Interdisciplinary Application of Residue Theorem in Trigonometric and Fraction Functions

Yeheng Xiao[a]
School Malvern College Qingdao, Qingdao, China

Keywords: Residue Theorem, Complex Variable Function, Definite Integral.

Abstract: In complex function theory, there exists a fundamental concept known as the residue theorem., lies at the intersection of pure mathematics and diverse scientific applications. This theorem elegantly connects the integrand function over a precisely defined closed geometric contour in the complex plane to the sum of its residues at isolated singularities within that contour. In mathematics, it serves as a potent means for the evaluation of otherwise intractable complex integrals, offering new insights into the behavior of complex - valued functions, from analyzing the distribution of zeros and poles to studying function singularities. Its applications span multiple scientific disciplines. In physics, it simplifies calculations in quantum mechanics, electromagnetism, and statistical physics, providing crucial solutions for problems like scattering amplitudes and electromagnetic field distributions. In engineering, it aids in signal processing and control system design, especially when dealing with Laplace and Fourier transforms. Hence, this paper aims to calculate representative definite integrals with the help of Residue theorem, paving the way for connecting its applications in interdisciplinary field.

1 INTRODUCTION

Complex variable functions are an important branch in mathematics and are widely applied in many fields such as physics, engineering, computer science, and finance (Churchill & Brown, 2014). The following are the main application scenarios of complex variable functions. In the field of mathematics, residue theorem relates closely to the complex variable functions. The residue theorem is a useful tool to calculate integrals associated with complex - variable functions The residue theorem is an important tool for the calculation of integrals of complex variable functions.

In the course of researching complex variable functions, for the calculation of the integrals of some functions that have singular points in a closed region, Cauchy's integral theorem cannot be directly applied (Bak & Newman, 2010). However, the residue theorem provides an effective method. It links the residues of the function at each singular point within the region enclosed by a closed curve with the integral of the function along that closed curve. That is, the integral of the function along the closed curve is equal to $2\pi i$ times the sum of the residues of the function at each singular point within the region enclosed by the closed curve, which greatly simplifies the calculation of the integrals of complex variable functions (Stein & Shakarchi, 2003). The residue theorem deepens the understanding of the singular points of complex variable functions. The singular points of complex variable functions are divided into types such as removable singular points, poles, and essential singular points. Through calculating the residues of the function at singular points, the residue theorem enables a more in-depth study of the properties and characteristics of singular points. For example, by calculating the residues, the author can determine the type of a singular point and understand the local properties of the function near the singular point (Chi, 2024).

The theoretical framework of complex variable functions serves as the foundation for the residue theorem. The analyticity of complex variable functions, Cauchy's integral formula and other theories are the basis for the derivation and proof of the residue theorem. The properties of analytic functions ensure the establishment of the residue

[a] https://orcid.org/0009-0006-4658-8411

Xiao, Y.
Interdisciplinary Application of Residue Theorem in Trigonometric and Fraction Functions.
DOI: 10.5220/0013814900004708

In *Proceedings of the 2nd International Conference on Innovations in Applied Mathematics, Physics, and Astronomy (IAMPA 2025)*, pages 109-113
ISBN: 978-989-758-774-0

theorem and provide various methods for calculating residues, such as using the Laurent series expansion. At the same time, some basic concepts and methods of complex variable functions, such as contour integrals and isolated singular points, are also the prerequisites and keys for the application of the residue theorem. The residue theorem promotes the development and application of the theory of complex variable functions. The emergence of the residue theorem has promoted the extensive application of the theory of complex variable functions in other fields. It has played an important role in aspects such as calculating real integrals, solving differential equations, and studying fluid mechanics and electromagnetics. These applications not only expand the research scope of complex variable functions but also further enrich and perfect the theoretical system of complex variable functions.

