Application of Residue Theorem on Some Different Types of Integrals

Yida Wu[a]
Jinan Foreign Language School, Jinan, China

Keywords: Residue Theorem, Singularities, Complex Number, Definite Integral.

Abstract: Complex analysis in the complex plane is an important branch of integration in the field of mathematics, and it is also an efficient mathematical tool to study and analyse the behaviour of complex variables. Complex functions are used in many scientific fields such as physics, computer science, and engineering. Complex analysis can solve many problems that are difficult to be solved by integrals of real variables alone or the solutions are very complex. Many problems in physics, chemistry, and engineering can be efficiently solved using complex analysis. This paper mainly introduces how to solve some specific types of integrals by using the residue theorem skilfully, and how to simplify the complexity of calculation and integration by using the residue theorem. Moreover, this paper illustrates the basic application of the residue theorem in detail through several examples. The discussion in this paper is helpful to popularize the idea of calculating and solving these types of integrals, and promote the application of these types of integrals in solving practical problems.

1 INTRODUCTION

Integral calculus serves as a foundational pillar of advanced mathematics and plays an indispensable role in interdisciplinary domains grounded in mathematical frameworks (Bak & Newman, 2010). When addressing practical problems in applied disciplines, scholars often encounter scenarios requiring holistic solutions. While elementary real integrals can be resolved through conventional techniques, such as computing integrals and applying the Newton-Leibniz formula, many specialized forms of integrals prove intractable via these classical approaches. This limitation obstructs the application of the Newton-Leibniz framework, creating significant challenges for research in affected fields. To overcome this, mathematicians turn to the residue theorem, a cornerstone of complex analysis, as a transformative tool for evaluating such integrals.

Central to this methodology is the concept of residues, defined as coefficients of the minus-power term in the Laurent series expansion. Residues enable the computation of integrals involving isolated singularities in which the functions exhibit undefined or divergent behavior. The residue theorem simplifies these calculations by reducing contour integrals to a summation of residues enclosed within a specified path. This innovation not only circumvents the need for indefinite integrals but also streamlines the evaluation of previously unsolvable integrals, marking a paradigm shift in definite integral computation (Zhu et al, 2022). Beyond theoretical mathematics, the residue theorem holds profound implications for mathematical physics, underpinning advancements in electromagnetism, quantum mechanics, fluid dynamics, and other fields reliant on complex variable functions.

The basic idea of calculating the integral by using the residual theorem is as follows: First, the transformation function transforms the real variable along the closed loop curve into the integral of the complex variable; Then, the problem is transformed to solve the residual values at isolated singularities in the closed loop. Finally, the solution of the product function is obtained by using the residual theorem. The purpose is to summarize the rest theorem systematically and understand its application, and to calculate the integral of this important theorem.

2 METHOD AND THEOREMS

2.1 Cauchy-Goursat Theorem

Supposed that f(z) is a complex function, and let curve C be a simple, closed positively oriented

[a] https://orcid.org/0009-0007-6783-1879

Wu, Y.
Application of Residue Theorem on Some Different Types of Integrals.
DOI: 10.5220/0013815000004708

In *Proceedings of the 2nd International Conference on Innovations in Applied Mathematics, Physics, and Astronomy (IAMPA 2025)*, pages 114-118
ISBN: 978-989-758-774-0

contour curve. If the expression is all analytic inside the curve, then the integral of this function along this curve is 0 (Fan, 2022). Namely,

$$\oint_c f(z)\,dz = 0 \qquad (1)$$

Let $f(z)$ denote a complex function defined within the annular region bounded by two concentric circles centered at z_0. The radius of the two concentric circles is R_1 and R_2 ($R_2 > R_1$). If the function is all analytic within the area ($R_1 < |z - z_0| < R_2$), then the function at the point z could express into Loran series uniform. Namely,

$$f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n + \sum_{n=1}^{\infty} b_n\,(z - z_0)^{-n} \qquad (2)$$

Laurent expansion is a generic form of Taylor expansion. If the function is fully resolved in that region, then the second part vanishes. That is, $b_n = 0$.

