Optimizing Sensor Deployment Strategy for Tracking Mobile Heat Source Trajectory

Thanh Phong Tran[1][a], Laetitia Perez[2][b], Laurent Autrique[2][c], Edouard Leclercq[1][d], Syrine Bouazza[1][e] and Dimitri Lefevbre[1][f]

[1]*GREAH, Université Le Havre Normandie, 25 Street Philippe Lebon, Le Havre, France*

[2]*LARIS, University of Angers, 62 Av. Notre Dame du Lac, Angers, France*

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Abstract: Previous studies have investigated inverse problems in physical systems described by partial differential equations, particularly for identifying unknown parameters of mobile heat sources. An iterative minimization of a quadratic cost function, based on the conjugate gradient method, has shown reliable results in identifying heat densities and trajectories both offline and online. Although fixed sensor arrays can be effective, covering the full operating range of a moving heat source requires a large number of sensors, leading to inefficiencies and waste. A more efficient approach uses fewer mobile sensors mounted on autonomous robots. However, this introduces challenges in robot control, ensuring optimal positioning, coordination, and collision avoidance. To address this, we propose a method that combines sensitivity-based sensor placement with robot assignment algorithms such as the Hungarian Algorithm and Multi-Agent Path Finding. This enables effective tracking of the heat source's trajectory while optimizing sensor deployment. The approach not only increases overall sensitivity of the sensor network but also improves identification performance with reduced latency and higher accuracy.

NOMENCLATURE

c	specific heat capacity, $Jkg^{-1}K^{-1}$
$s^i(t)$	basis function for piecewise linear functions
$\overline{s}(t)$	vector of basis function $s^i(t)$
t, t_f	time and final time, s
t_{id}	identification time, s
t_d	delay time, s
$x(t), y(t)$	space variable, m
$\overline{x}(t), \overline{y}(t)$	vector related to space variable, m
σ_{res}	standard deviation of temperature residual, K
$\sigma_{\delta d}$	standard deviation of trajectory estimation, m
μ_{delay}	average delay on the identification, s
μ_{res}	average temperature residual, K
$\mu_{\delta d}$	average trajectory estimation errors, m
$\phi(t)$	heat flux density, Wm^{-2}
$\overline{\phi}(t)$	vector related to heat flux density, Wm^{-2}
η	precision of heat flux discontinuity
λ	thermal conductivity, $Wm^{-1}K^{-1}$
ρ	la masse volumique, kgm^{-3}
θ	temperature, K
h	heat transfer coefficient, $Jkg^{-1}K^{-1}$
e	plate thickness, m
l	plate dimension (width, length), m
$\overrightarrow{n}$	unit external outward-pointing vector
n	number of robot-sensors
N	number of time interval for identification
N_{vs}	number of virtual sensors
N_t	number of identification resolution
r	heat flux radius, m
τ	related to time discretization, s

[a] https://orcid.org/0000-0002-5596-8524
[b] https://orcid.org/0000-0001-6340-0317
[c] https://orcid.org/0000-0002-7611-4923
[d] https://orcid.org/0000-0003-2840-1378
[e] https://orcid.org/0000-0001-5040-3690
[f] https://orcid.org/0000-0001-7060-756X

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1 INTRODUCTION

In recent years, the study of partial differential equations (PDEs) has received increasing attention, supported by advances in computational tools and powerful computing systems. PDEs play a crucial role across a wide range of fields, from military applications (such as aerospace and national security) to science and daily life, including physics, biology, finance, and engineering. Researchers have increasingly focused on modeling complex problems and phenomena using systems of linear or nonlinear, single-order or higher-order PDEs, and on developing methods to solve them effectively (Hussein and Rusul, 2020). Moreover, the solution of inverse problems related to PDEs has become a growing area of interest, as it often requires sophisticated techniques, including both classical mathematical approaches and modern methods involving artificial intelligence and machine learning (Berg and Nyström, 2021; Aarset et al., 2023).

In the course of researching methods to identify the parameters of mobile heat sources, specifically heat density and movement trajectory, the authors have developed an identification approach that combines the Gradient Conjugate Method (GCM) with an iterative procedure. This approach is based on minimizing a cost function derived from the comparison between temperature data collected by thermal sensors and data generated from theoretical models. This inverse heat conduction problem-solving framework is well-known as ill-posed in the Hadamard sense, based on the GCM involves three key components: the direct problem, the adjoint problem, and the sensitivity problem (Fakih et al., 2024). The authors have successfully performed both offline and online identification of the heating flux or the trajectory, as well as simultaneous identification of both parameters for one or multiple heat sources, whether fixed or mobile. The identification algorithms employ iterative methods using sliding windows, either with fixed size or adaptively adjusted, in combination with future value prediction techniques.