2 METHODS

2.1 Complex Numbers and Functions

A complex number is of the form $z = x + iy$ where x and y are real numbers, and $i = \sqrt{1}$. A complex function f(z) maps complex numbers to complex numbers. One will review the basic operations on complex numbers and functions, such as addition, multiplication, and differentiation (Qiu, 2020).

For a function $f(z)$ that is analytic at an isolated singularity z, its Laurent series expansion is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad (1)$$

The residue is defined as the coefficient of term $(z - z_0)^{-1}$ in the Laurent series, that is $\mathrm{Res}(f, z_0) = a_{-1}$ In practical calculations, for different types of isolated singularities, there are different methods for calculating the residue. If z_0 is an mth order pole of $f(z)$, then the formula

$$Res(f, z_0) = \frac{m}{(m-1)!} \lim_{z \to z_0} f(z) \quad (2)$$

can be used to calculate the residue. For example, for the function$f(z) = \frac{z}{(z-1)^2(z+2)}$, z = 1 is a second-order pole and z = - 2 is a first-order pole. For z = 1, according to the above formula, first let $g(z)\frac{z}{z+2}$, then $Res(f, z_0) = \frac{1}{9}$; for z=-2 $Res(f, z_0) = -\frac{2}{9}$.

Singularities are points where a complex function is not analytic. Multiple kinds of singularities are present, and among them are removable singularities, poles, and essential singularities. The behavior of a function near its singularities is crucial for understanding the residue concept.

2.2 Residue Theorem

The residue theorem establishes a connection between the integral of a function along a closed curve and the residues of the function at the isolated singularities inside the curve. Let f(z) be analytic in the region D enclosed by a simple closed curve C except for a finite number of isolated singularities $z_1 z_2 \ldots,\ z_n$, and continuous in the closed region $\overline{D} = D \cup C$ except at these singularities. Then one has

$$\oint_c f(z)dz = 2\pi \mathrm{i} \sum_{k=1}^{n} Res(f, z_k) \quad (3)$$

Below is the General Steps for Calculating Integrals Using the Residue Theorem (Trefethen & Weideman, 2014).

The first is to determine the integration path. Select an appropriate closed integration path C, which is usually constructed according to the characteristics of the integrand function and the integration interval. For example, for the integral $\int_{-\infty}^{\infty} f(x)\, dx$on the real axis, a semi-circular path (with radius R) in the upper half-plane is often added, so that the integration path C = $C_R + \gamma$ (γ is the line segment $[-R, R]$ on the real axis) forms a closed curve.

The second is to analyze the singularities of the integrand function. Find all the isolated singularities of the integrand function f(z) inside the integration path C, and determine the types of the singularities (removable singularities, poles, or essential singularities). Then, according to the type of the singularity, use the corresponding method to calculate the residue at each singularity.

The third is to calculate the sum of the residues. Add up the residues at all the singularities inside the integration path C to obtain $\sum_{k=1}^{n} Res(f, z_k)$.

The fourth is to apply the residue theorem to find the value of the integral. According to the residue theorem shown in Eq. (3), one can calculate the numerical value of the integral taken along the closed - loop curve C Then, by analyzing the limit situation of the integral on the supplementary path (such asC_R) when$R = \infty$, the value of the original integral on the real axis can be obtained (Xu & Fan, 2024). For example, when $\lim R \to \infty \oint_c f(z)dz = 2\pi \mathrm{i} \sum_{k=1}^{n} Res(f, z_k)$.

The residue theorem has many applications. In physics, Cauchy's residue theorem is used in various areas. For instance, in quantum field theory, it is used to calculate scattering amplitudes and Green's functions. In statistical mechanics, it can be applied to evaluate partition functions. The author will briefly introduce some of these applications and explain how the theorem is used in these contexts.

In Quantum Mechanics In quantum mechanics, the residue theorem is used to calculate the Green's functions. The Green's function is a powerful tool for solving the Schrödinger equation and understanding the propagation of quantum waves. For example, in the study of scattering problems, the Green's function can be expressed as a complex integral. By applying the residue theorem, people can evaluate this integral and obtain the scattering amplitude, which is a key quantity in understanding the interaction between particles.