2.2 Definition About Residues and Residue Theorem

Suppose that f(z) is a complex function defined in a area containing finitely many singularities and C is a contour curve enclosing all these singularities within the region (Qiu, 2020). In such cases, the integral of $f(z)$ along C can be solved by residue theorems. The $\int_C f(z)dz$ equals to the adduct of all residues of f(z) ($\mathrm{Res}_{z=z_0} f(z)$) by 2πi.

$$\int_C f(z)dz = 2\pi i \sum_{z_k} Res_{z=z_k} f(z) \qquad (3)$$

in which c_n are calculated by

$$c_n = \frac{1}{2\pi i}\oint_c \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \qquad (4)$$

Here, C represents arbitrary sealing contour lying totally within the domain of integration and traversing counterclockwise around z_0. This contour integral is evaluated by parameterizing the path C and integrating the resulting expression with respect to the parameter (Zhou et al, 2022). Additionally, the residue at z_0 corresponds to the coefficient of the $(z - z_0)^{-1}$ term in the Laurent series expansion off(z), i.e., $\mathrm{Res}_{z=z_0} f(z) = c_{-1}$.

There are several different Classification of singularities (Zhang et al, 2023). The first is removable singularity. In this type of singularities, though there is no definition at this point, the value exists at the area near the point. In other terms, the there are no negative power terms in the Laurent expansion. The second is pole. In this type of singularities, when a point approaches a singularity, the value of the function is infinite. In addition, there are limited power terms in the Laurent expansion. The third is essential singularity. In this type, the value of this point is oscillating, unstable and tends to any value in any complex number.

3 APPLICATIONS

3.1 $1/(1 + x^n)$-type Integral

In this type, n is an integer and $n > 2$.

First, the author will assume $n = 3$ (Zhou & Huang, 2022). By using the residue theorem, the integral could be expressed to closed loop integral in the complex plane. The function has three single poles, $z_1 = e^{\pi i/3}, z_2 = e^{\pi i}, z_3 = e^{5\pi i/3}$. Then, the author will construct an anticlockwise curve C_{r_1} with argument is 2π/3 and radius r_1 is 1/3. This curve only cover pole z_1. According to the residue theorem:

$$I = \int_0^r \frac{dx}{1 + x^3} + \int_{C_r} \frac{dz}{1 + z^3} + \int_l \frac{d\zeta}{1 + \zeta^3} = 2\pi i \,\mathrm{Res}\, f(z_1) \qquad (5)$$

As r → ∞ the first term of the integral corresponds to the target integral. Meanwhile, the second term vanishes (approaches zero), and the third term is directly related to the target integral through a symmetry or transformation.

For the second term, substituting $z = Re^{i\theta}$, (with r→∞) and observing that $|f(z)z| < \varepsilon \to 0$,people conclude that this term becomes negligible in the limit. Thus, the integral could express into:

$$\left|\int_{C_r} f(z)dz\right| = \left|\int_{C_r} zf(z)dz/z\right| \le \int_{C_r} |zf(z)|\frac{|dz|}{|z|} < \int_0^{2\pi/3} \varepsilon\frac{rd\theta}{r} = \frac{2\pi}{3}\varepsilon \to 0 \qquad (6)$$

In the third term of the integral, let $\zeta = \rho e^{2\pi i/3}$, then the integral could express into:

$$\int_l \frac{d\zeta}{1 + \zeta^3} = \int_\infty^0 \frac{e^{2\pi/3}d\rho}{1 + \rho^3} = -e^{2\pi/3}\int_0^\infty \frac{dx}{1 + x^3} = -e^{2\pi/3}F(x) \qquad (7)$$

Here, $\rho = x, F(x) = \int f(x)dx$. The singular residue on the right can be obtained by L'Hospital's rule: $-\frac{1}{3}e^{\pi i/3}$. Substituting this result one can obtain that

$$F(x) = \int_0^\pi \frac{dx}{1 + x^3} = \frac{2\pi i}{1 - e^{2\pi i/3}} = \frac{\pi}{3}\frac{2i}{e^{\pi i/3} - e^{-\pi i/3}} = \frac{\pi}{3}csc\frac{\pi}{3} \qquad (8)$$

Next, the author will assume $n = 4$. When $n = 4$, the function has four singularities in the complex plane, they are $z_k = e^{\pi i(2k-1)/4}$ ($k = 1, 2, 3, 4$). Select a 1/4 great circular loop with a positive real axis, and the loop surrounds only z_1. According to the residue theorem, the integration satisfies the function. Let $\vartheta = \rho e^{\pi i/2}$, the third term of the function is

$$\int_l \frac{d\vartheta}{1+\vartheta^4} = \int_\infty^0 \frac{e^{\pi/2}}{1+\rho^4}$$
$$= -e^{\pi i/2}\int_0^\infty \frac{dx}{1+x^4} = -e^{\pi i/2}F(x) \quad (9)$$

