Exploring Integrals Involving Logarithmic Function by Different Methods

Siyang Wang[a]
Shenghua Zizhu Academy, Shanghai, China

Keywords: Definite Integral, Logarithmic Function, L'Hôpital's Rule.

Abstract: This paper focuses on the theoretical development and applications of integral and logarithmic functions, with the research background covering the historical evolution of mathematical analysis and modern interdisciplinary demands. From the foundational work to modern scientific and engineering applications, the integral serves as a core tool for quantifying continuous variables. The logarithmic function, due to its data compression and computational simplification characteristics, has become a universal language across multiple disciplines. The paper emphasizes addressing the systematization of integral methods for logarithmic functions, involving the solution of complex integral expressions and the handling of improper integrals. The research method combines classical mathematical tools with innovative techniques: through integration by parts, linear variable substitution and the residue theorem. L'Hôpital's rule and partial fraction decomposition are utilized to handle limits and integrals with logarithms in the denominator. The research indicates that the combination of integral and logarithmic functions provides mathematical support for fields. The significance of the paper lies in integrating theory and application, strengthening the universality of integral techniques, and building a rigorous framework for modelling complex continuous systems and solving practical problems, promoting the in-depth application of mathematical tools in scientific and technological innovation.

1 INTRODUCTION

The concept of integration originated in ancient civilizations. Archimedes of ancient Greece found a method to calculate the areas of curves and volumes, approaching the exact solution through infinite subdivision. Liu Hui's "circle-cutting method" and the volume formula for spheres by Zu Chongzhi and his son in China also contained the idea of integration. Newton and Leibniz established calculus in the 17th century, defining integration as the inverse operation of differentiation and introducing the symbol ∫, laying the foundation for modern integration theory. In the 19th century, Cauchy and Riemann refined the strict mathematical definition of integration, making it a core tool for analyzing continuous variables (Atkinson & Han, 2012).

Integration is widely applied in science and engineering. In physics, it is used to calculate the work done which done by a variable force and the distribution of electric fields. In engineering, it helps determine the center of gravity and moment of inertia of structures (Lu, 2025). In probability theory, integration is employed to find the expected value of continuous random variables, while in artificial intelligence, it is utilized to optimize loss functions. In finance, stochastic integration is used to simulate stock price fluctuations, and in environmental science, integration models are employed to predict the spread of pollutants. From classical mechanics to quantum computing, integration remains a bridge for quantifying continuous changes and connecting mathematics with reality, driving human understanding of the complex world and technological innovation (Stewart, 2015).

The logarithmic function was established by the Scottish mathematician John Napier in 1614, aiming to solve the complex multiplication and division problems in astronomy (Dautov, 2021). His work proposed the use of logarithms as an effective tool for simplifying calculations, converting multiplication and division into addition and subtraction, and significantly enhancing efficiency with the aid of logarithmic tables (Arfken et al, 2013). Subsequently,

[a] https://orcid.org/0009-0009-2465-0730

Wang, S.
Exploring Integrals Involving Logarithmic Function by Different Methods.
DOI: 10.5220/0013814800004708
Paper published under CC license (CC BY-NC-ND 4.0)
In *Proceedings of the 2nd International Conference on Innovations in Applied Mathematics, Physics, and Astronomy (IAMPA 2025)*, pages 104-108
ISBN: 978-989-758-774-0
Proceedings Copyright © 2025 by SCITEPRESS – Science and Technology Publications, Lda.

Henry Briggs developed common logarithms with base 10 and compiled detailed logarithmic tables, promoting their application in fields such as navigation and engineering. In the 18th century, Euler clarified the inverse relationship between logarithms and exponents and introduced the natural logarithm (with base e), laying the foundation for calculus and scientific analysis.

The rest of this paper is organized as follows. Section 2 describes the methodology and concepts, Section 3 presents the integral results, and Section 4 concludes the paper.

2 METHODS

2.1 Integrating Logarithmic Functions

In modern applications, the logarithmic function permeates multiple disciplines. The first is in scientific measurement, decibels (sound intensity), pH values (acidity and alkalinity), and the Richter scale (earthquake energy) all use logarithmic scales to compress a wide range of data. The second is in economics and biology, compound interest models and population growth are often described by exponential functions (Malyavin, 2022). Taking the logarithm of these functions linearizes the analysis. The third is in information technology, algorithm complexity (such as binary search $O(\log n)$) and data compression rely on logarithms to simplify problem scales. The final is in engineering and astronomy, signal attenuation and star brightness calculations both require logarithmic conversions to enhance data processing efficiency. The logarithmic function has evolved from a practical computing tool to a core language in scientific research, continuously pushing the boundaries of human cognition and technological development. (Smith, 1998).

