Δ-Y Transformations in Manipulator's Stiffness Analysis

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Abstract: The paper proposes a Δ-Y transformations technique for stiffness modelling of over-constrained manipulators with internal cross-linkages. It allows representing complex structures as a serial-parallel equivalent one that can be easily handled by the VJM-based method. To derive desired analytical expressions for the equivalent serial-parallel structure, the MSA-based stiffness modelling approach is employed first, which allows describing the stiffness response for both the Δ and Y structures operating with VJM-type stiffness matrices. Further, the desired relations between equivalent Δ-Y and Y-Δ stiffness matrices are obtained. The example of stiffness modelling of a non-rigid Gough-Stewart platform with multiple cross-linkages demonstrates the benefits of the proposed technique.

1 INTRODUCTION

Stiffness modelling is a hot topic in robotics, essential both for the robot manipulation accuracy improvement and human-robot collaboration enhancement (Wu *et al.*, 2022, Hussain *et al.*, 2021, Yue *et al.*, 2022, Blumberg *et al.*, 2021). It enables the estimation of mechanical deflections in the manipulator components, resulting in slight changes to the actual configuration. Based on the computed deflections, the related compliance error compensation techniques help to reduce the impact of the external forces on the manipulator's end-effector and improve the end-effector accuracy (Nguyen *et al.*, 2022, Gonzalez *et al.*, 2022, Kim & Min, 2020, Klimchik, Pashkevich, *et al.*, 2013, Kim, 2023). Currently, because of practical advantages, the most commonly used stiffness modelling approaches in robotics are Virtual Joint Modelling (VJM) and Matrix Structural Analysis (MSA) (Gosselin & Zhang, 2002, Pashkevich *et al.*, 2009, Majou *et al.*, 2007, Quennouelle & Gosselin, 2008, Deblaise *et al.*, 2006, Klimchik, Pashkevich, *et al.*, 2019). They are relatively simple from the computational point of view but require substantial efforts for related stiffness model development and estimation of its parameters. The modelling accuracy for both VJM and MSA methods can be enhanced by relying on the CAD-based FEA identification technique (Klimchik *et al.*, 2024). Considering mathematical fundamentals, the VJM is efficient for stiffness modelling of pure serial-parallel structures, which can be decomposed into equivalent serial ones (Görgülü *et al.*, 2020, Hu *et al.*, 2019). In contrast, the MSA struggles with serial structures but can handle complex cross-linkages (Deblaise *et al.*, 2006, Klimchik, Chablat, *et al.*, 2019, Soares Júnior *et al.*, 2015, Detert & Corves, 2017, Klimchik *et al.*, 2018). It was proved that the VJM is the best approach for non-linear stiffness analysis (Zhao *et al.*, 2022, Pashkevich *et al.*, 2011). For these reasons, integrating cross-linkages in the VJM is a crucial problem.

There were some attempts to integrate closed loops into VJM methods (Klimchik, Wu, *et al.*, 2013, Klimchik *et al.*, 2017). But they are not capable of handling cross-linkages. To overcome this problem, we propose a Δ-Y stiffness model transformation

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technique that allows us to present cross-linkages as an equivalent serial-parallel mechanical structure while preserving the original mechanical properties of the system.

2 STIFFNESS MODELLING VIA Δ-Y TRANSFORMATIONS

2.1 Stiffness Model of an Elastic Link

VJM represents each elastic component as a superposition of a rigid element (between the nodes $u \to v$), which describes the geometry of the perfect component, and an elastic component at the right end of the link (node *v*), which represents the mechanical flexibility of the corresponding body, as shown in 0, where the node *u* is fixed to the base or previous component. This model is mathematically expressed as a linear matrix equation.:

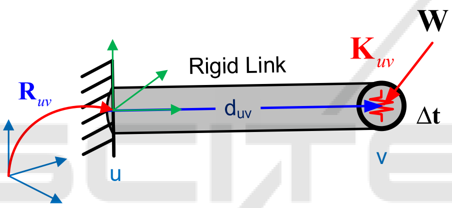

Figure 1: VJM-based stiffness model of a flexible link.

$$\mathbf{W} = \mathbf{K}_{uv} \cdot \Delta\mathbf{t} \tag{1}$$

relating the 6-dimensional wrench $\mathbf{W}$ consisting of three force components and three moment components applied to node *v* and the corresponding displacement $\Delta\mathbf{t}$ is a 6-dimensional vector consisting of three linear displacements and three angular displacements. Here, 6×6 stiffness matrix $\mathbf{K}_{uv}$ must be expressed in the global coordinate system, while the VJM usually operates with the stiffness matrix $\mathbf{K}_\theta$ obtained in the local coordinate system. The latter demands a relevant transformation $\mathbf{K}_\theta \to \mathbf{K}_{uv}$

$$\mathbf{K}_{uv} = \begin{bmatrix} \mathbf{R}_{uv} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{R}_{uv} \end{bmatrix}_{6\times6} \cdot \mathbf{K}_\theta \cdot \begin{bmatrix} \mathbf{R}_{uv} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{R}_{uv} \end{bmatrix}_{6\times6}^{T} \tag{2}$$

depending on the 3×3 rotation matrix $\mathbf{R}_{uv}$ which defines the link *uv* orientation with respect to the global coordinate system. It should be noted that in classical VJM, the transformation (2) is incorporated in the manipulator Jacobian, but it should be applied straightforwardly here.