In statistical physics, the partition function is a central concept. It is often expressed as an integral over a complex contour. The residue theorem can be employed to evaluate this integral and obtain the thermodynamic properties of the system, such as the free energy, entropy, and specific heat. This allows people to study the equilibrium and non - equilibrium behaviours of physical systems.

3 APPPLICATION

3.1 Integrals of Trigonometric Functions

Let $z = e^{i\theta}$, then $\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2i}$, and $d\theta = dz/iz$ (Hang et al, 2023). When θ varies from 0 to 2π, z makes a positive circuit along the unit circle |z| = 1 in the complex plane. Thus, the original integral $\int_0^{2\pi} R(\cos\theta, \sin\theta)d$ is transformed into the contour integral of a complex function

$$I = \oint_{|z|=1} R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right)\frac{dz}{iz} \qquad (4)$$

Then, by using the Cauchy residue theorem, people can calculate the residues of the integrand at the isol ated singular points inside the unit circle |z| = 1, and further obtain the value of the integral (Zeng, 2020).

For example, to calculate

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \ (|a| < 1) \qquad (5)$$

Let $z = e^{i\theta}$, the integrand is then transformed into

$$I = \oint_{|z|=1} \frac{2}{(a + 2)z + az^2} dz, \qquad (6)$$

and in what follows one can introduce a new function $f(z) = \frac{2}{az^2+(a+2)z}$. The singular points of $f(z)$ are $z_1 = 0$ and z_2 is not inside the unit circle. $z_1 = 0$ is a first-order pole, and $\text{Res}[f(z), 0] = \lim_{z\to 0} z f(z) = \frac{2}{a+2}$. According to residue theorem,

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1 - a^2}} \qquad (7)$$

Next, the author will evaluate the integral $\int_C f(z)dz$.

(a) Evaluation of the integral for the semicircle z = $2e^{i\theta}(0 \le \theta \le \pi)$.

First, the author shall express z and dz in terms of θ. Given z = $2e^{i\theta}$, then dz=$2ie^{i\theta}d\theta$. Substitute z into the function f(z): $f(z) = \frac{z+2}{z}$=1+$\frac{2}{z}$. Substituting z = $2e^{i\theta}$, one gets $f(z) = 1 + \frac{2}{2e^{i\theta}} = 1 + e^{-i\theta}$. Now, the task is to calculate the contour integral $\int_C f(z)dz$, i.e., $\int_C f(z)dz = \int_0^{\pi}\left(1 + e^{-i\theta}\right) \cdot 2ie^{i\theta}d\theta$. Expand the integrand: $\left(1 + e^{-i\theta}\right) \cdot 2ie^{i\theta} = 2ie^{i\theta} + 2i$ and integrate it term - by – term, then

$$\int_0^{\pi} \left(2ie^{i\theta} + 2i\right)d\theta = 2i\int_0^{\pi} e^{i\theta}\, d\theta + 2i\int_0^{\pi} d\,\theta \qquad (8)$$

For $\int_0^{\pi} e^{i\theta}\, d\theta$, using the formula $\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C(here a = i)$, one has $\int_0^{\pi} e^{i\theta}\, d\theta = \left[\frac{1}{i}e^{i\theta}\right]_0^{\pi}$. Since $\frac{1}{i} = -i$, then $\int_0^{\pi} e^{i\theta}\, d\theta = -i(e^{i\pi} - e^0) = -i(-1 - 1) = 2i$. $\int_0^{\pi} d\theta = \pi$. So, $2i\int_0^{\pi} e^{i\theta}\, d\theta + 2i\int_0^{\pi} d\theta = 2i \cdot 2i + 2i\pi = -4 + 2\pi i$.

(b) Evaluation of the integral for the semicircle $z = 2e^{i\theta}(\pi \le \theta \le 2\pi)$.

