - \tan^{-1} a$.

To convert $I(a)$ back to I, $a = 0$ will be taken and the function will look like

$$I(0) = \frac{\pi}{2} - \tan^{-1} 0 = \frac{\pi}{2}, \quad (8)$$

and $\pi/2$ will be the final result of the integral $\int_0^\infty \frac{\sin x}{x} dx$ (Nahin, 2015).

The author now considers another example

$$I = \int_0^1 \frac{x-1}{\ln x} dx, \quad (9)$$

which involves the use of Feynman's trick in definite integral. To use Feynman's integral method to tackle this question, a parameter, a, need to be first introduced to the integral and the integral will then become

$$I(a) = \int_0^1 \frac{x^a - 1}{\ln x} dx \quad (10)$$

Note that the original integral correspond to $I(1)$. Then, differentiate $I(a)$ under the integral sign, with respect to a to get

$$\frac{dI}{da} = \frac{d}{da}\int_0^1 \frac{x^a - 1}{\ln x} dx = \int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a - 1}{\ln x}\right) dx \quad (11)$$

The derivative will then get simplified as $\frac{\partial}{\partial a} x^a = x^a \ln x$, and the $\ln x$ in the denominator will get canceled out:

$$\frac{dI}{da} = \int_0^1 x^a \, dx = \frac{1}{a+1} \quad (12)$$

To recover the function $I(a)$, $\frac{dI}{da}$ needs to be integrated

$$I(a) = \int \frac{1}{a+1} da = \ln(a+1) + C \quad (13)$$

The evaluation of $I(0)$ can be used to compute the value of C, $I(0) = \int_0^1 \frac{1-1}{\ln x} dx = 0$. Therefore, $C = 0$ and $I(a) = \ln(a+1)$. Then substitute $a = 1$ to obtain the original integral and the final result would be ln 2 (Nahin, 2015; Zill, 2009).

Another example is the evaluation of the Gaussian integral, which makes use of Feynman's trick in infinite integral. The Gaussian integral is essentially $\mathrm{I} = \int_{-\infty}^{\infty} e^{-ax^2} dx$. To compute this integral a parameter needs to be first introduced to the integral by defining

$$\mathrm{I}(a) = \int_{-\infty}^{\infty} e^{-ax^2} dx \quad (14)$$

Differentiate $I(a)$ with respect to a to get $I'(a) = \int_{-\infty}^{\infty} -x^2 e^{-ax^2} dx$. Then to recover $I(a)$, integrate $I'(a)$ using integration by parts, which will give the result, a first order differential equation that relates $I'(a)$ to $I(a)$:

$$I'(a) = -\frac{1}{2a} I(a) \quad (15)$$

To solve this equation, it has to be rewritten in the form of $\frac{dI}{da} = -\frac{1}{2a} I(a)$, and thus $\frac{da}{I(a)} = -\frac{1}{2a} da$. Then integrate both sides of the equation and the left hand side of the equation will become $\ln|I(a)| + c$ and the right side of the equation becomes $-\frac{1}{2}\ln|a| + c$. Combine the two together, it is found that $\ln|I(a)| = -\frac{1}{2}\ln|a| + C$, or alternatively,

$$|I(a)| = e^{-\frac{\ln|a|}{2}} e^C. \quad (16)$$

Since $e^{-\frac{\ln|a|}{2}} = |a|^{-\frac{1}{2}} = \frac{1}{\sqrt{|a|}}$, $|I(a)| = \frac{e^C}{\sqrt{|a|}}$, $I(a) = \frac{C'}{\sqrt{a}}$, where $C' = \pm e^C$ is a new constant. To determine the value of C', the initial condition shall be used. Substitute $a = 1$ into the original integral and $I(1) = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ will be obtained. Then substitute $a = 1$ into the solution, $I(1) = \frac{C'}{\sqrt{1}} = \sqrt{\pi}$, thus $C' = \sqrt{\pi}$, which means the final computation of the integral $I = \sqrt{\frac{\pi}{2}}$ (Zill, 2009).