Let us also present an alternative **MSA-based model** describing the elastic member composed of the rigid link and virtual spring, assuming that both ends of the link *u, v* are not fixed (see 0). Generally, such a model is represented in the form of a matrix equation as

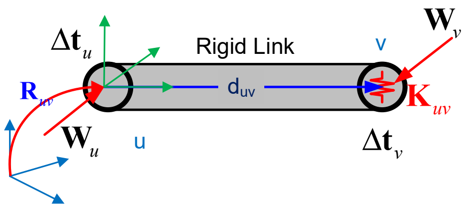

Figure 2: MSA-based stiffness model of a flexible link.

$$\begin{bmatrix} \mathbf{W}_u \\ \mathbf{W}_v \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}_{12\times12} \cdot \begin{bmatrix} \Delta\mathbf{t}_u \\ \Delta\mathbf{t}_v \end{bmatrix} \tag{3}$$

relating the 6-dimensional wrenches $(\mathbf{W}_u, \mathbf{W}_v)$ applied to the nodes *u*, *v* and the corresponding displacements $(\Delta\mathbf{t}_u, \Delta\mathbf{t}_v)$. It is clear that for the considered physical model (rigid link + virtual spring), the sub-matrices $\mathbf{K}_{11}, \mathbf{K}_{12}, \mathbf{K}_{21}, \mathbf{K}_{22}$ can be expressed via the spring stiffness matrix $\mathbf{K}_{uv}$ and link geometry vector $\mathbf{d}_{uv}$. Corresponding derivations are presented in (Klimchik, Pashkevich, *et al.*, 2019) and yield the following expression with a symmetrical matrix of the size 12×12

$$\begin{bmatrix} \mathbf{W}_u \\ \mathbf{W}_v \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{uv}^{-T}\mathbf{K}_{uv}\mathbf{D}_{uv}^{-1} & -\mathbf{D}_{uv}^{-T}\mathbf{K}_{uv} \\ -\mathbf{K}_{uv}\mathbf{D}_{uv}^{-1} & \mathbf{K}_{uv} \end{bmatrix}_{12\times12} \cdot \begin{bmatrix} \Delta\mathbf{t}_u \\ \Delta\mathbf{t}_v \end{bmatrix} \tag{4}$$

It includes 6×6 geometric transformation matrix (Klimchik *et al.*, 2024)

$$\mathbf{D}_{uv} = \begin{bmatrix} \mathbf{I}_{3\times3} & -(\mathbf{d}_{uv}\times) \\ \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} \end{bmatrix}_{6\times6} \tag{5}$$

defining translation from the node *u* to the node *v* expressed in the global coordinate system, which includes a 3×3 skew-symmetric matrix $(\mathbf{d}_{uv}\times)$ derived from the vector $\mathbf{d}_{uv}$ $(u \to v)$ in the following way

$$(\mathbf{d}\times) = \begin{bmatrix} 0 & -d_z & d_y \\ d_z & 0 & -d_x \\ -d_y & d_x & 0 \end{bmatrix}_{3\times3}; \quad \mathbf{d} = \begin{pmatrix} -d_x \\ -d_y \\ -d_z \end{pmatrix}_{3\times1}, \tag{6}$$

as well as 3×3 identity and zero matrices $\mathbf{I}_{3\times3}$, $\mathbf{0}_{3\times3}$. It has been proven that the matrix $\mathbf{D}_{uv}$ inversion leads to a simple change of the vector $\mathbf{d}_{uv}$ direction

$$\mathbf{D}_{uv}^{-1}=\begin{bmatrix}\mathbf{I}_{3\times3} & (\mathbf{d}_{uv}\times)\\ \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3}\end{bmatrix} \tag{7}$$

that yield the following properties

$$\mathbf{D}_{vu}=\mathbf{D}_{uv}^{-1} \tag{8}$$

Similar properties are observed in the transformed matrices

$$\begin{aligned}\mathbf{D}_{uv}^{T}&=\begin{bmatrix}\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3}\\ (\mathbf{d}_{uv}\times) & \mathbf{I}_{3\times3}\end{bmatrix}\\ \mathbf{D}_{uv}^{-T}&=\begin{bmatrix}\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3}\\ -(\mathbf{d}_{uv}\times) & \mathbf{I}_{3\times3}\end{bmatrix}\end{aligned} \tag{9}$$