Logarithmic function calculus is a significant part of calculus, with its core revolving around the natural logarithmic function $\ln x$ and its extended forms. In terms of derivatives, the derivative of the natural logarithmic function $\ln x$ is $\frac{1}{x}$, which is its most notable property. For logarithmic functions with general Ibases $\log_a x$, they can be transformed into natural logarithmic forms through the change-of-base formula, and their derivatives are $\frac{1}{x \ln a}$. Regarding integrals, $\int \ln \chi \, dx = \ln \chi - \chi + C$, while the integral $\int \frac{1}{x} dx = \ln|x| + C$ reveals the intrinsic connection between the natural logarithm and the reciprocal function.

At the application level, logarithmic differentiation is a key tool, suitable for simplifying the differentiation process of power-exponential functions (such as $y = f(x)^{g(x)}$ or complex product functions. By taking the logarithm of the function, multiplication is transformed into addition, and then the chain rule is used for differentiation, which can efficiently handle complex expressions like $y = \frac{x^2\sqrt{x+1}}{e^x}$. Additionally, logarithmic functions are often used in integration by substitution, for example, when dealing with $\int \frac{1}{x \ln x}$, setting $u = \ln x$ simplifies it to $\ln|\ln x| + C$. Understanding these basic properties lays a mathematical foundation for analyzing exponential growth, probability models, and engineering problems.

Integration by parts is a fundamental technique in calculus derived from the product rule. It transforms the integral of a product of functions into simpler terms using the formula $\int u \, dv = uv - \int v \, du$. This method is particularly useful for integrals involving products of algebraic, exponential, logarithmic, or trigonometric functions. Strategic selection of u (to differentiate) and dv (to integrate) is key, often guided by the LIATE rule (Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential) (Han et al, 2024). By reducing complex integrals to manageable forms, it enables solutions to problems like $\int x \, e^x \, dx$ or $\int ln(x) \, dx$, making it indispensable in advanced mathematics and applied sciences.

The indefinite integral of the natural logarithmic function $\ln x$ can be derived through integration by parts

$$\int \ln x \, dx = x \ln x - x + C \qquad (1)$$

For derivation process, let $u = lnx$, $dv = dx$, then $du = \frac{1}{x} dx$, $v = x$. Using the integration by parts formula $\int udv = uv - \int vdu$, so it is calculated that $\int \ln \chi \, dx = \chi \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C$.

For $\log_a x \, dx$,by using the change-of-base formula $\log_a x = \frac{\ln x}{\ln a}$, the integral is transformed into

$$\int \log_a x \, dx = \frac{1}{\ln a}(x \ln x - x) + C \qquad (2)$$

The linear variable substitution is another way to do it. For $\int \ln(ax+b) \, dx$, let $t = ax + b$, so $dt = a \, dx$, the integral is transformed into

$$\frac{1}{a}\int \ln t \, dt = \frac{1}{a}(t \ln t - t) + C$$
$$= \frac{ax+b}{a}\ln(ax+b) - \frac{ax+b}{a} + C.$$

SCIENCE AND TECHNOLOGY PUBLICATIONS

2.2 Other Methods and Techniques

The L'Hôpital's rule is a useful method used to calculate limits. Suppose one has two functions $f(x)$ and $g(x)$ which are differentiable in an open interval containing a point a (except possibly at a itself), and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \; or \; \pm\infty$.

The author considers the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$. By Cauchy's Mean Value Theorem, for any x in a neighborhood of a, there exists a point c between x and a such that $\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}$. As $x \to a, c \to a$. If the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists or is $\pm\infty$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$. This is the essence of L'Hôpital's rule, providing a powerful tool to evaluate indeterminate forms (Zhu, 2023).

For the Lagrange theorem, for a given planar arc between two endpoints which there is at least one point at which the tangent to the arc is parallel to the line through its endpoints (Li, 2023), i.e.,

$$\frac{f(b)-f(a)}{b-a} = f'(c) \tag{3}$$

The Residue Theorem in complex analysis simplifies computing contour integrals of meromorphic functions. For a function f(z) with isolated singularities inside a closed contour C, the integral around C equals $2\pi i$ multiplied by the sum of residues within C. A residue, extracted from the Laurent series coefficient of $(z-z_0)^{-1}$ encapsulates local behavior near singularities. This theorem transforms intricate integrals into manageable residue calculations, crucial for evaluating real integrals, analyzing wave propagation, and solving differential equations. Its power lies in linking global integration to localized singularity data, making it indispensable in physics, engineering, and mathematical research.

