Complex Analysis Approach to Solving Problems with Trigonometric Functions

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Abstract: Nowadays, many problems are involved with the integrals. Whether in the field of physics or mathematics, if people want to solve some intricate integration, there is no doubt that one should use the residue theorem. This paper mainly focuses on how to solve some complex integration by using the residue theorem. By enumerating some examples about the integrations in complex plane, the author shall show the effects on different field to prove the prominence of residue theorem. By this paper, it will embody and learn to the greatest extent possible. The meaning of this paper is that it shows the importance of residue theorem, and give person another tool to solve some involute integrations in complex plane. What is more, this paper can show the importance of residue, which not only can apply in mathematics, but also can use in considerable fields. In addition, it can help people analyse the earthquake, electricity and a huge variety of difficult integrations.

1 INTRODUCTION

In the era of rapid development, a huge variety of integration should be solved, regardless of in mathematics or physics, even in geography. In order to solve these difficult problems, one must use residue to reduce the difficulty of integrations. For example, in multi-dimensional and multi-point earthquake response spectrum, there are considerable problems such as lower rate of calculation and this deviates from the original simplicity and high efficiency of the response spectrum method and. Analyzing multi-dimensional and multi-point earthquake response spectrum and based on wavelet transform, a seismic simulation algorithm is proposed. This method can fit the standard response spectrum, but does not fully consider the spatial effect factors. This method also has some problems in duration, but applying the residue can reduce the time of calculation to a certain degree (Zhao et al, 2022).

In addition, the residues not only help researcher analyze the earthquake, but also can help investigator finish the calculation and analysis of flexible DC transmission system (Sakhaei, 2025). Use the residue theorem to calculate the αβ component vector of the AC voltage on the complex plane and estimate the angular frequency (Kaitlin & Patricia, 2022). The residue theorem can be used to calculate the integral of the complex function $f(z)$ along a closed path C containing several singular points. Comparing with the conventional method, it can provide higher efficiency. The complex function $f(z)$ can be designed according to the expected interference and noise suppression characteristics of the frequency and phase angle estimator. What is more, it also can simplify the formula for the path integral and the control structure of the estimator is designed (Liu, 2023). In Physics when researching for topological phase, it can judge a phase is either mediocre or topological in a system by depending on topological invariants in a system. However, when calculating the topological invariants, it needs to definite the transformation of Hamiltonian from lattice space to momentum space, thus in this process, it must use the residue theorem to deal with this problem. It can transform the loop integration problem of an analytic function into the problem of finding the sum of the residues of the singular points of the integrand (Meng & Guan, 2023).

In this paper, the author will describe the definition of the residue theorem and its formular at first. At the same time, the work will give some

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examples about some problems which can transform to use the residue theorem to make it easier to accomplish. It will also tell the applications about residue theorem in variance field, mainly showing how to use the residue theorem in trigonal functions. Finally, it shows a conclusion of the residue theorem with some meaningful value and its future development and use.

2 METHODS

2.1 Residue Theorem

In complex analysis, Residue Theorem is a paramount tool to exploit the path integral of an analytic function along a closed curve. What is the meaning of Residue? The meaning of it is that it refers to the integral of an analytic function along any positive simple closed curve surrounding an isolated singular point in a circular domain divided by $2\pi i$. The formula of this theorem is given by

$$\oint_c f(z)dz = 2\pi i \sum_{k=1}^{n} Res[f(b_k)] \qquad (1)$$

In this formula, the part of $\mathrm{Res}[f(b_k)]$ means that the residue of $f(z)$ is in b_k and it is identical with the coefficient, a_{-1}, in $(z-b_k)^{-1}$ Laurent Expansion in neighbourhood of b_k. The Laurent expansion plays an essential role in complex function, due to the fact that it is a form of series in isolated singularity. It can give the value of residue directly, so that people can use it in some specific condition rather than the residue theorem (Wu, 2011).

When it comes to the Laurent expansion, it must think about the singularity (Labora & Labora, 2025). The singularity can be divided into many categories, but in residue field, the essential singularity and removable singularity usually be used in a wide range of problems. Especially removable singularity, the residue of it is zero. In addition, the pole is another paramount factor. For example, the first-order pole in Laurent expansion, the maximum negative power term is $(z-z_0)^{-1}$. Otherwise, the essential singularity has infinitely many negative powers. Such as $\mathrm{z} = 0$ is an essential singularity of $\mathrm{f(z)} = e^{\frac{1}{z}}$. When calculating the residue, one firstly needs to identify the types of singularity. For different types of singularity, different methods can be used. The application of Residue theorem is in many respects. For instance, Rational fraction expansion and trigonometric functions are nee to use residue theorem.