Based on these properties, the following important matrix multiplication rules were derived

$$\mathbf{D}_{ij}^{-1}\mathbf{D}_{ik}=\mathbf{D}_{kj}^{-1};\qquad \mathbf{D}_{ik}^{T}\mathbf{D}_{ij}^{-T}=\mathbf{D}_{kj}^{-T} \tag{10}$$

which are convenient for the mathematical derivations presented below. It is also worth mentioning that in eq. (4) the rank of 12×12 matrix is equal to 6, which is in good agreement with the physical properties of link representations. In fact, the lines of this block matrix are linearly dependent and satisfy an obvious relation

$$\mathbf{W}_u+\mathbf{D}_{uv}^{-T}\mathbf{W}_v=\mathbf{0}_{6\times1} \tag{11}$$

that in the adopted notation expresses the static equilibrium condition, resulting in a rank deficiency of 6. In the following subsection, the obtained model will be used to derive stiffness models of complex structures.

2.2 Stiffness Models of Δ-Structures

Using the elastic link stiffness model (4) let us derive the stiffness models for Δ- and Y-structures, which are presented in 0. Each of them consists of three elastic components connected either at the corners or a single central node.

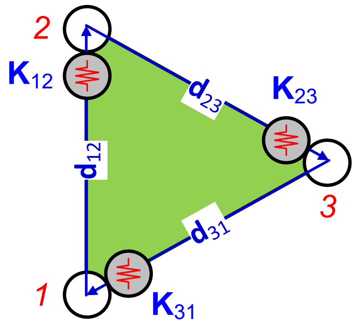

Figure 3: VJM-based stiffness models for Δ-structure and their parameters.

For the **Δ-structure**, the stiffness model of the separate elastic links (1,2) (2,3) (3,1) can be written as follows:

$$\begin{bmatrix}\mathbf{W}_1^{(12)}\\ \mathbf{W}_2^{(12)}\end{bmatrix}=\begin{bmatrix}\mathbf{D}_{12}^{-T}\mathbf{K}_{12}\mathbf{D}_{12}^{-1} & -\mathbf{D}_{12}^{-T}\mathbf{K}_{12}\\ -\mathbf{K}_{12}\mathbf{D}_{12}^{-1} & \mathbf{K}_{12}\end{bmatrix}\cdot\begin{bmatrix}\Delta\mathbf{t}_1^{(12)}\\ \Delta\mathbf{t}_2^{(12)}\end{bmatrix} \tag{12}$$

$$\begin{bmatrix}\mathbf{W}_2^{(23)}\\ \mathbf{W}_3^{(23)}\end{bmatrix}=\begin{bmatrix}\mathbf{D}_{23}^{-T}\mathbf{K}_{23}\mathbf{D}_{23}^{-1} & -\mathbf{D}_{23}^{-T}\mathbf{K}_{23}\\ -\mathbf{K}_{23}\mathbf{D}_{23}^{-1} & \mathbf{K}_{23}\end{bmatrix}\cdot\begin{bmatrix}\Delta\mathbf{t}_2^{(23)}\\ \Delta\mathbf{t}_3^{(23)}\end{bmatrix} \tag{13}$$

$$\begin{bmatrix}\mathbf{W}_3^{(31)}\\ \mathbf{W}_1^{(31)}\end{bmatrix}=\begin{bmatrix}\mathbf{D}_{31}^{-T}\mathbf{K}_{31}\mathbf{D}_{31}^{-1} & -\mathbf{D}_{31}^{-T}\mathbf{K}_{31}\\ -\mathbf{K}_{31}\mathbf{D}_{31}^{-1} & \mathbf{K}_{31}\end{bmatrix}\cdot\begin{bmatrix}\Delta\mathbf{t}_3^{(31)}\\ \Delta\mathbf{t}_1^{(31)}\end{bmatrix} \tag{14}$$

Further, taking into account that total wrenches $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ are expressed as

$$\begin{aligned}\mathbf{W}_1&=\mathbf{W}_1^{(12)}+\mathbf{W}_1^{(31)}\\ \mathbf{W}_2&=\mathbf{W}_2^{(12)}+\mathbf{W}_2^{(23)}\\ \mathbf{W}_3&=\mathbf{W}_3^{(31)}+\mathbf{W}_3^{(23)}\end{aligned} \tag{15}$$

and the node displacements satisfy the following constraints

$$\begin{aligned}\Delta\mathbf{t}_1^{(12)}&=\Delta\mathbf{t}_1^{(31)}\\ \Delta\mathbf{t}_2^{(12)}&=\Delta\mathbf{t}_2^{(23)}\\ \Delta\mathbf{t}_3^{(31)}&=\Delta\mathbf{t}_3^{(23)}\end{aligned} \tag{16}$$

the desired stiffness model can be re-written in the form of a single matrix equation