Cauchy's Integral Theorem states that for a holomorphic function in a simply connected domain, the integral over any closed contour is zero. Established by Augustin-Louis Cauchy, it is central to complex analysis. These principles enable efficient evaluation of complex integrals, residue calculus, and solutions to partial differential equations, forming the cornerstone of analytic function theory.

3 APPLICATIONS

The first example is a basic integral

$$I = \int x \ln x \, dx \tag{4}$$

One can solve the integral by using the integration by parts. Assume that $u = \ln x$, so $du = \frac{1}{x}dx$; assume $dv = xdx$, so $v = \frac{1}{2}x^2$. By substituting into the formula for integration by parts, the solution is that

$$\begin{aligned}\int udv &= uv - \int vdu \\ &= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}dx \\ &= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}xdx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C\end{aligned} \tag{5}$$

The second example is the multiplication of polynomials and logarithms

$$I = \int x^2 \ln x \, dx \tag{6}$$

One can also solve the question by using the integration by parts. Assume $u = \ln x$, so $du = \frac{1}{x}dx$; assume $dv = x^2 dx$, so $v = \frac{x^3}{3}$. By substituting into the formula for integration by parts, the solution is that

$$\begin{aligned}\int x^2 \ln x \, dx &= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}dx \\ &= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 \, dx \\ &= \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C\end{aligned} \tag{7}$$

The third example is integration with a logarithm in the denominator, i.e.,

$$I = \int \frac{1}{x \ln x} dx \tag{8}$$

To solve the question, one can use the method of variable substitution (Liu & Liu, 2024). Assume $t = \ln x$, so $dt = \frac{1}{x}dx$, and thus $dx = xdt = e^t dt$. The original integration can turn into the final solution, which is

$$\int \frac{1}{e^t \cdot t} \cdot e^t dt = \int \frac{1}{t}dt = \ln|t| + C = \ln|\ln x| + C$$

The fourth example is a combination of logarithms and fractions, i.e.,

$$I = \int \frac{\ln(1+x)}{x^2} dx \tag{9}$$

Likewise, one can solve the formula by using the integration by parts. Assume that $u = \ln(1+x)$, so $du = \frac{1}{1+x}dx$; assume that $dv = \frac{1}{x^2}dx$, so $v = -\frac{1}{x}$. Substituting the equation into the formula, it is thus found that

$$\int \frac{\ln(1+\chi)}{\chi^2}dx = -\frac{\ln(1+\chi)}{\chi} + \int \frac{1}{\chi(1+\chi)}dx$$

Given that the fraction decomposition is $\frac{1}{x(1+x)} = \frac{1}{x} - \frac{1}{1+x}$, the integral on the right-hand side is

$$\int \frac{1}{\chi(1+\chi)} dx = \ln|\chi| - \ln|1+\chi| + C$$

Therefore, the final solution is that

$$I = -\frac{\ln(1+\chi)}{\chi} + \ln\left|\frac{\chi}{1+\chi}\right| + C \qquad (10)$$

The fifth integral is of absolute value

$$I = \int \ln|\chi|\, dx \qquad (11)$$

To solve the equation, one can use the case-by-case discussion. First, when $x > 0$, the integration is the same with $\int \ln x dx$, the solution is $\chi \ln \chi - \chi + C$. Second, when $x < 0$, let $t = -x$, so $\ln|x| = \ln|t|$ and $dx = -dt$. The integration turns into

$$-\int \ln t\, dt = -(t \ln t - t) + C$$
$$= \chi \ln(-\chi) - \chi + C$$

Regardless of whether x is positive or negative, the solution of integration can be presented as

$$I = \chi \ln|\chi| - \chi + C \qquad (12)$$

The sixth definite Integrals and improper Integrals is

$$I = \int_0^1 \ln \chi\, dx \qquad (13)$$

This is an improper integral (with $x = 0$ as a singular point). One can calculate the limit which is $\lim_{a\to 0} \int_a^1 \ln x\, dx = \lim_{a\to 0} [x \ln x - x]_a^1$. Substituting the upper and lower limits, it is thus found that $\lim_{a\to 0}[(1 \cdot 0 - 1) - (a \ln a - a)] = -1 - \lim_{a\to 0} a \ln a + 0$. Calculating the limit by using L'Hôpital's rule, which is

$$\lim_{a\to 0} a \ln a = \lim_{a\to 0} \frac{\ln a}{\frac{1}{a}} = \lim_{a\to 0} \frac{\frac{1}{a}}{-\frac{1}{a^2}} = \lim_{a\to 0}(-a) = 0$$