The proposed thermal sensor configurations include both fixed and mobile sensors, the latter being deployed on autonomous robots. Notably, the selection and control of mobile sensors have been identified as critical factors that significantly influence the efficiency of the parameter identification process in terms of accuracy, computational cost, and response delay. Over the years, the authors have developed methods for determining optimal sensor location and selection strategies for mobile robots based on the sensitivity problem. The robot navigation strategies proposed so far are heuristic, in which robots prioritize tasks and move toward the nearest target location (Chakraa et al., 2023; Chakraa et al., 2025). However, the collision problem is not addressed in this paper but will be in future work.

This paper is structured into the following four parts. The first part will briefly present the research problem and the context of the physical system in which the mathematical modeling of the direct problem and the formulation of the inverse problem, dedicated to identifying the trajectory of a moving heat source will be presented. In the second part, the methodology of quasi-online identification using a method of selecting sensor positions based on sensitivity problem combined with strategy for deploying sensor network will be presented. The numerical results will be considered to discuss strategies for deploying the sensor network in Section 4. The last section will represent concluding remarks of this study.

2 MODELING AND INVERSE PROBLEM FORMULATION

2.1 Physical System Presentation

The heat conduction equation is a partial differential equation that describes the distribution of heat in a given object over time. Once this temperature distribution is known, the conductive heat flux at any point in the material or on its surface can be calculated using Fourier law. The general form of the 3D heat conduction equation describes how temperature varies in a three-dimensional space over time. The general equation is:

$$\rho C \frac{\partial \theta(x,y,z,t)}{\partial t} - \lambda \Delta \theta(x,y,z,t) = Q(x,y,z,t) \quad (1)$$

where $Q(x,y,z,t)$ is the internal heat generation per unit volume (W/m^3), and Δ is Laplace operation, see Eq. 4. This mathematical model of the diffusion equation in three-dimensional (Eq. 1) can be reduced into a similar two-dimensional pattern (Eq. 6) within certain limits that did not change the physical properties of heat transfer process (Tran, 2018a). From now on, the temperature as a function of space and time will be denoted by $\theta(x,y,t)$ and expressed in Kelvin (K).

In this study, a mobile heat source is modeled as a disk which moves along a trajectory S on a square aluminum plate with dimensions $\Omega = L \times L \times e \subset \mathbb{R}^3$, where L represents the side length and e the thickness. The boundary of this domain is denoted $\partial\Omega \subset \mathbb{R}^2$. Spatial coordinates within the reduced 2D domain are

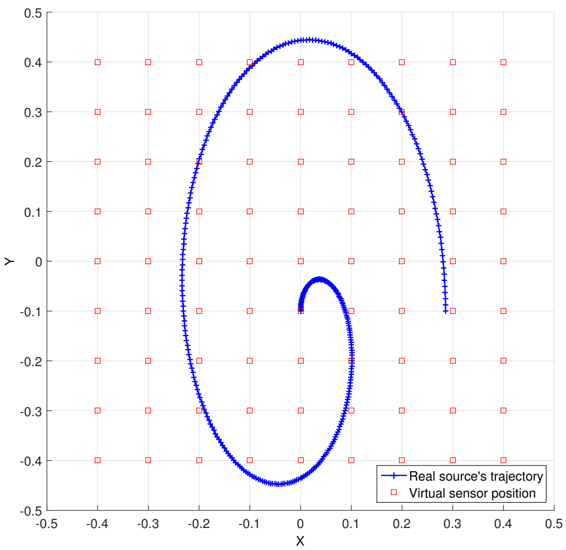

Figure 1: Heat source trajectory and mesh of sensors.

given by $(x,y) \subset]-L/2, L/2[$ in meters, while the temporal variable t belongs to the interval $T = [0, t_f]$, where t_f marks the final observation time, expressed in seconds.

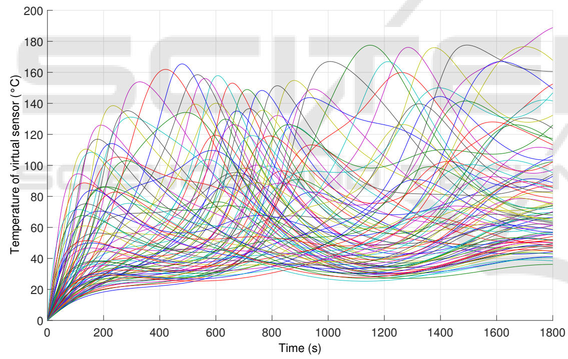

Figure 2: Time and spatial temperature evolution.