$$\begin{bmatrix}\mathbf{W}_1\\ \mathbf{W}_2\\ \mathbf{W}_3\end{bmatrix}=\begin{bmatrix}\mathbf{K}_{11}^{(\Delta)} & \mathbf{K}_{12}^{(\Delta)} & \mathbf{K}_{13}^{(\Delta)}\\ \mathbf{K}_{21}^{(\Delta)} & \mathbf{K}_{22}^{(\Delta)} & \mathbf{K}_{23}^{(\Delta)}\\ \mathbf{K}_{31}^{(\Delta)} & \mathbf{K}_{32}^{(\Delta)} & \mathbf{K}_{33}^{(\Delta)}\end{bmatrix}_{18\times18}\cdot\begin{bmatrix}\Delta\mathbf{t}_1\\ \Delta\mathbf{t}_2\\ \Delta\mathbf{t}_3\end{bmatrix} \tag{17}$$

where

$$
\begin{aligned}
\mathbf{K}_{11}^{(\Delta)} &= \mathbf{D}_{12}^{-T}\mathbf{K}_{12}\mathbf{D}_{12}^{-1} + \mathbf{K}_{31} \\
\mathbf{K}_{12}^{(\Delta)} &= -\mathbf{D}_{12}^{-T}\mathbf{K}_{12} \\
\mathbf{K}_{13}^{(\Delta)} &= -\mathbf{K}_{31}\mathbf{D}_{31}^{-1} \\
\mathbf{K}_{21}^{(\Delta)} &= -\mathbf{K}_{12}\mathbf{D}_{12}^{-1} \\
\mathbf{K}_{22}^{(\Delta)} &= \mathbf{D}_{23}^{-T}\mathbf{K}_{23}\mathbf{D}_{23}^{-1} + \mathbf{K}_{12} \\
\mathbf{K}_{23}^{(\Delta)} &= -\mathbf{D}_{23}^{-T}\mathbf{K}_{23} \\
\mathbf{K}_{31}^{(\Delta)} &= -\mathbf{D}_{31}^{-T}\mathbf{K}_{31} \\
\mathbf{K}_{32}^{(\Delta)} &= -\mathbf{K}_{23}\mathbf{D}_{23}^{-1} \\
\mathbf{K}_{33}^{(\Delta)} &= \mathbf{D}_{31}^{-T}\mathbf{K}_{31}\mathbf{D}_{31}^{-1} + \mathbf{K}_{23}
\end{aligned}
\tag{18}
$$

It can be proved that the rank of 18×18 matrix is equal to 12, which agrees with the physical properties of the considered Δ-structure. In fact, the lines of this block matrix are linearly dependent and satisfy an obvious relation

$$
\mathbf{W}_1 + \mathbf{D}_{12}^{-T}\mathbf{W}_2 + \mathbf{D}_{31}^{T}\mathbf{W}_3 = \mathbf{0}_{6\times1} \tag{19}
$$

that in the adopted notation expresses the static equilibrium condition, resulting in a rank deficiency of 6.

2.3 Stiffness Models of Y-Structures

For the **Y-structure**, presented in 0, the stiffness model of the separate elastic links (1,0) (2,0) (3,0) can be written as follows

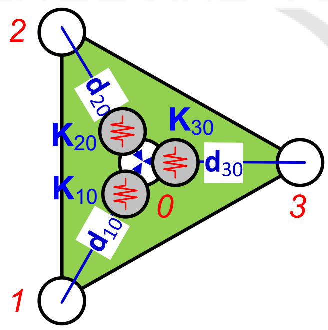

Figure 4: VJM-based stiffness models for Y-structure and their parameters.

$$
\begin{bmatrix} \mathbf{W}_1^{(10)} \\ \mathbf{W}_0^{(10)} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{10}^{-T}\mathbf{K}_{10}\mathbf{D}_{10}^{-1} & -\mathbf{D}_{10}^{-T}\mathbf{K}_{10} \\ -\mathbf{K}_{10}\mathbf{D}_{10}^{-1} & \mathbf{K}_{10} \end{bmatrix} \cdot \begin{bmatrix} \Delta\mathbf{t}_1^{(10)} \\ \Delta\mathbf{t}_0^{(10)} \end{bmatrix} \tag{20}
$$

$$
\begin{bmatrix} \mathbf{W}_2^{(20)} \\ \mathbf{W}_0^{(20)} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{20}^{-T}\mathbf{K}_{20}\mathbf{D}_{20}^{-1} & -\mathbf{D}_{20}^{-T}\mathbf{K}_{20} \\ -\mathbf{K}_{20}\mathbf{D}_{20}^{-1} & \mathbf{K}_{20} \end{bmatrix} \cdot \begin{bmatrix} \Delta\mathbf{t}_2^{(20)} \\ \Delta\mathbf{t}_0^{(20)} \end{bmatrix} \tag{21}
$$

$$
\begin{bmatrix} \mathbf{W}_3^{(30)} \\ \mathbf{W}_0^{(30)} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{30}^{-T}\mathbf{K}_{30}\mathbf{D}_{30}^{-1} & -\mathbf{D}_{30}^{-T}\mathbf{K}_{30} \\ -\mathbf{K}_{30}\mathbf{D}_{30}^{-1} & \mathbf{K}_{30} \end{bmatrix} \cdot \begin{bmatrix} \Delta\mathbf{t}_3^{(30)} \\ \Delta\mathbf{t}_0^{(30)} \end{bmatrix} \tag{22}
$$