The final solution is therefore

$$\int_0^1 \ln \chi\, dx = -1 \qquad (14)$$

The seventh is a higher-order logarithmic integral

$$I = \int (\ln \chi)^2\, dx \qquad (15)$$

It can be solved by applying integration by parts twice. For the first integration by parts, one can assume $u = (\ln \chi)^2$, so $du = 2 \ln \chi \cdot \frac{1}{x} dx$; assume $dv = dx$ so $v = \chi$. The integration turns into that $\chi(\ln \chi)^2 - \int \chi \cdot 2\chi \cdot \frac{1}{x} dx = \chi(\ln \chi)^2 - 2\int \ln \chi\, dx$. Next, one can use the integration by parts the second times to calculate $\int \ln x\, dx$, and the solution is that $\int \ln x\, dx = x \ln x - x + C$. Combining the solution together, it is that

$$\int (\ln \chi)^2\, dx = \chi(\ln \chi)^2 - 2(\chi \ln \chi - \chi) + C$$
$$= \chi(\ln \chi)^2 - 2\chi \ln \chi + 2\chi + C \qquad (16)$$

4 CONCLUSIONS

This article systematically explores the historical evolution, core methods, and interdisciplinary applications of integrals and logarithmic functions. The main content is divided into three parts: Firstly, by reviewing the development of integral theory from Archimedes' "method of exhaustion" to the establishment of Newton-Leibniz calculus and then to the rigorous definition by Cauchy-Riemann, it clarifies the core position of integrals in quantifying continuous variables. At the same time, the development of logarithmic functions from Napier's simplification of astronomical calculations to Euler's introduction of natural logarithms demonstrates their evolution. The combination of the two highlights the practical significance of mathematical tools in fields. The second part focuses on the mathematical methods for integrating logarithmic functions, systematically deriving the integral formulas for the natural logarithm and the general logarithmic function. Additionally, the application of L'Hôpital's rule and partial fraction decomposition demonstrates the synergy between limit analysis and algebraic techniques. The third part verifies the effectiveness of the method through seven typical examples, covering definite integrals, improper integrals, and higher-order integrals. These examples not only consolidate the theoretical derivation but also reveal the universality of the recursive solution rules and the handling of absolute values. The research results show that the combination of integrals and logarithmic functions provides mathematical support for the modeling of continuous systems.

Although this article systematically reviews the methodological framework for integration and logarithmic functions, there remains room for expansion. Additionally, the paper focuses mainly on classical integration techniques and pays less attention to the auxiliary role of modern computational tools, such as symbolic computation software. The value of these tools in verification and accelerating the solution process could be further explored. Future research directions can be developed from the following perspectives: Exploring the combination of logarithmic functions and fractional calculus to address the modeling needs of nonlocal problems and extending integral methods to stochastic differential equations or high-dimensional optimization problems in machine learning to enhance the adaptability of theoretical tools.

REFERENCES

Arfken, G., Weber, H. J., & Harris, F. E. (2013). Mathematical Methods for Physicists (7th ed.). Elsevier.

Atkinson, K. E., & Han, W. (2012). Theoretical Numerical Analysis: A Functional Analysis Framework (3rd ed.). Springer.

Dautov, R. R. (2021). On some classes of improper integrals with logarithmic functions. Lobachevskii Journal of Mathematics, 42(9), 1567–1575.

Han, S. X., Hu, Y., & Huang, Y. Z. (2024). Asymptotic Formula of Definite Integral Expanded at Two Points. College Mathematics, 40(03), 76–81.

Li, T. Z. (2023). A class of definite integrals of rational fractions of trigonometric functions. Research on College Mathematics, 26(06), 55–57.

Liu, X. P., & Liu, C. P. (2024). A Note on the Primitive Function of Definite Integral. Studies in College Mathematics, 27(06), 5–7+41.

Lu, Q. (2025). A Brief Discussion on the Calculation of Integrals. Technology Wind, (03), 110–112+134.

Malyavin, M. V. (2022). Logarithmic p-series and logarithmic p-integrals in Mathematical analysis. Journal of Capital Normal University (Natural Science Edition), 44(1), 60–67.

Stewart, J. (2015). Calculus: Early Transcendentals (8th ed.). Cengage Learning.

Zhu, Y. B., Li, J. H., & Chai, H. Y. (2023). On the limits of a class of definite integrals. Research on College Mathematics, 26(06), 16–19.

SCITEPRESS
SCIENCE AND TECHNOLOGY PUBLICATIONS