The integral is solved by substituting all the above results into formula:

$$F(x) = \int_0^\pi \frac{dx}{1+x^4} = \frac{2\pi i}{1-e^{\pi i/2}}\mathrm{Re\,s}\,f(z_1)$$
$$= \frac{\pi}{4}\frac{2i}{e^{\pi i/4}-e^{-\pi i/4}} = \frac{\pi}{4}csc\frac{\pi}{4} \quad (10)$$

In the same way, one could choose a 1/2 large semi-circular loop containing the positive real axis. The loop surrounds two singularities z_1 and z_2, and they can also be selected by positive real axis, 3/4 great arc C_{R3} and a closed loop consisting of a ray l_3 with an argument principal value of $3\pi/2$. Clearly, the loop surrounds the three singularity points z_1, z_2 and z_3. The final value of these integrals of different curve is same, according to the residue theorem.

3.2 $R(\cos\theta, \sin\theta)$ -type Integral

The function is characterized as a rational form in real variables. By using the Euler's formula, the integral of the function could be transformed into $z = e^{i\theta}$, $\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2}$, $d\theta = \frac{dz}{zi}$ (Loney, 2001). Then the integral of the function along the curve could be transformed into this form:

$$F = \oint_{|z|=1} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{zi} \quad (11)$$

The integral can then be significantly streamlined through an application of the residue theorem. $\int_0^{2\pi} R(cos\,\theta, sin\,\theta)d\theta = 2\pi i\sum_c \mathrm{Res}[f(z)]$.

There is an example which is helpful to this paper to introduce the method for solving this type of integration (Shen, 2017), i.e.,

$$F = \int_0^{2\pi} \frac{cos\,2\,\theta}{5-4\,cos\,\theta}d\theta \quad (12)$$

The point with in the unit circle C could be defined as $z = e^{i\theta}(0 \le \theta \le 2\pi)$. After applying the Euler's formula, it can be translated into $z^2 = -isin2\theta$. $cos\,2\,\theta = \frac{z^2+z^{-2}}{2}$, $sin\,2\,\theta = \frac{z^2-z^{-2}}{2i}$, $d\theta = \frac{dz}{iz}$.

The integral could be converted into this form, $f(z) = \frac{z^2+z^{-2}}{2iz[5-2(z+z^{-1})]} = \frac{(z^4+1)i}{2z^2(z-2)(2z-1)}$. The function has the singularities that is $z_1 = 0$ and two simple poles, $z_2 = 1/2$ and $z_3 = 2$ (not inside). So, it only to calculate the residues z_1 and z_2. This is because

$$Res\left[\frac{1}{2}, f(z)\right] = \lim_{x\to 1/2}\left(z-\frac{1}{2}\right)f(z) = -\frac{17i}{24} \quad (13)$$

It then turns out to be

$$F = 2\pi i\left(\frac{5i}{8}-\frac{17i}{24}\right) = \frac{\pi}{6} \quad (14)$$

There is another example:

$$F = \int_0^{2\pi} \cos^{2n}\theta\, d\theta \quad (15)$$

in which $n \in N$. Since $\cos\theta = (z+z^{-1})/2$, thus the mentioned integral can be recast into

$$F = \int_C \left(\frac{z+z^{-1}}{2}\right)^{2n}\frac{dz}{zi} = \frac{-i}{2^{2n}}\int_C \frac{(z^2+1)^{2n}}{2^{2n+1}}dz \quad (16)$$

Clearly, this integral has a $(2n+1)$-order singularity at $z = 0$. Thus, it is calculated that

$$F = 2\pi i\frac{-i}{2^{2n}}\mathrm{Res}[0, f(z)]$$
$$= \frac{\pi}{2^{2n-1}}\lim_{z\to 0}\left[\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}[(z^2+1)]^{2n}\right]$$
$$= \frac{\pi}{2^{2n-1}}C_{2n}^{n}$$

3.3 $P(x)/Q(x)$-type Integral

In this subsection, the author will consider the integral of the form

$$F = \frac{P(x)}{Q(x)}\int_0^{2\pi} cos^{2n}\,\theta\, d\theta \quad (17)$$

This is an example to introduce the solution of this type integral

$$I = \int_{-R}^{\infty} \frac{xe^{2ix}}{x^2-1}dx \quad (18)$$

After applying the partial fraction (Wang & Li, 2016), the singularities of the function would be obvious

$$I = \lim_{\substack{R\to\infty \\ r_1,r_2\to 0^+}} \int_{-R}^{-1-r_1} \frac{xe^{2ix}}{x^2-1}dx$$
$$+ \int_{-1+r_1}^{1-r_2} \frac{xe^{2ix}}{x^2-1} + \int_{1+r_2}^{R} \frac{xe^{2ix}}{x^2-1} \quad (19)$$