Again, $z = 2e^{i\theta}$, $dz = 2ie^{i\theta}d\theta$, and $f(z) = 1 + \frac{2}{z} = 1 + e^{-i\theta}$. Calculate the contour - integral $\int_C f(z)dz$:

$$\int_C f(z)dz = \int_{\pi}^{2\pi} \left(1 + e^{-i\theta}\right) \cdot 2ie^{i\theta}d\theta \qquad (9)$$

Expand the integrand $\left(1 + e^{-i\theta}\right) \cdot 2ie^{i\theta} = 2ie^{i\theta} + 2i$ and integrate term - by – term, then $\int_{\pi}^{2\pi}\left(2ie^{i\theta} + 2i\right)d\theta = 2i\int_{\pi}^{2\pi} e^{i\theta}\, d\theta + 2i\int_{\pi}^{2\pi} d\,\theta$.

For $\int_{\pi}^{2\pi} e^{i\theta}\, d\theta$, using $\int e^{i\theta}\, d\theta = -ie^{i\theta} + C$, one has $\int_{\pi}^{2\pi} e^{i\theta}\, d\theta = -i(e^{i\cdot 2\pi} - e^{i\pi}) = -i(1 + 1) = -2i$. So, it is calculated that $2i\int_{\pi}^{2\pi} e^{i\theta}\, d\theta + 2i\int_{\pi}^{2\pi} d\theta = 2i \cdot (-2i) + 2i\pi = 4 + 2\pi i$.

(c) Evaluation of the integral for the circle $z = 2e^{i\theta}(0 \le \theta \le 2\pi)$.

People can use the results from parts (a) and (b). Since the circle $z = 2e^{i\theta}(0 \le \theta \le 2\pi)$is composed of the two semicircles from parts (a) and (b), then

$$\int_C f(z)dz = \int_0^{\pi} \left(1 + e^{-i\theta}\right) \cdot 2ie^{i\theta}d\theta$$

$$+\int_{\pi}^{2\pi}\left(1+e^{-i\theta}\right)\cdot 2ie^{i\theta}d\theta \qquad (10)$$

From part (a), it is $\int_0^{\pi}\left(1+e^{-i\theta}\right)\cdot 2ie^{i\theta}d\theta = -4+2\pi i, and from part (b), \int_{\pi}^{2\pi}\left(1+e^{-i\theta}\right)\cdot 2ie^{i\theta}d\theta = 4+2\pi i$. Then it is found that $\int_C f(z)dz = (-4+2\pi i)+(4+2\pi i) = 4\pi i$.

3.2 Integrals of Fractional Function

The author shall define

$$f(t) = \frac{az^3+bz^2+cz+d}{z^4-1} \qquad (11)$$

with $a=6, b=i+1, c=16, d=i-1$ (Li et al, 2021). The task is to evaluate the integrals $\int_r f(z)\,\mathrm{d}z$ with $\gamma_1(t) = i+\frac{e^{it}}{2}, 0\le t\le 2\pi$ $\quad \gamma_2(t) = \frac{i-1}{2}+\sqrt{2}e^{it}, 0\le t\le 2\pi$ $\gamma_3(t) = 1+5e^{it}, 0\le t\le 2\pi$.

Firstly, people can factor the denominator z^4-1. People know that $z^4-1=(z-1)(z+1)(z-i)(z+i)$ by the difference - of - powers formula $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1})$. Here, a = z, b = 1 and n = 4. So,

$$f(z) = \frac{6z^3+(i+1)z^2+16z+(1-i)}{(z-1)(z+1)(z-i)(z+i)} \qquad (12)$$

For the contour $\gamma_1(t) = i+\frac{e^{it}}{2}, 0\le t\le 2$. The center of the contour γ_1 is $z_0 = i$ and the radius $r = 1/2$. The singularities of $f(z)$ are the roots of $z^4-1=0$, i.e., $z=1, z=-1, z=i, z=-i$.