For this moving heat source, we apply a time-dependent heat density flux $\phi(t)$ (in Wm^{-2}) to the surface of the plate. This density flux is concentrated over a fixed, homogeneous circular area D of radius r, centered at the point $I(x_I(t), y_I(t))$(Tran, 2018b). Heat source total heating flux is defined by:

$$\Phi(x,y,t) = \begin{cases} \phi(t) & \text{if } (x,y) \in D(I(t), r) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and could be expressed continuously and differentiably as in Eq. 5 (see Table 2), where the parameter $\eta \in \mathbb{R}^+$ is related to the heat flux discontinuity at the disk boundary. Without loss of generality, the time interval $T = [0, t_f] = \bigcup_{d=0}^{N_t - 1} [t_d, t_{d+1}]$ is divided into N_t segments, with $t_d = \tau d$ and $\tau = t_f / N_t$. Thus, the coordinates of trajectory are discretized as $x(t) = \sum x^d s^d(t)$, $y(t) = \sum y^d s^d(t)$, and then $x^d = x(t_d)$, $y^d = y(t_d)$. The basis of hat functions for time discretization is $\forall d = 0, \ldots, N_{t-1}$:

$$s^d(t) = \begin{cases} 1 + t/\tau - d & \text{if } t \in [t_{d-1}, t_d] \\ 1 - t/\tau + d & \text{if } t \in [t_d, t_{d+1}] \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The temperature evolution of virtual sensors on the alunimium will be represented in Fig. 2. The numerical solution of system (Eq. 6) can be achieved by implementing the finite element method of Comsol Multiphysics interfaced with Matlab using parameters in Table 1.

Table 1: Value of input parameters.

Symbol, Definition	Value
Volumetric heat capacity, ρC	$2.421e6\ J/(m^3 K)$
Natural convection, h	$10\ \mathrm{W}/(m^2\mathrm{K})$
Thermal conductivity, λ	$237\ \mathrm{W}/(m^2\mathrm{K})$
Initial temperature, θ_0	$294.15K$
Final time, t_f	1800 s
Thickness, e	$2e{-}3$ m

2.2 Inverse Problem Formulation

In order to identify the trajectory $I(x_I(t), y_I(t))$ of the mobile heat source from the measured temperature changes using n sensors on the plate, an inverse problem can be formulated and solved by minimizing a quadratic criterion (Eq. 7).

An iterative conjugate gradient regularization method was implemented to identify unknown parameters (Tran et al., 2017; Vergnaud et al., 2015; Tran, 2018b; Fakih et al., 2024). The algorithm of this method consists of iteratively solving three well-posed problems in the Hadamard sense: direct problem, adjoint problem, and sensitivity problem.

- A direct problem (Eq. 6) gives the spatio-temporal evolution of temperature $\theta(x,y,t)$ to calculate the criterion $J(\theta, \Phi)$ (see Eq. 7), so that it helps to judge the quality of the estimates at iteration k by using this stop condition $J(\theta, \Phi) < J_{stop}$.
- An adjoint problem (Eq. 8) gives the spatio-temporal evolution of temperature adjoin function $\psi(x,y,t)$ to calculate the cost function gradients by unknown parameters $\overrightarrow{\nabla J}(\theta, x^k)$ and $\overrightarrow{\nabla J}(\theta, y^k)$ (see Eq. 9), then to define the descent direction of unknown parameters $\overrightarrow{d}_x^{k+1}$ and $\overrightarrow{d}_y^{k+1}$ (Eq. 10).
- A sensitivity problem (Eq. 11) gives the spatio-temporal evolution of variation of temperature $\delta\theta(x,y,t)$ to calculate the descent depth γ^{k+1} (Eq. 12) in the descent direction.

Table 2: Literature of models/equations for unknown parameter identification based on CGM.