3 APPLICATIONS OF FEYNMAN'S INTEGRAL

3.1 Application in Precision Physics

Modern particle physics is becoming extremely precise and reliant on theoretical predictions for the analysis and interpretation of experimental results, which depends on the calculation of multi-loop corrections to physical observables. However, with the aid of Feynman's integral trick, the evaluation of multi-loop integral can be significantly simplified as the trick combines denominators of propagators in loop integrals into a single term (Wang et al, 2021). In general, the purpose of using Feynman's trick is to transform the multi-propagator integral into a single denominator integral, enable momentum shifts and dimensional regularization and therefore simplifies divergent integrals for renormalization.

Loop integral in quantum field theory refers to the corrections to processes, for example, particle interactions via virtual particles. These integrals typically have the following three characteristics: have multiple denominators, are divergent and have great dependencies on momentum. Hence, the parameterization character of Feynman's integral trick can combine the denominators and make the calculation process much easier. Consider the following loop example (Griffiths, 2018),

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)[(k+q)^2 - m^2]} \quad (17)$$

Next, apply Feynman's integral trick and combine the denominators by setting the parameter $x\epsilon[0,1]$ (Zill, 2009), it is found that $\frac{1}{(k^2-m^2)[(k+q)^2-m^2]} = \int_0^1 dx \frac{1}{[(k+xq)^2-\Delta^2]}$, where $\Delta = m^2 - xq^2(1-x) - i\varepsilon$. Then shift the momentum by taking $l = k + xq$,

$$I = \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 - \Delta)^2} \quad (18)$$

This shift takes out all the cross terms, leaving the integral to be solely dependent on l^2. To handle the divergence property of the integral, wick rotation to the Euclidean space ($k^0 \to ik_E^0$) needs to be carried out $\int d^4k \ \to i \int d^4k_E$. The denominator will become $l_E^2 + \Delta$. Last, carry out dimensional regularization and evaluate the integral in $d = 4 - \varepsilon$ dimensions

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^2} \propto \frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}} \Delta^{2-\frac{d}{2}}}. \quad (19)$$

This isolates divergence as the poles in ε, which are canceled during renormalization. Feynman's integral trick plays a vital role in many aspects of precision physics and here are the examples of two areas where Feynman's integral trick are used. Firstly, Feynman's integral trick is used in Higg Boson production(LHC) as in processes like gluon-gluon fusion, it is crucial for simplifying complex loop integrals and enabling precise theoretical predictions. The Higgs boson is predominantly produced via gluon-gluon fusion ($gg \to H$) at the LHC. This involves a loop of virtual particles (e.g., top quarks) due to the Higgs' strong coupling to heavy particles (Anastasiou, 2014). The process of loop integral complexity requires evaluating loop integrals with propagators involving the top quark mass (mt) and external momenta. Thus, at next-to-leading order (NLO) or next-to-next-to-leading order (NNLO), Feynman parametrization manages integrals with additional propagators and phase-space constraints (Schwartz, 2014). This reduces theoretical uncertainties in Higgs cross-section predictions to ~1−2%~1−2%, critical for LHC precision tests. Secondly, Feynman's integral trick is also extremely useful in hadronic vacuum polarization (HVP) (Aoyama et al, 2020). HVP is a quantum effect where virtual quark-antiquark pairs polarize the vacuum, modifying the photon propagator. This contributes to key observables like the muon's anomalous magnetic moment ($g - 2$) and precision tests of the Standard Model. The HVP tensor involves loop integrals with quark propagators and photon propagators. These integrals are divergent and require regularization and renormalization. Therefore, with Feynman's parameterization, more accurate results can be obtained with ease (Aoyama et al, 2020). Precise HVP calculation has a significant impact on precise physics as it is crucial for resolving the ~3.7σ~ 3.7σ discrepancy between Standard Model predictions and Fermilab/BNL experiments and refining predictions for ZZ-boson masses, Higgs couplings, and lepton universality.