People check which singularities lie inside the contour γ_1. The distance between a point z and the center i of the contour γ_1 is given by |z - i|. For $z=1, |1-i| = \sqrt{1+(-1)^2} = \sqrt{2} > \frac{1}{2}$. For $z=-1, |-1-i| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2} > \frac{1}{2}$. For $z=i, |i-i| = 0 < \frac{1}{2}$. For $z=-i, |-i-i| = |-2i| = 2 > \frac{1}{2}$. By the residue theorem, $\int_{\gamma_1} f(z)dz = 2\pi i\cdot \underset{z=i}{Res}f(z)$. To find the residue at $z=i$, people can use the formula $\underset{z-z_0}{Res}\frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}$ where $g(z) = 6z^3+(i+1)z^2+16z+(1-i)$ and $h(z) = z^4-1$.

Obviously, it is easy to find $h'(z) = 4z^3$. Then $h'(i) = 4i^3 = -4i$. $g(i) = 6i^3+(i+1)i^2+16i+(1-i) = -6i-(i+1)+16i+(1-i) = (-6-1+16-1)i+(-1+1) = 8i$. So, $\underset{z=i}{Res}f(z) = \frac{g(i)}{h'(i)} = \frac{8i}{-4i} = -2$. Then $\int_{\gamma_1} f(z)dz = 2\pi i\times(-2) = -4\pi i$.

Secondly, for the contour $\gamma_2(t) = \frac{i-1}{2}+\sqrt{2}e^{it}, 0\le t\le 2$, the center of the contour γ_2 is $z_0 = \frac{i-1}{2}$ and the radius $r=\sqrt{2}$. Calculate the distances from the singularities $z=1, z=-1, z=i, z=-i$ to the center $z_0 = \frac{i-1}{2}$. $|1-\frac{i-1}{2}| = \frac{\sqrt{9+1}}{2} = \frac{\sqrt{10}}{2} > \sqrt{2}$. $|-1-\frac{i-1}{2}| = \frac{\sqrt{1+1}}{2} = \frac{\sqrt{2}}{2} < \sqrt{2}$. $|i-\frac{i-1}{2}| = \frac{\sqrt{1+1}}{2} = \frac{\sqrt{2}}{2} < \sqrt{2}$. $|-i-\frac{i-1}{2}| = \frac{\sqrt{1+9}}{2} = \frac{\sqrt{10}}{2} > \sqrt{2}$. The singularities inside the contour γ_2 are $z=-1$ and $z=i$. By the residue theorem,

$$\int_{\gamma_2} f(z)dz = 2\pi i\left(\underset{z=-1}{Res}f(z)+\underset{z=i}{Res}f(z)\right) \qquad (13)$$

For the residue at $z=-1$: $h'(z) = 4z^3, h'(-1) = -4$. In addition, $g(-1) = 6(-1)^3+(i+1)(-1)^2+16(-1)+(1-i) = -6+(i+1)-16+(1-i) = -20$. Thus, $\underset{z=-1}{Res}f(z) = \frac{g(-1)}{h'(-1)} = \frac{-20}{-4} = 5$. Here, one has already found $\underset{z=i}{Res}f(z) = -2$. So, $\int_{\gamma_2} f(z)dz = 2\pi i(5-2) = 6\pi i$.

Thirdly, for the contour $\gamma_3(t) = 1+5e^{it}, 0\le t\le 2$, the center of the contour γ_3 is $z_0 = 1$ and the radius $r = 5$. All the singularities $z=1, z=-1, z=i, z=-i$ lie inside the contour γ_3. By the residue theorem, $\int_{\gamma_3} f(z)dz = 2\pi i\left(\underset{z=1}{Res}f(z)+\underset{z=1}{Res}f(z)+\underset{z=i}{Res}f(z)+\underset{z=-i}{Res}f(z)\right)$.