As presented in (Beddiaf et al., 2012; Beddiaf et al., 2014; Fakih et al., 2024)

1. Laplace operator

$$\Delta\theta(x,y,t) = \frac{\partial^2\theta(x,y,t)}{\partial x^2} + \frac{\partial^2\theta(x,y,t)}{\partial y^2} + \frac{\partial^2\theta(x,y,t)}{\partial z^2} \tag{4}$$

2. Total heat flux

$$\Phi(x,y,t) = \frac{\phi(t)}{\pi} arccot\Big(\eta\sqrt{(x-x_I(t))^2+(y-y_I(t))^2} - r\Big) \tag{5}$$

As presented in (Tran et al., 2017; Tran, 2018b; Vergnaud et al., 2014; Vergnaud et al., 2015; Vergnaud et al., 2016; Vergnaud et al., 2020; Fakih et al., 2024)

3. Direct problem

$$\begin{cases} \rho C\dfrac{\partial\theta(x,y,t)}{\partial t} - \lambda\Delta\theta(x,y,t) = \dfrac{\Phi(x,y,t) - 2h(\theta(x,y,t)-\theta_0)}{e} & \text{on } \Omega\times T \\ \theta(x,y,0) = \theta_0(x,y) & \text{on } \Omega \\ -\lambda\dfrac{\partial\theta(x,y,t)}{\partial\vec{n}} = 0 & \text{on } \partial\Omega\times T \end{cases} \tag{6}$$

4. Cost function

$$J(\theta,\Phi) = \frac{1}{2}\int_T \sum_{n=1}^{N_c}\big(\theta(C_n,t,\Phi) - \hat{\theta}(C_n,t)\big)^2 dt \quad \text{at sensors} \quad C_n \tag{7}$$

5. Adjoint problem

$$\begin{cases} \rho C\dfrac{\partial\psi(x,y,t)}{\partial t} - \lambda\Delta\psi(x,y,t) = \mathrm{E}(x,y,t) + \dfrac{2h\psi(x,y,t)}{e} & \text{on } \Omega\times T \\ \psi(x,y,t_f) = 0 & \text{on } \Omega \\ -\lambda\dfrac{\partial\psi(x,y,t)}{\partial\vec{n}} = 0 & \text{on } \partial\Omega\times T \end{cases} \tag{8}$$

6. Gradient of unknown parameters (where $\omega(x,y,t) = \sqrt{(x-x_I(t))^2+(y-y_I(t))^2}$)

$$\overrightarrow{\nabla J}(\theta,x^k) = -\int_0^{t_f}\int_\Omega \frac{\mu\phi(t)}{\pi}\cdot\frac{(x-x_I(t))}{\omega(x,y,t)\,(1+\mu^2(\omega(x,y,t)-r)^2)}\cdot s_x^d(t)\cdot\frac{\psi(x,y,t)}{e}\,d\Omega\,dt \quad (a)$$

$$\overrightarrow{\nabla J}(\theta,y^k) = -\int_0^{t_f}\int_\Omega \frac{\mu\phi(t)}{\pi}\cdot\frac{(y-y_I(t))}{\omega(x,y,t)\,(1+\mu^2(\omega(x,y,t)-r)^2)}\cdot s_y^d(t)\cdot\frac{\psi(x,y,t)}{e}\,d\Omega\,dt \quad (b) \tag{9}$$

7. Descent direction

$$\overrightarrow{d}_x^{k+1} = -\overrightarrow{\nabla J}(\theta,x^k) + \frac{\left\|\overrightarrow{\nabla J}(\theta,x^k)\right\|^2}{\left\|\overrightarrow{\nabla J}(\theta,x^{k-1})\right\|^2}\overrightarrow{d}_x^k \text{ and } \overrightarrow{d}_y^{k+1} = -\overrightarrow{\nabla J}(\theta,y^k) + \frac{\left\|\overrightarrow{\nabla J}(\theta,y^k)\right\|^2}{\left\|\overrightarrow{\nabla J}(\theta,y^{k-1})\right\|^2}\overrightarrow{d}_y^k \tag{10}$$

8. Sensitivity problem

$$\begin{cases} \rho C\dfrac{\partial\delta\theta(x,y,t)}{\partial t} - \lambda\Delta\delta\theta(x,y,t) = \dfrac{\delta\Phi(x,y,t) - 2h\delta\theta(x,y,t)}{e} & \text{on } \Omega\times T \\ \delta\theta(x,y,0) = 0 & \text{on } \Omega \\ -\lambda\dfrac{\partial\delta\theta(x,y,t)}{\partial\vec{n}} = 0 & \text{on } \partial\Omega\times T \end{cases} \tag{11}$$

9. Descent depth

$$\gamma^{k+1} = \frac{\displaystyle\int_T\sum_{n=1}^{N_c}\Big(\theta(C_n,t,\overrightarrow{\Phi}^k) - \hat{\theta}(C_n,t)\Big)\delta\theta(C_n,t,\overrightarrow{\Phi}^k)dt}{\displaystyle\int_T\sum_{n=1}^{N_c}\delta\theta(C_n,t,\overrightarrow{\Phi}^k)^2 dt} \tag{12}$$