2.2 Applications

The residue theorem is not only used in mathematics, but also can be applied in physics. For instance, in Quantum Mechanics, the residue theorem probably facility to calculate of integration in Green's function, especially in energy level calculation (Lin, 2015). What is more, it can also determine location of energy levels and resonance states in a system rapidly by using residue theorem. Otherwise, it will clarify the integration of oscillating function in electromagnetic fields. In addition, in statistical mechanics, it is a predominant way to solve some complex integration, specifically in Gaussian Integral and Fourier Transform. In addition, considerable critical phenomenon problems can be solved by utilizing the residue theorem. In the field of signal processing, the residue is the priority due to the fact that it could make it easy to a certain extent. For example, in filter frequency response, it can transform the integration in complex plane.

2.3 Trigonometric Functions

The trigonometric functions are essential too. It can help people to solve some problems in triangle. The sine, cosine and tangent are always used. They have some equations. For example, $\sin^2\alpha + \cos^2\alpha = 1$, $\sin 2\alpha = 2\sin\alpha\cos\alpha$, $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha$ (Liu, 2019). These equations can help people to transfer the difficult function to easy one. Because it is a function, it also has the same properties of function. For instance, they have the cycle, but they are not same. The tangent function is π. The sine and cosine are 2π. Otherwise, the parity is not same too. The parity of sine and tangent are odd function, and the cosine function is even function.

The Odd/Even Symmetry can help people simplify the calculations. In addition, the sine and cosine functions are Bounded Functions, nevertheless the tangent is not bounded functions. What is more, symmetry is different too. For example, the sine function is symmetric about the origin, and the cosine function is symmetric about the y-axis, due to even function. The tangent function is specific one. It is symmetric about the origin, and the image is symmetric about each $\frac{\pi}{2} + k\pi$ point.

3 EXAMPLES

3.1 Basic Setups

The first example is to evaluate the integral

$$\mathrm{I} = \oint_{|z|=1} \frac{z+1}{z^2} dz \tag{2}$$

The author will use both direct contour integration method and also the calculus of residues to solve this integral. In this first method, one can parameter the integration in a unit circle with $|z| = 1$. Depending on Euler's formula, $z = \rho e^{i\theta}$, it can transform to $z = e^{i\theta}$. The angle θ is from 0 to 2π, thus $dz = ie^{i\theta} d\theta$. So, the integration can transform to

$$\begin{aligned}\mathrm{I} &= \oint_{|z|=1} \frac{z+1}{z^2} \mathrm{dz} \\ &= \int_0^{2\pi} \frac{e^{i\theta}+1}{e^{2i\theta}} \cdot ie^{i\theta} \mathrm{d}\theta \\ &= \mathrm{i} \int_0^{2\pi} \left(e^{-i\theta}+1\right) \mathrm{d}\theta \end{aligned} \tag{3}$$

The author calculates respectively $\int_0^{2\pi} e^{-i\theta} d\theta = 0$ and $\int_0^{2\pi} 1 d\theta = 2\pi i$. Therefore,

$$I = i \cdot (2\pi + 0) = 2\pi i. \tag{4}$$

In the second method, there is a second-order pole in $z = 0$ of an integrand $f(z) = \frac{z+1}{z^2}$. So, one can calculate the residue as

$$\begin{aligned}\mathrm{Res}(f,0) &= \lim_{z\to 0} \frac{d}{dz}\left(z^2 \cdot \frac{z+1}{z^2}\right) \\ &= \lim_{z\to 0} \frac{d}{dz}(z+1) = 1 \end{aligned} \tag{5}$$

Depending the residue theorem, one can calculate the points as

$$2\pi i \cdot Res(f,0) = 2\pi i \cdot 1 = 2\pi i \tag{6}$$

The residue theorem not only can solve some integration in complex variable function, but also can complete some real integral especially in trigonometric functions. In a study of real integral which is about trigonometric functions, it has a common form:

$$I = \int_0^{2\pi} R(\cos\theta, \sin\theta) d\theta\,. \tag{7}$$

The definition of $R(\cos\theta, \sin\theta)$ is a rational function in a unit circle with a range of $|z| = 1$. There are considerable replacements to make real integrals be transformed into closed integrals around the unit circle. Let $z = e^{i\theta}$, then $\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2i}$, $d\theta = \frac{dz}{iz}$. The initial integration can be transformed:

$$\int_0^{2\pi} R(\cos\theta, \sin\theta) d\theta = \oint_{|z|=1} f(z) \frac{dz}{iz} \tag{8}$$