Let $I_1 = \int_{S_{r_1}} \frac{ze^{2zi}}{z^2-1}dz$, $I_2 = \int_{S_{r_2}} \frac{ze^{2zi}}{z^2-1}dz$, $I_R = \int_{C_R} \frac{ze^{2zi}}{z^2-1}dz$ .Then, $f(z) = \frac{ze^{2zi}}{z^2-1}$ is all holomorphic inside the curve. After using the Cauchy integral theorem, $\int_{-R}^{-1-r_1} \frac{xe^{2xi}}{x^2-1}dx + \int_{-1+r_2}^{1-r_2} \frac{xe^{2xi}}{x^2-1}dx + \int_{1+r_2}^{R_1} \frac{xe^{2xi}}{x^2-1}dx + I_1 + I_2 + I_R = 0$. By virtue of the

Jordan Lemma, it works that when $z \to 0, z/z^2 - 1 \to 0$. Thus $\lim_{R\to\infty} I_R = 0$.Because f has the simple poles on $z = \pm 1$, $\lim_{r_1 \to 0^+} I_1 = -i\pi \text{Res}(f, -1) = -i\pi \lim_{z\to -1}(z+1)f(z) = \frac{-i\pi e^{-2i}}{2}$. Then, $\lim_{r_2\to 0^+} I_2 = -i\pi \text{Res}(f, 1) = \frac{-\pi e^{2i}}{2}$. Therefore,

$$P.V. \int_{-\infty}^{\infty} \frac{xe^{2ix}}{x^2-1} dx = \frac{i\pi e^{-2i}}{2} + \frac{i\pi e^{2i}}{2} = i\pi \cos(2)$$

There is another example:

$$I = \int_{-\infty}^{\infty} \frac{\sin 2x}{x^2+x+1} dx \quad (20)$$

Firstly, the partial fraction is applied on this function to simplify the structure of the function (Zhou & Wu, 2018). Then, it is clearly to notice that the function $f = \frac{1}{z^2+z+1}$ does not have a singularity. Also, it is found that $\lim_{z\to 0} \frac{1}{z^2+z+1} = 0$ but the integral has simple poles which are $z_1 = e^{\frac{2\pi i}{3}}$ and $z_2 = e^{-\frac{2\pi i}{3}}$. According to the virtue of the Jordan Lemma, the function could be transformed into that:

$$I = Im \int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+x+1} dx = Im \oint_C \frac{e^{2ix}}{z^2+z+1} dz$$

The third example of the function in this type is that

$$I = \int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2} dx \quad (21)$$

At first, this is because the integral is an even function, $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2} dx$. It possesses singularities at $z = \pm 3i$ (second-order poles) and $z = \pm 3i$ (simple poles), while remaining holomorphic everywhere else on the complex plane. To apply the residue theorem, one can construct a semi-circular contourC_r in the higher half-plane with radius $r > 3$. This contour encloses all singularities $z = \pm 3i$ and $z = \pm 3i$ within the region bounded by: the real axis part $[-r, r]$, the upper semicircle C_r defined by $|r| = r$. By integrating $f(z)$ counter-clockwise around this boundary, the residue theorem transforms the integral into the following form:

$$I = \int_{-r}^{r} f(x)dx + \int_{C_r} f(z)dz$$
$$= 2\pi i\left[\text{Res}(2i, f(z)) + \text{Res}(3i, f(z))\right] \quad (22)$$

The residues at the points are $Res[2i, f(z)] = \lim_{z\to 2i} \frac{d}{dz} \frac{z^2}{(z^2+9)(z+2i)^2} = \frac{-13i}{200}$, and $Res[3i, f(z)] = \lim_{z\to 3i}(z-3i) \frac{z^2}{(z-3i)(z+3i)(z^2+4)} = \frac{3i}{50}$, thus, the total integral $\int_{-r}^{r} f(x)dx = \frac{\pi}{100} - \int_{C_r} f(z)dz$.