10. Updating new value

$$x^{k+1} = x^k - \gamma^{k+1}d_x^{k+1}, \text{ and } y^{k+1} = y^k - \gamma^{k+1}d_y^{k+1} \tag{13}$$

The key challenges in tracking mobile heat source are accurately determining the locations of the sensors to collect precise and sensitive temperature data, and developing an efficient strategy for moving the robot-sensors to ensure timely data acquisition. Algorithm 1 allows us to select the most relevant positions to move n robot-sensors $c_{i=1,2,\ldots,n}$ whose over the time interval $\tau_m = [\tau_m^-, \tau_m^+]$ by maximizing the Euclidean norm of temperature variation $L_i^2 = \left\|\delta\theta\left(c^i,t\right)\right\|$ (see, Eq. 14) over sliding time intervals τ_m (Tran, 2018b). This algorithm returns a list of goal positions G for the robot-sensors.

Algorithm 1: Method for selecting the sensor's next positions.

Data: initialisation n, L^2, $\tau_m \leftarrow [\tau_m^-, \tau_m^+]$
Result: calculate the most sensible positions;
solve sensitivity problem $\delta\theta(c_i,t)$
calculate Euclidean norm L_i^2

$$L_i^2 = \sqrt{\sum_{\tau_m} \left(\delta\theta\left(c_i,t\right)\right)^2} \; \forall i = 1,2,\ldots,N_{vs} \tag{14}$$

while $i < n$;
do
    choose a sensor c_i of the largest value of L_i^2;
    $G \leftarrow G + c_i$
    remove this sensor i in the list
    $i \leftarrow i+1$
    return G (List of goal positions).
end

Next, we study how to assign the computational positions to the n robot-sensors. To do this, we solve a Linear Assignment Problem (LAP) using the Hungarian algorithm (Chakraa et al., 2025; Rinaldi et al., 2024; Chopra et al., 2017; Ismail and Sun, 2017) which will be introduced in the next section. The task now is to find the optimal solution to deploy the robots (called the mobile sensor network) from their current positions in R to the goal positions in G.

3 DEPLOYING SENSOR NETWORK STRATEGY

3.1 Problem Statement

In multi-robot mobility systems, efficiently allocating the movement sequence of each robot in the group plays an important role in minimizing the total amount of resources used, such as energy consumption or travel time to the destination (Zhang et al., 2023; Luo et al., 2023; Smith and Jones, 2023; Doe and Roe, 2025). In this paper, we propose an algorithm to solve the problem of optimally assigning n mobile robots to n predefined target locations so that the total travel distance is minimized. Each set of current positions of the robots and the set of targets are represented as position coordinates in a 2D Cartesian plane. The cost of assigning a given robot to a target is defined as the *Euclidean* distance between their corresponding coordinates.

3.2 Mathematical Formulation of Assignment Algorithm

Let $R = \{r_1(x_1^r, y_1^r), r_2(x_2^r, y_2^r), \ldots, r_n(x_n^r, y_n^r)\}$ denote the set of robots; $G = \{g_1(x_1^g, y_1^g), g_2(x_2^g, y_2^g), \ldots, g_n(x_n^g, y_n^g)\}$ denote the set of goals. And, let d_{ij} be the cost (distance) for robot i to reach goal position j, defined as:

$$d_{ij} = \sqrt{(x_i^r - x_j^g)^2 + (y_i^r - y_j^g)^2} \tag{15}$$

The binary decision variable $b_{ij} \in \{0,1\}$ is defined by :

$$b_{ij} = \begin{cases} 1 & \text{if robot } i \text{ is assigned to goal } j \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

The applied objective of the LAP algorithm is to assign n robots to n goals in a way that minimizes the total assignment cost. The mathematical formulation of the LAP is given as follows:

$$f = \min \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} \cdot b_{ij} \tag{17}$$

The constraints presented in (18) guarantee a one-to-one correspondence between robots and goals, such that each robot is assigned to exactly one goal and vice versa.

$$\sum_{j=1}^{n} b_{ij} = 1, \text{ and } \sum_{i=1}^{n} b_{ij} = 1 \quad \forall i,j \in \{1,2,\ldots,n\} \tag{18}$$

