Applications of Cauchy's Residue Theorem in Complex Functions of Fractional Form

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Abstract: This paper explores the applications of Cauchy's Residue Theorem in complex analysis, focusing on its utility in evaluating complex integrals around closed contours. The study begins with an introduction to the Laurent series and Taylor series, which are foundational for understanding the Residue Theorem. The Residue Theorem uses these series to find "residues", which is a fancy term for coefficients that capture the behavior at singularities. Summing these residues gives the integral’s value instantly. The paper then delves into the theorem's theoretical framework, illustrating its application through several examples, including functions with simple poles, higher-order poles, and removable singularities. For simple poles (basic singularities), calculating residues is straightforward. For harder cases (like higher-order poles), it needs the use of derivatives. The results demonstrate the theorem's effectiveness in simplifying complex integral calculations, particularly in cases involving trigonometric and rational functions. The research highlights the theorem's significance in both theoretical mathematics and practical applications, such as physics and engineering. The paper concludes with a discussion on the potential for further exploration and the implications of these findings for advanced mathematical studies.

1 INTRODUCTION

Functions of complex variables are the primary subject of study in complex analysis, a mathematical branch (Ahlfors, 1979). One of the most powerful tools in this field is Cauchy theorems, which enables a precise calculation for integrating by summing the coefficients obtained through Laurent series expansions at critical points. This theorem is particularly useful in solving real integrals that are otherwise difficult to compute using standard techniques. Among its key results, Cauchy's Residue Theorem stands out as a "mathematical supertool" that transforms intricate integrals into simple algebraic computations by leveraging singularities, representing points where functions behave abnormally (Shen & Li, 2016). This theorem bridges pure theory and applied mathematics, offering unified solutions to problems ranging from electromagnetic field calculations to signal processing.

The importance of Cauchy's Residue Theorem extends beyond pure mathematics. It has significant applications in physics, engineering, and other sciences where complex integrals frequently arise. In physics, it provides exact solutions to problems in fluid dynamics and quantum mechanics. Engineers rely on it for signal processing algorithms and control system design. Even in pure mathematics, it aids in number theory through the study of zeta functions. The Residue Theorem's significance is well-documented in both historical and modern contexts (Shen, 2017). Ahlfors noted its role in 19th-century function theory development, while contemporary researchers like Stein & Shakarchi emphasize its utility in evaluating Fourier and Laplace transforms. In physics, Peskin et al demonstrate how residues simplify Feynman path integrals for particle interactions. In this essay, Cauchy’s Residue Theorem could be extended to solve higher order singularities.

Section breakdown is organized in this arrangement. Section 2 introduces the Laurent series and Taylor series, which are essential for understanding the Residue Theorem. Section 3 presents several examples demonstrating the theorem's application, including functions with

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simple poles, higher-order poles, and removable singularities. This paper addresses both gaps by presenting visualizable examples and systematic methods for handling second-order pole. There are examples in Section 3 showing how to apply the theorem to second order poles. Final section synthesizes key discoveries and present forward-looking propositions.

2 METHOD AND THEORY

2.1 Laurent Series & Taylor Series

Before getting into Cauchy's Theorem, it is necessary to mention the two series that it is based on.

The mathematical apparatus of Laurent series constitutes a generalization of Taylor series to accommodate singularities in complex function representation (Zhou, 2022). While a Taylor series can only represent functions exhibiting differentiable behavior within a neighborhood of dot, Laurent series can represent functions that have singularities, such as poles or essential singularities, within the region of interest. This gives Laurent expansion unique advantages in handling functions with singularities.

The Laurent series of a holomorphic expression $\Gamma(\tau)$ within annulus of τ_0 shows:

$$\Gamma(\tau) = \sum_{n=0}^{\infty} \alpha_n (\tau - \tau_0)^n + \sum_{n=1}^{\infty} \beta_n (\tau - \tau_0)^{-n} \quad (1)$$

where α_n are complex-valued coefficients, and the series expansion involves negative coefficients. Its convergence holds in an annular region, mathematically characterized as the open set between two circles with identical centers but distinct radii. Geometrically, an annulus refers to the ring-shaped domain encircled by an inner circumference with a radius of r_1 and an outer circumference with a radius of r_2, both centered at the same point.