3.2 Examples

The author now calculates this integration

$$\mathrm{I} = \int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta} d\theta \tag{9}$$

Let $z = e^{i\theta}$,$\cos\theta = \frac{z+z^{-1}}{2}$,$d\theta = \frac{dz}{iz}$, and the path of integration in complex plane is a unit circle with $|z| = 1$. The initial integration has changed to $\oint_{|z|=1} \frac{\frac{z+z^{-1}}{2}}{5+4\cdot\frac{z+z^{-1}}{2}} \cdot \frac{dz}{iz}$, and simplify denominator and numerator:

$$\oint_{|z|=1} \frac{z^2+1}{2\mathrm{iz}\cdot(2z^2+5z+2)} \mathrm{dz} \tag{10}$$

The roots of $2z^2 + 5z + 2$ are respectively $z = -\frac{1}{2}$ and $z = -2$. However, in a unit circle with $|z| = 1$ only have $z = 0$ and $z = -\frac{1}{2}$ to be poles. When z is equal to 0 (it is a first order pole):

$$\mathrm{Res}(f,0) = \lim_{z\to 0} z \cdot \frac{z^2+1}{2iz\cdot(2z^2+5z+2)} = \frac{1}{4i} \tag{11}$$

When z is equal to $-\frac{1}{2}$, then $\mathrm{Res}\left(f, -\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \cdot \frac{z^2+1}{2iz\cdot(2z^2+5z+2)} = -\frac{5}{12i}$. By using the residue theorem, the total residue is: $\frac{1}{4i} - \frac{5}{12i} = -\frac{1}{6i}$ (Xu & Fan, 2024). The result of integration is: $2\pi i \cdot \left(-\frac{1}{6\pi}\right) = -\frac{\pi}{3}$. However, people should test the result, and depending on standard integration formula one can get:

$$I = \int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta} d\theta = -\frac{\pi}{3} \tag{12}$$

Next, the author calculates the integration:

$$\mathrm{I} = \int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} (|a| < 1) \tag{13}$$

Let $z = e^{i\theta}$ and author uses some replacement to transform the initial integration. The replacement is $\cos\theta = \frac{z+z^{-1}}{2}$, $d\theta = \frac{dz}{iz}$ and the unit circle of $|z| = 1$. The final integration is

$$\begin{aligned}\mathrm{I} &= \oint_{|z|=1} \frac{1}{1+a\cdot\frac{z+z^{-1}}{2}} \cdot \frac{dz}{iz} \\ &= \oint_{|z|=1} \frac{2}{az^2+2z+a} \cdot \frac{dz}{i} \end{aligned} \tag{14}$$

The author now analyses the singularity. The root of $az^2 + 2z + a = 0$ is $z = \frac{-1\pm\sqrt{1-a^2}}{a}$ but only $z_2 = \frac{-1+\sqrt{1-a^2}}{a}$ is in this unit circle, due to the fact that $|a| < 1$,and test the length of z_2 , $|z_2| < 1$. The integrand is:$f(z) = \frac{2}{i(az^2+2z+a)}$. The residue in the pole z_2 is given by $\mathrm{Res}(f, z_2) = \lim_{z\to z_2} (z - z_2) \cdot$

$\frac{2}{ia(z-z_2)(z-z_1)} = \frac{2}{ia(z_2-z_1)}$. Taking $z_2 - z_1 = \frac{2\sqrt{1-a^2}}{a}$ into the residue, it is found that $\mathrm{Res}(f, z_1) = \frac{1}{i\sqrt{1-a^2}}$. By using the residue theorem, it is found that

$$2\pi i \cdot Res(f, z_1) = 2\pi i \cdot \frac{1}{i\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}} \quad (15)$$

The author also evaluates the integral

$$I = \int_0^{\pi} \frac{1}{a - b\cos\theta} d\theta, a > b > 0 \quad (16)$$

One can use residue theorem to transform to complex integral. The author expands the range of integral from 0 to 2π.