As $r \to \infty$, this contour integral over C_r vanishes, i.e., $\int_{C_r} f(z)dz| = 0$. For any point z on the semi-circular contour C_r, it is observed that $|z^2| = |z|^2$. Applying the triangle inequality $|z + w| \geq ||z| - |w||$, people can derive the following estimate, so that by this equation, $|\int_{C_r} f(z)dz| = |\int_{C_r} \frac{z^2}{(z^2-9)(z^2+4)}| \leq \frac{r^2}{(r^2-9)(r^2+4)^2} L(C_r)$, where $L(C_r) = \pi r$ is the length of the semicircle C_r. Thus, it is derived that

$$|\int_{C_r} f(z)dz| = \left|\int_{C_r} \frac{z^2}{(z^2-9)(z^2+4)}\right|$$
$$\leq \frac{r^2}{(r^2-9)(r^2+4)^2} \quad (23)$$

As $r \to \infty$, the right-hand side of the inequality approaches zero, implying that the contour integral $\int_{C_r} f(z)dz$ vanishes. Consequently, the Cauchy principal value of the integral over the real line is:

$$P.V. \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{100} \quad (24)$$

Since the integrand $f(x) = \frac{x^2}{(x^2+9)(x^2+4)}$ is an even function, the principal value simplifies to twice the integral from 0 to ∞, i.e.,

$$\int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+4)} dx = \frac{\pi}{200} \quad (25)$$

When $r \to \infty$, the right-hand goes to 0, rendering $\int_{C_r} f(z)dz = 0$. The principal part is therefore $P.V. \int_{-\infty}^{\infty} f(x)dx = \frac{x}{100}$. As the integral is even, one can find that $\int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+4)} dx = \frac{\pi}{200}$.

4 CONCLUSION

The residue theorem plays a very important role in dealing with complex function problems. In this paper, the definition and application range of the residue theorem are given in detail, and the basic reserve knowledge about the residue theorem is introduced, such as the classification of singularities, Laurent expansion, and the definition of residue. In this paper, the author focuses on the application of the residue theorem to some specific integrals. This paper describes in detail how to convert the object integral into a complex function form which can be used by the residue theorem, and then greatly reduces the difficulty of real integration by using the residue theorem. The residue theorem is a powerful tool for dealing with the integration of complex functions. Types of integrals that may be difficult to solve with only real variables can be solved by clever application of the residue theorem. The residue theorem offers a systematic approach to evaluating complex integrals by leveraging the singularities of the integrand. This method involves identifying the singular points of the function within the contour and computing their associated residues to determine the integral's value. This method generally simplifies the computational difficulty of the original method and provides a novel and concise approach to processing integrals. This paper discusses the application of residue theorems for certain kinds of integrals, which helps to extend the application of residue theorems, and promotes the application of residue theorems in solving practical problems.

REFERENCES

Bak, J., & Newman, D. J. (2010). Complex analysis (3rd ed.). Springer.

Fan, M. C. (2022). Solutions to several improper integrals from The American Mathematical Monthly using residue theorem. Journal of Huizhou College, 42(06), 72-78.

Loney, N. W. (2001). Use of the residue theorem to invert Laplace transforms. Chemical Engineering Education, 35(1), 22-24.

Qiu, W. G. (2020). Infinite sum and residue theorem. University Physics, 39(09), 31-33.

Shen, Y. W. (2017). Residue theorem and its applications. Journal of Tonghua Normal College, 38(06), 24-26.

Wang, Y. L., & Li, C. J. (2016). Calculate a special kind of generalized integral by using residue theorem. Journal of Advanced Science, 36(10), 23-24.

Zhou, C. M., & Wu, L. (2018). The difference of calculating contour integration between the Cauchy integral formula and the residue theorem. Journal of Ningxia Normal College, 39(10), 94-97.

Zhou, W. P., Liu, Y. F., & Song, T. L. (2022). Two types of infinite integrals solved by the residue theorem. Physics and Engineering, 32(01), 56-59.

Zhou, Y. Q., & Huang, W. T. (2022). Revisiting two types of infinite integrals solved by the residue theorem. College Physics, 41(09), 24-27+42.

Zhang, Z. X., Du, X. R., & Ma, Y. L. (2023). Generalization of a class of generalized integrals using the residue theorem. Physics and Engineering, 33(04), 12-17.

Zhu, X. X., Zhang, B., & Yang, H. J. (2022). Complex Functions and Integral Transforms. Beijing, China: Tsinghua University Press. pp. 65-70.