Laurent's formulation/expansion can be split into two parts, the principal part and the analytic part. This principal series representation includes a component featuring inverse powers of $(z - z_0)$, specifically the summation from $n = -\infty$ to $n = 0$ of all powers. This part captures the function's behavior in the vicinity of the singularity z_0 is of particular interest. Containing non-negative terms of $(z - z_0)$, the series are summation of all powers where n ranges over all natural numbers starting from zero. This part behaves like a Taylor series and represents the analytic part of the function, the coefficients are determined by the following formula, thereby facilitating accurate calculations.

$$\alpha_n = \frac{1}{2\pi i} \oint_C \frac{\Gamma(\xi)}{(\xi - z_0)^{n+1}} d\xi (n = 0, \pm 1, \pm 2). \quad (2)$$

Of note is that Laurent series still works if z_0 is an isolated singularity. The residue of the function at the specified point is precisely determined by the corresponding coefficient.

By making $\Gamma(\tau)$ differentiable on $(z - z_0) < R$, then all $b_1 = b_2 = b_3 \cdots\cdots b_n = 0$, the Laurent series is then weakened (reduced) to become a Taylor series. For example, people can focus on the Laurent expansion, $\Gamma(\tau) = \frac{1}{\tau - 2}$ within annulus of $\tau = 0$. For $|\tau| < 2$, it is found that $\Gamma(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{2^{n+1}}$, but for $|\tau| > 2$, $\Gamma(\tau) = \sum_{n=1}^{\infty} \frac{2}{\tau^{n+1}}$. As a special case, when The elimination of all principal part terms transforms the Laurent expansion turns into a Taylor series, which states if $\Gamma(\tau)$ exhibits the property of complex-differentiability in $|z - z_0| < R$, then for any z in that region, one has that $\Gamma(\tau) = \sum_{n=0}^{\infty} \alpha_n (\tau - \tau_0)^n$, and the coefficient in this power series given by $a_n = \frac{f^{(n)}(\tau_0)}{n!}$ $(n = 0,1,2,3, \cdots n)$. Especially, $a_0 = f(z_0)$, $a_1 = f'(\tau_0)/1$, $a_2 = f''(\tau_0)/2$, and so on. It is a series like a polynomial but of infinite degree.

2.2 Cauchy's Theorem

This paper starts by talking about Cauchy-Goursat Theorem, which states that if the function $f(z)$ exhibits differentiability at each point within the interior of a simple closed curve C, then

$$\oint_C^{\infty} f(z)dz = 0 \quad (3)$$

The theorem reveals that the line integral of an analytic complex function around any closed contour always equals zero (Lin, 2021). This is a pivotal finding in complex analysis, demonstrating that if a function is holomorphic within a simply connected region, its line integral becomes independent of the specific integration path chosen. Second, about Cauchy's renowned contour integral relation, which states that if given a closed curve with f(z) being analytic in its entirety, then

$$f(z_0) = \oint_C \frac{f(z)}{z - z_0} dz \quad (4)$$

This theorem is derived from the Cauchy-Goursat Theorem (Mitrinović & Kečkić, 1984). It reveals that the integral of an analytic function along a closed contour can be represented using the values of the function within the contour. This formula is particularly useful for calculating the values of analytic functions. Finally, all above leads to the Cauchy's Theorem. Suppose C is a simple closed contour, and $f(z)$ is analytic at all points within C, with the exception of the singularities $(z_1, z_2, z_3 \cdots z_n)$ then,

$$\oint_C \Gamma(\tau)d\tau = 2\pi i \mathrm{Res}(\Gamma, \tau_1) + \mathrm{Res}(\Gamma, \tau_2) + \cdots + \mathrm{Res}(\Gamma, \tau_n) \quad (5)$$

Here, $\mathrm{Res}(\Gamma, \tau_n)$ denotes the residue of $\Gamma(\tau)$ at the singularity z_k. The residue can be regarded as the coefficient of $1/(z - z_k)$in the Laurent expansion.

The Residue Theorem serves as a pivotal instrument within the realm of complex function theory. It offers an approach to evaluate contour integrals in the complex plane, especially when dealing with functions that possess isolated singularities. This theorem enables the computation of integrals of complex functions along closed contours by aggregating the residues of the function at its singular points enclosed by the contour.