Further, taking into account that total wrenches $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ are expressed as

$$
\begin{aligned}
\mathbf{W}_0 &= \mathbf{W}_0^{(10)} + \mathbf{W}_0^{(20)} + \mathbf{W}_0^{(30)} \\
\mathbf{W}_1 &= \mathbf{W}_1^{(10)} \\
\mathbf{W}_2 &= \mathbf{W}_2^{(20)} \\
\mathbf{W}_3 &= \mathbf{W}_3^{(30)}
\end{aligned}
\tag{23}
$$

and the node displacements satisfy the following constraints

$$
\Delta\mathbf{t}_0^{(10)} = \Delta\mathbf{t}_0^{(20)} = \Delta\mathbf{t}_0^{(30)} \tag{24}
$$

the desired stiffness model can be rewritten in the form of a single matrix equation

$$
\begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \mathbf{W}_3 \\ \mathbf{W}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11}^{(Y0)} & \mathbf{0}_{6\times6} & \mathbf{0}_{6\times6} & \mathbf{K}_{14}^{(Y0)} \\ \mathbf{0}_{6\times6} & \mathbf{K}_{22}^{(Y0)} & \mathbf{0}_{6\times6} & \mathbf{K}_{24}^{(Y0)} \\ \mathbf{0}_{6\times6} & \mathbf{0}_{6\times6} & \mathbf{K}_{33}^{(Y0)} & \mathbf{K}_{34}^{(Y0)} \\ \mathbf{K}_{41}^{(Y0)} & \mathbf{K}_{42}^{(Y0)} & \mathbf{K}_{43}^{(Y0)} & \mathbf{K}_{44}^{(Y0)} \end{bmatrix} \cdot \begin{bmatrix} \Delta\mathbf{t}_1 \\ \Delta\mathbf{t}_2 \\ \Delta\mathbf{t}_3 \\ \Delta\mathbf{t}_0 \end{bmatrix} \tag{25}
$$

where

$$
\begin{aligned}
\mathbf{K}_{11}^{(Y0)} &= \mathbf{D}_{10}^{-T}\mathbf{K}_{10}\mathbf{D}_{10}^{-1} \\
\mathbf{K}_{14}^{(Y0)} &= -\mathbf{D}_{10}^{-T}\mathbf{K}_{10} \\
\mathbf{K}_{22}^{(Y0)} &= \mathbf{D}_{20}^{-T}\mathbf{K}_{20}\mathbf{D}_{20}^{-1} \\
\mathbf{K}_{24}^{(Y0)} &= -\mathbf{D}_{20}^{-T}\mathbf{K}_{20} \\
\mathbf{K}_{33}^{(Y0)} &= \mathbf{D}_{30}^{-T}\mathbf{K}_{30}\mathbf{D}_{30}^{-1} \\
\mathbf{K}_{34}^{(Y0)} &= -\mathbf{D}_{30}^{-T}\mathbf{K}_{30} \\
\mathbf{K}_{41}^{(Y0)} &= -\mathbf{K}_{10}\mathbf{D}_{10}^{-1} \\
\mathbf{K}_{42}^{(Y0)} &= -\mathbf{K}_{20}\mathbf{D}_{20}^{-1} \\
\mathbf{K}_{43}^{(Y0)} &= -\mathbf{K}_{30}\mathbf{D}_{30}^{-1} \\
\mathbf{K}_{44}^{(Y0)} &= \mathbf{K}_{10} + \mathbf{K}_{20} + \mathbf{K}_{30}
\end{aligned}
\tag{26}
$$

It can be proved that the rank of 24×24 matrix is equal to 18, which agrees with the physical properties of the considered Y-structure. In fact, the lines of this block matrix are linearly dependent and satisfy an obvious relation

$$
\mathbf{W}_0 + \mathbf{D}_{10}^{T}\mathbf{W}_1 + \mathbf{D}_{20}^{T}\mathbf{W}_2 + \mathbf{D}_{30}^{T}\mathbf{W}_3 = \mathbf{0}_{6\times1} \tag{27}
$$

the desired stiffness model can be rewritten in the form of a single matrix equation

To simplify further derivations, let us present both models in a similar way, with the matrices of the same dimensions of 18×18. For this purpose, let us eliminate the redundant variable $\Delta\mathbf{t}_0$ from the linear matrix equation (25). Taking into account that in the