3.2 Application in Perturbation Theory

Feynman's integral trick that involves the introduction of an auxiliary parameter can significantly simplify the complex calculation in perturbation theory and energy correction in quantum chemistry. It can simplify the matrix elements in perturbation theory as Calculating matrix elements of the perturbing Hamiltonian H′ between unperturbed states often involves challenging integrals. Therefore, by introducing an auxiliary parameter into the integral and differentiate the integral with respect to the parameter, the integral can be reduced to a much simpler form (Sakurai & Napolitano, 2021). For example, introduce the parameter λ into an integral and it becomes

$$\int \psi_m^* \hat{\mathrm{H}}' \psi_n d\tau = \frac{d}{d\lambda} \int \psi_m^* e^{-\lambda \hat{\mathrm{H}}'} \psi_n d\tau |\lambda = 0 \quad (20)$$

The outcome of this is that it avoids direct computation of complex integrals and enables systematic computation of first-and-higher-order energy correction. Feynman's trick is also a useful tool in variational perturbation theory, which is the combination of variational principle and perturbation theory and its purpose is to approximate the ground state energy of a system with Hamiltonian $\hat{\mathrm{H}} = \hat{\mathrm{H}}_0 + \lambda\hat{V}$, where $\hat{\mathrm{H}}_0$ is solvable and $\lambda\hat{V}$ is a perturbation. It addresses two main issues: solving singular or high dimensional integrals, and parameter optimization as trial wavefunctions depend on variational parameters.

3.3 Application in Electromagnetism

Feynman's integral trick, or Feynman parameterization, is a critical tool in quantum electrodynamics (QED) for simplifying loop integrals, such as those encountered in calculating the electron self-energy (Liu et al, 2023). The electron self-energy correction corresponds to a one-loop Feynman diagram where an electron emits and reabsorbs a virtual photon. This process introduces a divergent integral, which Feynman's trick helps manage by restructuring the integrand. The divergent integral for the self-energy correction takes the form:

$$\sum(p) \propto \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(p-k+m)\gamma_\mu}{(k^2-m^2+i\varepsilon)[(p-k)^2+i\varepsilon]}$$

where k is the loop momentum.

Feynman's trick simplifies this by introducing a Feynman parameter x to combine the denominators:

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA+(1-x)B]^2} \quad (21)$$

Applying this identity merges the propagators into a single quadratic denominator:

$$\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(p-k+m)\gamma_\mu}{[k^2-2xpk+x(p^2-m^2)+i\varepsilon]^2}.$$

Shifting the momentum $k \to k + xp$ linearizes the denominator, simplifying the integral to a scalar form. The divergence is then isolated into terms like $\int d^4 k/k^4$, which can be regularized using dimensional regularization or cut off methods. This restructuring reveals the ultraviolet (UV) divergence as a pole in $\epsilon = 4 - d$ (in dimensional regularization), which is absorbed into renormalization constants for the electron mass and charge. Feynman's method not only streamlines calculations but also clarifies how divergences relate to measurable quantities, enabling precise predictions like the Lamb shift or the electron's anomalous magnetic moment. This approach exemplifies how Feynman's trick turns intractable integrals into structured problems, bridging formal theory and experimental reality in QED.

4 CONCLUSION

Feynman's integral trick is vital in the field of calculus as not only does it solve many complex integral problems, but it is also revolutionary due to the uniqueness and innovativeness of its concept, which can possibly inspire future innovations. In this paper, the examples used are all classic integrals that have been pre-discussed. This paper explains the fundamental concepts and related knowledge regarding Feynman's integral technique and how to apply it in practical questions, with the aid of detailed worked examples. Furthermore, this paper has also exploited various different fields where the application of Feynman's integral technique is required, such as physics and engineering, where Feynman's integral technique simplifies loop calculation. Indeed, simply by introducing a parameter, differentiate with respect to the parameter and then restoring the integral, computation of complex integrals becomes much more intuitive and easier. The Feynman's integral technique offers a valuable approach for solving challenging integrals. In conclusion, this paper elaborates on how to approach integrals by Feynman's integral technique and how it can be applied in various situations.

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