In previous studies, we have successfully demonstrated the identification of the heating flux and the moving trajectory of the heat source with various sensor configurations (from 1 to 9 sensors and more). These studies have shown that the smaller the number of sensors (e.g. $n=1$), the less the observed data has been and the poor accuracy has been achieved. On the contrary, the larger the number of sensors (e.g. $n=9$ and more), the larger the collected data set will be, including unreliable noisy data, making the computation time important and even affecting the accuracy. Particularly for mobile sensor systems, the smaller the number of sensors, the easier it will be to manage and control. In this study, we illustrate the method with n = 6 robot-sensors as an example. Consider the set of robot-sensor current positions *R*={*(0.2,-0.1), (0.2,0), (0.3,-0.1), (0.3,0), (0.4,-0.1), (0.4,0)*} and the set of robot-sensor next positions *G* = {*(0.3,0.1), (0.2,0.2), (0.3,0.2), (0.2,0.1), (0.3,0.3), (0.1,0.2)*} (Fig. 3). The cost matrix is calculated using the Euclidean distance (Eq. 15) and gives the following result:

$$D = \begin{bmatrix} 0.224 & 0.300 & 0.316 & 0.200 & 0.412 & 0.316 \\ 0.141 & 0.200 & 0.224 & 0.100 & 0.316 & 0.224 \\ 0.200 & 0.316 & 0.300 & 0.224 & 0.400 & 0.361 \\ 0.100 & 0.224 & 0.200 & 0.141 & 0.300 & 0.283 \\ 0.224 & 0.361 & 0.316 & 0.283 & 0.412 & 0.424 \\ 0.141 & 0.283 & 0.224 & 0.224 & 0.316 & 0.361 \end{bmatrix}$$

The optimal assignment is obtained by solving the above LAP using the Hungarian algorithm, also known as the Kuhn-Munkres algorithm. This algorithm efficiently finds a minimum-cost (Eq. 17) perfect matching in a weighted bipartite graph and runs in $O(n^3)$ time (Giordani et al., 2010). It ensures optimal solutions by continuously improving feasible labels and enhancing paths.

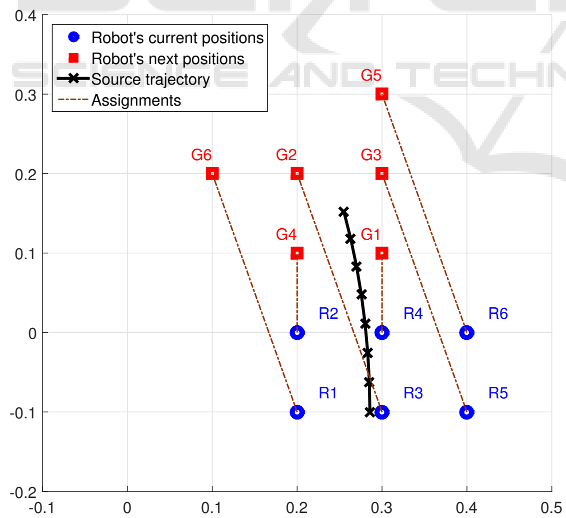

Figure 3: Presentation of LAP using the Hungarian algorithm.

This provides an optimal solution for assigning robot-sensor next positions to robots in scenarios where the costs are additive and independent. In Fig. 3, Robot 1 moves to position 6, and respectively Robot 2 moves to position 4, Robot 3 moves to position 2, Robot 4 moves to position 1, Robot 5 moves to position 3, and Robot 6 moves to position 5. The total calculated cost is 1.46. In this study, it is assumed that the robot-sensors move at the same speed and have the same accuracy. The above results show that the robot-sensors have different travel distances, so the travel time to the required locations is also different. Therefore, the priority order is that the robots that need to travel far will start first.

Accordingly, the problem of trajectory collisions needs to be considered in practice in order to ensure that multiple agents can avoid one another. Future work will focus on addressing this issue. This paper is limited to calculating the optimal travel distance.

3.3 Sensor Assignment Problem

A strategy for deploying sensor network using the LAP was proposed and applied to control the movement of *n* robot-sensors to measure the temperature in order to identify the moving trajectory of the mobile heat source. This deployment strategy is introduced in Algorithm 2.

Algorithm 2: Deploying sensor network method for CGM.