Δ-type structure, no wrench is applied to the node #0, i.e. $\mathbf{W}_0 = \mathbf{0}_{6\times1}$, the last line of (25) can written as

$$-\mathbf{K}_{10}\mathbf{D}_{10}^{-1}\cdot\Delta\mathbf{t}_1 - \mathbf{K}_{20}\mathbf{D}_{20}^{-1}\cdot\Delta\mathbf{t}_2 - \mathbf{K}_{30}\mathbf{D}_{30}^{-1}\cdot\Delta\mathbf{t}_3 + (\mathbf{K}_{10}+\mathbf{K}_{20}+\mathbf{K}_{30})\cdot\Delta\mathbf{t}_0 = \mathbf{0} \tag{28}$$

which yields the following expression for the deflections in the free node #0

$$\Delta\mathbf{t}_0 = \mathbf{K}_\Sigma^{-1}\mathbf{K}_{10}\mathbf{D}_{10}^{-1}\Delta\mathbf{t}_1 + \mathbf{K}_\Sigma^{-1}\mathbf{K}_{20}\mathbf{D}_{20}^{-1}\Delta\mathbf{t}_2 + \mathbf{K}_\Sigma^{-1}\mathbf{K}_{30}\mathbf{D}_{30}^{-1}\Delta\mathbf{t}_3 \tag{29}$$

where

$$\mathbf{K}_\Sigma = \mathbf{K}_{10} + \mathbf{K}_{20} + \mathbf{K}_{30} \tag{30}$$

After substitution $\Delta\mathbf{t}_0$ into the three remaining lines of the equation (25), one can obtain the reduced-size stiffness model of the Y-structure as

$$\begin{bmatrix}\mathbf{W}_1\\ \mathbf{W}_2\\ \mathbf{W}_3\end{bmatrix} = \begin{bmatrix}\mathbf{K}_{11}^{(Y)} & \mathbf{K}_{12}^{(Y)} & \mathbf{K}_{13}^{(Y)}\\ \mathbf{K}_{21}^{(Y)} & \mathbf{K}_{22}^{(Y)} & \mathbf{K}_{23}^{(Y)}\\ \mathbf{K}_{31}^{(Y)} & \mathbf{K}_{32}^{(Y)} & \mathbf{K}_{33}^{(Y)}\end{bmatrix}_{18\times18} \cdot \begin{bmatrix}\Delta\mathbf{t}_1\\ \Delta\mathbf{t}_2\\ \Delta\mathbf{t}_3\end{bmatrix} \tag{31}$$

where

$$\begin{aligned}
\mathbf{K}_{11}^{(Y)} &= \mathbf{D}_{10}^{-T}\mathbf{K}_{10}\mathbf{D}_{10}^{-1} - \mathbf{D}_{10}^{-T}\mathbf{K}_{10}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{10}\mathbf{D}_{10}^{-1}\\
\mathbf{K}_{12}^{(Y)} &= -\mathbf{D}_{10}^{-T}\mathbf{K}_{10}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{20}\mathbf{D}_{20}^{-1}\\
\mathbf{K}_{13}^{(Y)} &= -\mathbf{D}_{10}^{-T}\mathbf{K}_{10}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{30}\mathbf{D}_{30}^{-1}\\
\mathbf{K}_{21}^{(Y)} &= -\mathbf{D}_{20}^{-T}\mathbf{K}_{20}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{10}\mathbf{D}_{10}^{-1}\\
\mathbf{K}_{22}^{(Y)} &= \mathbf{D}_{20}^{-T}\mathbf{K}_{20}\mathbf{D}_{20}^{-1} - \mathbf{D}_{20}^{-T}\mathbf{K}_{20}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{20}\mathbf{D}_{20}^{-1}\\
\mathbf{K}_{23}^{(Y)} &= -\mathbf{D}_{20}^{-T}\mathbf{K}_{20}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{30}\mathbf{D}_{30}^{-1}\\
\mathbf{K}_{31}^{(Y)} &= -\mathbf{D}_{30}^{-T}\mathbf{K}_{30}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{10}\mathbf{D}_{10}^{-1}\\
\mathbf{K}_{32}^{(Y)} &= -\mathbf{D}_{30}^{-T}\mathbf{K}_{30}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{20}\mathbf{D}_{20}^{-1}\\
\mathbf{K}_{33}^{(Y)} &= \mathbf{D}_{30}^{-T}\mathbf{K}_{30}\mathbf{D}_{30}^{-1} - \mathbf{D}_{30}^{-T}\mathbf{K}_{30}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{30}\mathbf{D}_{30}^{-1}
\end{aligned} \tag{32}$$

It gives a representation for the Y-structure similar to the Δ-type one (17). It is obvious that both representations operate with symmetrical matrices of size 18×18 whose rank is equal to 12. Here, the rank deficiency of 6 is induced by the equilibrium condition (27), which for $\mathbf{W}_0 = \mathbf{0}_{6\times1}$ can be easily transformed into the form (19) after left-multiplication by $\mathbf{D}_{10}^{-T}$ and relevant transformations using the **D**-matrix properties (10).