In the context of complex analysis, the residue of a complex function $f(z)$ at an isolated singularity z_0 can be identified as the coefficient associated with the $\frac{1}{z-z_0}$ term within the Laurent series expansion of $f(z)$ centered at z_0. The calculation of residues typically depends on the type of singularity. For simple poles, the residue can be directly computed using limits, while for higher-order poles, a combination of derivatives and limits is required.

The connections of three theorems are the following. Initially, the Cauchy-Goursat Theorem serves as the cornerstone, demonstrating that the integral of an analytic function along a closed contour equals zero. This fundamental principle underpins the Cauchy Integral Formula. Subsequently, the Cauchy Integral Formula acts as an expansion of the Cauchy-Goursat Theorem, enabling the computation of the value of an analytic function within a given path through the application of an integral. Third, the Theorem further extends the Cauchy Integral Formula, enabling people to solve integrals over closed paths by computing the residues of the function at its singular points. When a function has singularities inside the path, the Cauchy-Goursat Theorem no longer applies, but the Cauchy's Residue Theorem remains valid.

3 RESULTS AND APPLICATIONS

In the following examples, there would be three main categories. The first category is about simple fractions; the second is about fractions which both numerators and denominators are complex; while the third is about function with removable singularity and second poles (Xu & Fan, 2024).

3.1 Example Application 1

In this section, Residue Theorem is applied to simple fractions in which only the denominator is complex while the numerator is rational number (He, 2021).

The initial approach involves examining the function $f(z) = \frac{1}{z^2+1}$. It has simple poles at $z = i$, $z = -i$. To compute the integral about a contour $C = 1 + 4e^{it}$ that contains these poles, one first finds the residues at these poles. For the pole at $z = i$:$\mathrm{Res}(f, i) = \lim_{z\to i}(z - i)\frac{1}{z^2+1} = \frac{1}{2i}$. For the pole at $z = -i$: $\mathrm{Res}(f, -i) = \lim_{z\to -i}(z + i)\frac{1}{z^2+1} = \frac{1}{-2i}$. Then by applying Cauchy's Theorem, it is found that

$$\oint_C \frac{1}{z^2 + 1} dz = 2\pi i \left(\frac{1}{2i} + \frac{1}{-2i}\right) = 0 \quad (6)$$

This result is consistent with the fact that f(z) is analytic all points except for $z = i, z = -i$, and the residues at these points cancel each other out.

The second is to consider function $\Gamma(\tau) = \frac{1}{\tau^2-1}$. It has simple poles at $\tau = 1$, $\tau = -1$. To compute the integral about a contour $C = 1 + 5e^{it}$ that encloses these poles, first find the residues at each pole (Lin & Gong, 2018). For the pole at $\tau = 1 : \mathrm{Res}(\Gamma, 1) = \lim_{\tau\to i}(\tau - 1)\frac{1}{\tau^2-1} = \frac{1}{2}$. For the pole at $\tau = 1$: $\mathrm{Res}(\Gamma, -1) = \lim_{\tau\to -i}(\tau + 1)\frac{1}{\tau^2-1} = -\frac{1}{2}$. The integral is found to be $\oint_C \frac{1}{\tau^2-1} d\tau = 0$. This result is in line with the fact that$\Gamma(\tau)$ is analytic except for two points at which the residues cancel each other out.

One can also consider the function $f(z) = \frac{1}{(z-1)^2}$. This function has simple poles at z =1. To compute the integral of around a contour $C = 1 + 6e^{it}$ that encloses these poles, first find the residues at the pole. For the pole at $z = 1$: $\mathrm{Res}(f, 1) = \lim_{z\to i}(z - 1)^2 \frac{1}{(z-1)^2} = 1$. Then by applying Cauchy's Residue Theorem, the integral is:

$$\oint_C \frac{1}{(z - 1)^2} dz = 2\pi i \quad (7)$$

This result is consistent with the fact that is analytic everywhere except at, and the residues at these points cancel each other out.

3.2 Example Application 2

Cauchy's Residue Theorem is applied to fractions with both numerators and denominators be complex.