Let $z = e^{i\theta}$, so it has changed to $\cos\theta = \frac{z+z^{-1}}{2}$ and $d\theta = \frac{dz}{iz}$. The initial integral transforms to: $I = \frac{1}{2i}\int_{|z|=1} \frac{2z}{-bz^2+2az-b} dz$. The roots of $-bz^2 + 2az - b$ are $z_1 = \frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1}$ and $z_2 = \frac{a}{b} - \sqrt{\left(\frac{a}{b}\right)^2 - 1}$. z_2 is in this unit circle due to $a > b > 0$. The author calculates the residue in z_2: $\mathrm{Res}\left(\frac{2}{-b(z-z_1)(z-z_2)}, z_2\right) = -\frac{1}{b(z_2-z_1)}$. So $I = \frac{\pi}{\sqrt{a^2-b^2}}$.

The author calculates the $I = \oint_{|Z|=2} \frac{1}{(z^2+1)^2} dz$. People can divide the $(z^2+1)^2$ into $(z-i)^2(z+i)^2$. So, there are two double poles, $z = i$ and $z = -i$. When $z = i$, $\mathrm{Res}(f, i) = -\frac{2}{2i^3} = \frac{1}{4i}$. When $z = -i$, $\mathrm{Res}(f, -i) = -\frac{2}{(-2\{i\})^3} = -\frac{1}{4i}$. Thus, the author will calculate the sum of these two residues, and it is equal to zero.

The author will calculate integral $I = \int_0^{2\pi} \frac{\cos 2\theta d\theta}{5 - 4\cos\theta}$. Let $z = e^{i\theta}$, so $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z+z^{-1}}{2}$, $\cos 2\theta = \frac{z^2+z^{-2}}{2}$. The integration will transform to

$$I = \oint_{|z|=1} \frac{\frac{z^2+z^{-2}}{2}}{5 - 2(z+z^{-1})} \cdot \frac{dz}{iz}$$
$$= -\frac{1}{2i}\oint_{|z|=1} \frac{z^4+1}{z^2(2z-1)(z-2)} dz \quad (17)$$

There are two poles, respectively, z=0 and z =0.5. The author will respectively calculate the residue. When z=0, $\mathrm{Res}(f, 0) = -\frac{5}{8i}$. When z=0.5, $\mathrm{Res}(f, 0.5) = \frac{17}{24i}$. So, the total residue is $\frac{1}{12i}$. Then, the author will use the residue theorem to calculate the integration:

$$I = 2\pi i \cdot \frac{1}{12i} = \frac{\pi}{6} \quad (18)$$

The author will show the detailed process of this integration:

$$I = \int_{-\infty}^{+\infty} \frac{x\cos x}{x^2 - 7x + 10} dx \quad (19)$$

By dividing the denominator $x^2 - 7x + 10$ into $(x-2)(x-5)$, the initial formula can be expressed into practical fractions: $\frac{\frac{5}{3}}{x-5} - \frac{\frac{2}{3}}{x-2}$. The author transforms the initial integration into: $I = \frac{5}{3}\int_{-\infty}^{+\infty} \frac{\cos x}{x-5} - \frac{2}{3}\int_{-\infty}^{+\infty} \frac{\cos x}{x-2} dx$. Using complex integrals and residue theorem, it is found that $\mathrm{P.V.}\int_{-\infty}^{+\infty} \frac{\cos x}{x-a} dx = -\pi\sin a$. When $a = 5$ and $a = 2$, $\int_{-\infty}^{+\infty} \frac{\cos x}{x-5} dx = -\pi\sin 5$, $\int_{-\infty}^{+\infty} \frac{\cos x}{x-2} dx = -\pi\sin 2$. So,

$$I = \frac{5}{3}(-\pi\sin 5) - \frac{2}{3}(-\pi\sin 2)$$
$$= \frac{\pi}{3}(2\sin 2 - 5\sin 5). \quad (20)$$

4 CONCLUSIONS

All in all, the residue theorem can provide people another idea to exploit the path integral of an analytic function along a closed curve. It can help researcher to solve the complex integration with a higher efficiency. In this paper, the author wants to show more higher efficiency method to solve a huge variety of mathematics. To this end, the author also hopes that there would be more and more new and easier method instead the conventional method. In this paper, it nevertheless has some defects. For instance, the example is easier than the real-world problems, due to the fact that it only depends on theory to solve the troubles in the mathematics instead of adding real conditions. It may have a little difference with the real-world problems. In the future, the author wants to study more theory about the residue to provide more and more convenient method to solve these difficult integrals. In addition, the author also hopes to study some real-world problem to realize the ideas of combining theory with practice.

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