Data: initialisation: R, G, $\tau_m = [\tau_m^-, \tau_m^+]$
Result: estimate the trajectory $\hat{x}_I(t), \hat{y}_I(t)$
while $J(\theta, \Phi) < J_{stop}$;
do
  apply Algorithm 1:
    $G \leftarrow G + c_i$;
  calculate Euclidean norm of cost matrix:
    $D \leftarrow d_{ij}$;
  apply algorithm LAP:
    $G_j \leftarrow R_i$;
  deploy robot-sensors from set of R to G ;
  collect temperature data: $\hat{\theta}(x,y,t)$;
  apply identification based on CGM:
    $(\hat{x}_I(t), \hat{y}_I(t))$;
  load next time interval: $\tau_m \leftarrow \tau_{m+1}$;
  return $\hat{x}_I(t), \hat{y}_I(t)$
end

The next positions of the robot-sensors are calculated by Algorithm 1. Next, the cost matrix is calculated by combining the distances between the robots of R and the next positions G based on the Euclidean norm and applying the LAP algorithm to find the optimal positions and assign each robot-sensor R_i to the best next position G_j such that the total cost is minimized. When the robots reach to positions G, they will collect temperature data $\hat{\theta}(x,y,t)$ and send them to the CGM algorithm to estimate current positions of the heat source. Finally, the algorithm will repeat the determination of the robot position for the next time interval.

4 NUMERICAL RESULTS AND DISCUSSION

This section presents numerical results to evaluate proposed algorithms (for selecting the sensor positions using the sensitivity problem, and for deploying the sensor network using LAP) and method of identification based on CGM for determining the unknown trajectory of a single moving heat source. The objective is to minimize the output error by accurately estimating the trajectory of the heat source. The data set of temperature within the domain is a solution of the system of PDEs and is thus the acquisition of measurements by sensors. The temperatures are noisy and distributed by a normal distribution $N(\mu,\sigma)$ to reflect actual measurement conditions.

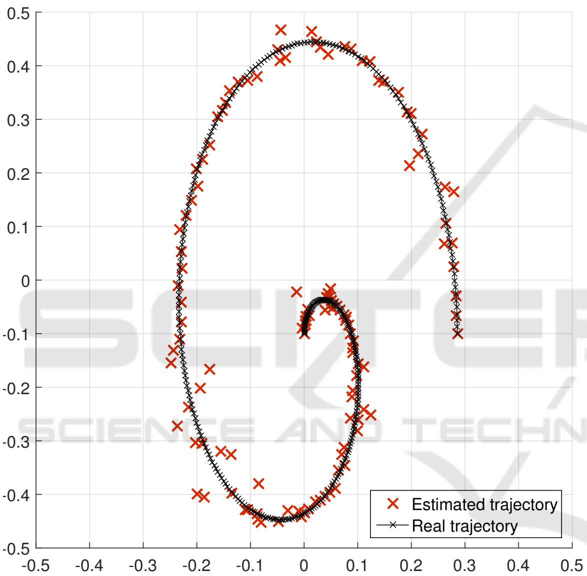

Figure 4: Presentation of estimated trajectory using the Gradient Conjugate Method.

The numerical results which were obtained by implementing the finite element method of Comsol Multiphysics 3.5 interfaced with Matlab R2012b, were performed on a personal computer with the following configuration: CPU Intel® Core™ i7-3520M CPU @ 2.90GHz, RAM 8.00 GB, OS Windows 11 (64-bit). These results demonstrate the implementation of the CGM in solving the inverse problem of determining the trajectory of a moving heat source $I(x_I(t),y_I(t))$, according to the defined stopping criterion. Accordingly, the trajectory of the studied source is determined after an identification time $t_{id} = 1,887s$. The numerical experimental time is 30 minutes. The final estimate of the trajectory of the heat source $\hat{I}(\hat{x}_I(t),\hat{y}_I(t))$ is shown in Figure 4.

In order to estimate the trajectory identification quality, we calculate the average of the temperature residual μ_{res}, the standard deviation of temperature residual σ_{res}, the average of the trajectory error $\mu_{\delta d}$, the standard deviation of the trajectory error $\sigma_{\delta d}$ and the maximum $max_{\delta d}$ of errors between estimated and real heat source trajectory considering a Gaussian noise $N(0,1)$ with mean $\mu = 0$ and standard deviation $\sigma = 1$ on measured temperature.

The error of the trajectory estimation is calculated using Eq. 19:

$$\delta d = \sqrt{\left(x_I(t)-\hat{x}_I(t)\right)^2 + \left(y_I(t)-\hat{y}_I(t)\right)^2} \tag{19}$$

The position errors of the identified trajectory compared to the real trajectory of the moving heat source are calculated and shown as Figure 5. Accordingly, the largest position error value is $max_{\delta d} = 3.767 \times 10^{-2}m$.