The author shall focus on the function $f(z) = z^t/z^2 + 1$. This function has singularities at $z = i, z = -i$. Then the author will calculate the residues at two singularities. The residues of a function $f(z)$ at a first-order pole $z = a$ is given by $\mathrm{Res}(f, a) = \lim_{z\to a}(z-a)f(z)$. Thus, it is calculated that $\mathrm{Res}(f, i) = \lim_{z\to i}(z-i)\frac{z^t}{(z-i)(z+i)} = \frac{e^{it\frac{\pi}{2}}}{2i}$ as well as $\mathrm{Res}(f, -i) = \lim_{z\to -i}(z+i)\frac{z^t}{(z-i)(z+i)} = -\frac{e^{it\frac{3\pi}{2}}}{2i}$. The final integral is

$$\oint_0^\infty \frac{z^t}{1+z^2} = 2\pi i\left(\frac{e^{it\pi/2}}{2i} - \frac{e^{3it\pi/2}}{2i}\right) = \pi\left(e^{it\pi/2} - e^{3it\pi/2}\right) \quad (8)$$

Another function is like this $f(z) = \frac{2z^2-z+1}{(2z-1)(z+1)}$. It has first-order poles at $z = -1, z = \frac{1}{2}$. To compute the integral about contour $C = 3 + 4e^{it}$ that contains these poles, first find the residues at each pole. It is found that $\mathrm{Res}(f, -1) = 2$ and $\mathrm{Res}\left(f, \frac{1}{2}\right) = 2/3$. The final integral could be calculated by using Cauchy's Residue Theorem

$$\oint_C \frac{2z^2-z+1}{(2z-1)(z+1)}dz = 2\pi i\left(2+\frac{2}{3}\right) = \frac{16\pi i}{3} \quad (9)$$

Finally, the author will consider a more complex function (with trigonometry functions) like this: $f(z) = \frac{1-\cos z}{z(z^2+1)}$. To calculate the integral associated with contour $C = i + 5e^{it}$, one can first calculate the three residues at $z = 0, z = i, z = -i$. It is found that $\mathrm{Res}(f, i) = \frac{1-\cos i}{-2}$, $\mathrm{Res}(f, -i) = \frac{1-\cos i}{-2}$, $\mathrm{Res}(f, 0) = \frac{1-\cos 0}{1} = 0$. Thus, using the Cauchy's Residue Theorem, the integral is

$$\oint_C f(z)dz = 2\pi i(0 + 1 - \cos i) = 2\pi i(1-\cos i) \quad (10)$$

3.3 Enhanced Application

In this section, Residue Theorem is applied to a function with removable singularity and second poles.

Consider the function like this

$$f(z) = \frac{z^2}{(z^2+\pi^2)^2 \sin z}. \quad (11)$$

To calculate the integral associated with contour $C = i + 6e^{it}$, it is found that it has one pole at $z = 0$, and $\mathrm{Res}(f, 0) = \lim_{z\to 0}(\frac{z}{\sin z})\lim_{z\to 0}(\frac{z}{(z^2+\pi^2)^2}) = 0$ (Labora & Labora, 2025). It is inferred that $z = i\pi$ is a pole of order 2. $\mathrm{Res}(f, i\pi) = -\frac{1}{4\pi\sinh\pi} + \frac{\cosh\pi}{4\pi\sinh\pi}$. Thus, it is calculated that

$$\oint_C f(z)\,dz = \frac{i}{2}\left(-\frac{1}{\sinh\pi} + \frac{\cosh\pi}{\sinh\pi}\right). \quad (12)$$

4 CONCLUSIONS

To summarize, this paper has explored the applications of Residue Theorem in complex analysis and has stated its usefulness in calculating complex integrals around closed contours. Through various examples, how the theorem simplifies the calculation of integrals involving functions with singularities is clearly shown. The results highlight the theorem's utility in both theoretical and applied contexts, including trigonometric function, fractional function, and others. Future research could explore the theorem's applications in more complex scenarios, such as functions with essential singularities or in higher-dimensional spaces. Additionally, further investigation into the computational aspects of the theorem could lead to more efficient algorithms for solving complex integrals. Overall, Cauchy's Residue Theorem remains a cornerstone of complex analysis, with wide-ranging implications for evaluating integrals with higher order poles.

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