2.4 Transformation of Y-Structure to Equivalent Δ-Structure

Now, let us derive expressions relating the parameters of Y- and Δ-structures with similar stiffness properties (see 0). To derive the desired expressions for $Y \to \Delta$ transformation, let us equate the upper off-diagonal components from equations (17) and (31), i.e. block-matrix elements

$$\begin{aligned}
\mathbf{K}_{12}^{(\Delta)} &= \mathbf{K}_{12}^{(Y)}\\
\mathbf{K}_{13}^{(\Delta)} &= \mathbf{K}_{13}^{(Y)}\\
\mathbf{K}_{23}^{(\Delta)} &= \mathbf{K}_{23}^{(Y)}
\end{aligned} \tag{33}$$

This yields the following equations

$$\begin{aligned}
-\mathbf{D}_{12}^{-T}\mathbf{K}_{12} &= -\mathbf{D}_{10}^{-T}\mathbf{K}_{10}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{20}\mathbf{D}_{20}^{-1}\\
-\mathbf{D}_{23}^{-T}\mathbf{K}_{23} &= -\mathbf{D}_{20}^{-T}\mathbf{K}_{20}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{30}\mathbf{D}_{30}^{-1}\\
-\mathbf{K}_{31}\mathbf{D}_{31}^{-1} &= -\mathbf{D}_{10}^{-T}\mathbf{K}_{10}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{30}\mathbf{D}_{30}^{-1}
\end{aligned} \tag{34}$$

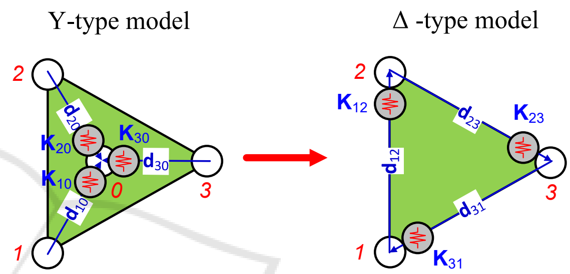

Figure 5: VJM-based Y-Δ transformation in the stiffness models.

that are easily solved for the desired Δ-structures parameters $\mathbf{K}_{12}$, $\mathbf{K}_{23}$, $\mathbf{K}_{31}$ (stiffness matrices)

$$\begin{aligned}
\mathbf{K}_{12} &= \mathbf{D}_{12}^{T}\mathbf{D}_{10}^{-T}\mathbf{K}_{10}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{20}\mathbf{D}_{20}^{-1}\\
\mathbf{K}_{23} &= \mathbf{D}_{23}^{T}\mathbf{D}_{20}^{-T}\mathbf{K}_{20}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{30}\mathbf{D}_{30}^{-1}\\
\mathbf{K}_{31} &= \mathbf{D}_{10}^{-T}\mathbf{K}_{10}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{30}\mathbf{D}_{30}^{-1}\mathbf{D}_{31}
\end{aligned} \tag{35}$$

Further, taking into account the symmetry of the stiffness matrices $\mathbf{K}_{ij} = \mathbf{K}_{ij}^T$ and specific properties of the **D**-matrix (10) allowing following simplifications

$$\begin{aligned}
\mathbf{D}_{20}^{-T} &= \mathbf{D}_{12}^{T}\mathbf{D}_{10}^{-T}\\
\mathbf{D}_{30}^{-T} &= \mathbf{D}_{23}^{T}\mathbf{D}_{20}^{-T}\\
\mathbf{D}_{10}^{-1} &= \mathbf{D}_{30}^{-1}\mathbf{D}_{31}
\end{aligned} \tag{36}$$

the above expressions (35) are reduced to a more convenient form

$$\begin{aligned}
\mathbf{K}_{12} &= \mathbf{D}_{20}^{-T}\cdot\mathbf{K}_{10}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{20}\cdot\mathbf{D}_{20}^{-1}\\
\mathbf{K}_{23} &= \mathbf{D}_{30}^{-T}\cdot\mathbf{K}_{20}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{30}\cdot\mathbf{D}_{30}^{-1}\\
\mathbf{K}_{31} &= \mathbf{D}_{10}^{-T}\cdot\mathbf{K}_{30}\mathbf{K}_\Sigma^{-1}\mathbf{K}_{10}\cdot\mathbf{D}_{10}^{-1}
\end{aligned} \tag{37}$$