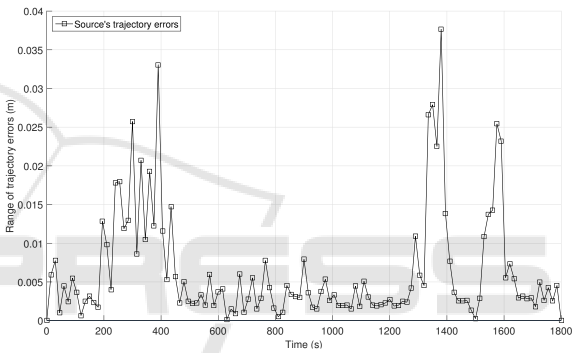

Figure 5: Presentation of trajectory estimation errors

As observed in Figure 5, the trajectory error exhibits notable peaks, particularly around the time points of 400 s and 1400 s. These fluctuations may be attributed to disturbances in the sensor network, especially when the next positions of the sensors are not accurately determined. These issues should be further investigated and addressed in future studies.

The average and the standard deviation of temperature residual are determined by:

$$\mu_{res} = \frac{1}{n}\sum_{i=1}^{n}\left(\theta(x,y,t,I)-\hat{\theta}(x,y,t)\right) \tag{20}$$

$$\sigma_{res} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\theta(x,y,t,I)-\hat{\theta}(x,y,t)\right)^2} \tag{21}$$

The results are the mean residual temperature $\mu_{res} = -0.252K$ and the standard deviation $\sigma_{res} = 0.763K$. Meanwhile, the noise on measured temperature has an average $\mu = 0$ and a standard deviation $\sigma = 1$, indicating that the proposed method is reliable and robust for estimating heat source trajectory. Furthermore, the statistical results show that the average trajectory error is $\mu_{\delta d} = 5.263 \times 10^{-5}m$

and the standard deviation of the trajectory error is $\sigma_{\delta d} = 5.964 \times 10^{-5}m$. The trajectory error clearly varies over time, as shown in Figures 4 and 5. Overall, the small average error demonstrates the reliability of the proposed methods.

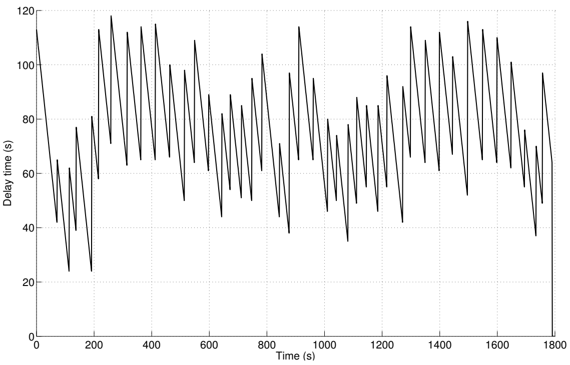

Figure 6: Presentation of estimation delay.

The main drawback of this method is the relatively long convergence time for online recognition. It becomes very important according to the complexity of the problem (and can also be affected by the number of parameters to be recognized, the number of heat sources, the number of robots in the sensor network,...).

The delay is defined as the time to obtain the value of the unknown estimated parameter from the end of the time interval of the sliding window. If the end time of the identification is t_3 for the time interval $\tau_m = [t_1, t_2]$ with $t_2 > t_1$, the delay of this process will be calculate by:

$$t_d = t_3 - t_2. \tag{22}$$

The delay of the moving heat source trajectory estimation is shown in Figure 6. Accordingly, the smallest delay time is $24s$ and the largest is $118s$. Thus, the average delay time of the entire heat source trajectory identification process is about $75s$. These results suggest that the proposed methods meet the requirements for tracking mobile heat sources. It has the potential for a quasi-online identification process.

5 CONCLUSION

This study proposes an efficient method to identify the trajectory of a single mobile heat source using the conjugate gradient method combined with an optimal deployment of mobile heat sensors. A sensitivity-based approach guided the sensor placement, and the Hungarian algorithm was used to assign the next position of robot-sensors to measurement locations with minimal travel cost. Numerical simulations demonstrate the effectiveness of the method. Using six mobile sensors with data collected every $15s$, the CGM successfully reconstructed the heat source trajectory with an average identification delay between $24 - 118s$. The identification was robust to noise, yielding an average residual temperature of $0.252K$ and a standard deviation residual temperature of $0.763K$. Although the method provides high accuracy, it has limitations in terms of computational time, especially for online or multiple-source scenarios. Furthermore, robot coordination assumes ideal motion without addressing real-world constraints such as collision avoidance or communication latency. Future research will aim to improve computational efficiency, integrate advanced motion planning, and validate the approach through real-world experiments with physical mobile robots in uncertain environments.

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