For the residue at z = 1: $h'(z) = 4z^3, h'(1) = 4$. $g(1) = 6\times 1^3+(i+1)\times 1^2+16\times 1+(1-i) = 6+(i+1)+16+(1-i) = 24$. $\underset{z=1}{Res}f(z) = \frac{g(1)}{h'(1)} = \frac{24}{4} = 6$. One found $\underset{z=1}{Res}f(z) = 5, \underset{z=i}{Res}f(z) = -2$. For the residue at $z=-i$: $h'(z) = 4z^3, h'(-i) = 4(-i)^3 = 4i$. $g(-i) = 6(-i)^3+(i+1)(-i)^2+16(-i)+(1-i) = 6i-(i+1)-16i+(1-i) = (-6-1-16-1)i+(-1+1) = -24i$. Hence, $\underset{z=-i}{Res}f(z) = \frac{g(-i)}{h'(-i)} = \frac{-24i}{4i} = -6$. Therefore, $\int_{\gamma_3} f(z)dz = 2\pi i(6+5-2-6) = 6\pi i$.

To conclude, for γ_1, the integral is $-4\pi i$; for γ_2, the integral is $6\pi i$; for γ_3, the integral is $6\pi i$.

4 CONCLUSION

The residue theorem is of utmost importance in mathematics, particularly in complex analysis. It serves as a powerful tool for evaluating complex integrals, providing an efficient method to calculate the values of integrals that would otherwise be extremely difficult or even impossible to solve using traditional real - variable methods. For numerical algorithms with the development of computer technology, more efficient numerical algorithms based on the residue theorem will be developed. These algorithms will be able to handle large - scale and high - dimensional integral problems, providing powerful computational tools for both theoretical and applied mathematics. It can cross disciplinary applications and for differential Equations: There will be more in - depth connections with complex - valued differential equations. The residue theorem can assist in solving certain types of differential equations by transforming them into integral problems and then using residue - based methods for solution. For physics the residue theorem applies in many fields such as Quantum Mechanics statistical physics, and the residue theorem is an indispensable tool in physics. Its applications in quantum mechanics, electromagnetism, and statistical physics have enabled people to solve complex problems and gain a deeper understanding of physical phenomena. As physics continues to advance, the residue theorem will likely find even more applications in new and emerging areas, further enhancing people's ability to describe and predict the behaviour of the physical world.

REFERENCES

Bak, J., & Newman, D. J. (2010). Applications of the residue theorem to the evaluation of integrals and sums. In Complex analysis (3rd ed., pp. 221-240).

Chi, J. C. (2024). Research on the gradual teaching of complex functions and integral transformations: Taking the solution of complex integrals with singularities as an example. Technology Wind, 2024(5), 97–99.

Churchill, R. V., & Brown, J. W. (2014). Complex variables and applications (9th ed.). McGraw-Hill.

Hang, Z. X., Du, X. R., & Ma, Y. L. (2023). Generalization of a class of generalized integrals using the residue theorem. Physics and Engineering, 33(4), 12–17.

Li, J. H., Cheng, X. H., & Su, G. Z. (2021). Generalization of a residue theorem in complex analysis. Studies in College Mathematics, 24(4), 31–32, 34.

Qiu, W. G. (2020). Infinite series summation and the residue theorem. College Physics, 39(9), 31–33.

Stein, E. M., & Shakarchi, R. (2003). Complex analysis. Princeton University Press.

Trefethen, L. N., & Weideman, J. A. C. (2014). The exponentially convergent trapezoidal rule. SIAM Review, 56(3), 385-458.

Xu, Z. P., & Fan, M. C. (2024). Solving three definite integrals involving inverse trigonometric functions using the residue theorem. Research on College Mathematics, 27(06), 69-73.

Zeng, Q. (2020). Exploring the application of residue theorem in solving different types of integrals. Technology Innovation and Application, 2020(11), 175-176.