In an alternative way, expressions (37) can be rewritten with respect to compliances and presented as

$$
\begin{aligned}
\mathbf{K}_{12}^{-1} &= \mathbf{D}_{20}\left(\mathbf{K}_{10}^{-1}+\mathbf{K}_{20}^{-1}+\mathbf{K}_{20}^{-1}\mathbf{K}_{30}\mathbf{K}_{10}^{-1}\right)\mathbf{D}_{20}^{T} \\
\mathbf{K}_{23}^{-1} &= \mathbf{D}_{30}\left(\mathbf{K}_{20}^{-1}+\mathbf{K}_{30}^{-1}+\mathbf{K}_{30}^{-1}\mathbf{K}_{10}\mathbf{K}_{20}^{-1}\right)\mathbf{D}_{30}^{T} \\
\mathbf{K}_{31}^{-1} &= \mathbf{D}_{10}\left(\mathbf{K}_{10}^{-1}+\mathbf{K}_{30}^{-1}+\mathbf{K}_{10}^{-1}\mathbf{K}_{20}\mathbf{K}_{30}^{-1}\right)\mathbf{D}_{10}^{T}
\end{aligned}
\tag{38}
$$

which are similar to expressions from electrical engineering, where the resistance corresponds to the compliance matrices and relevant transformation equations from Y to Δ circuits are expressed as follows.

$$
\begin{aligned}
R_{12} &= R_{10}+R_{20}+\frac{R_{10}R_{20}}{R_{30}} \\
R_{23} &= R_{20}+R_{30}+\frac{R_{20}R_{30}}{R_{10}} \\
R_{31} &= R_{10}+R_{30}+\frac{R_{30}R_{10}}{R_{20}}
\end{aligned}
\tag{39}
$$

where R_{12}, R_{23}, R_{31} are the Δ-circuit resistances and R_{10}, R_{20}, R_{30} are the resistances for the Y-circuit.

2.5 Transformation of Δ-Structure to Equivalent Y-Structure

For the inverse transformation, for $\Delta \rightarrow Y$ transformation (see 0), let us consider the above-derived equations (37) but solve them for $\mathbf{K}_{10}$, $\mathbf{K}_{20}$, $\mathbf{K}_{30}$. For convenience, these equations can be rewritten as

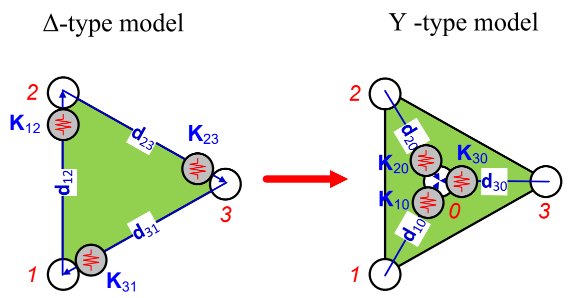

Figure 6: VJM-based Δ-Y transformation in the stiffness models.

$$
\begin{aligned}
\mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{20} &= \mathbf{K}_{10}(\mathbf{K}_{10}+\mathbf{K}_{20}+\mathbf{K}_{30})^{-1}\mathbf{K}_{20} \\
\mathbf{D}_{30}^{T}\mathbf{K}_{23}\mathbf{D}_{30} &= \mathbf{K}_{20}(\mathbf{K}_{10}+\mathbf{K}_{20}+\mathbf{K}_{30})^{-1}\mathbf{K}_{30} \\
\mathbf{D}_{10}^{T}\mathbf{K}_{31}\mathbf{D}_{10} &= \mathbf{K}_{10}(\mathbf{K}_{10}+\mathbf{K}_{20}+\mathbf{K}_{30})^{-1}\mathbf{K}_{30}
\end{aligned}
\tag{40}
$$

and further transformed to.

$$
\begin{aligned}
\mathbf{K}_{10}\mathbf{D}_{20}^{-1}\mathbf{K}_{12}^{-1}\mathbf{D}_{20}^{-T}\mathbf{K}_{20} &= \mathbf{K}_{10}+\mathbf{K}_{20}+\mathbf{K}_{30} \\
\mathbf{K}_{20}\mathbf{D}_{30}^{-1}\mathbf{K}_{23}^{-1}\mathbf{D}_{30}^{-T}\mathbf{K}_{30} &= \mathbf{K}_{10}+\mathbf{K}_{20}+\mathbf{K}_{30} \\
\mathbf{K}_{30}\mathbf{D}_{10}^{-1}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\mathbf{K}_{10} &= \mathbf{K}_{10}+\mathbf{K}_{20}+\mathbf{K}_{30}
\end{aligned}
\tag{41}
$$

which yields the following equalities.

$$
\begin{aligned}
\mathbf{K}_{10}\mathbf{D}_{20}^{-1}\mathbf{K}_{12}^{-1}\mathbf{D}_{20}^{-T}\mathbf{K}_{20} &= \mathbf{K}_{20}\mathbf{D}_{30}^{-1}\mathbf{K}_{23}^{-1}\mathbf{D}_{30}^{-T}\mathbf{K}_{30} \\
\mathbf{K}_{20}\mathbf{D}_{30}^{-1}\mathbf{K}_{23}^{-1}\mathbf{D}_{30}^{-T}\mathbf{K}_{30} &= \mathbf{K}_{30}\mathbf{D}_{10}^{-1}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\mathbf{K}_{10} \\
\mathbf{K}_{10}\mathbf{D}_{20}^{-1}\mathbf{K}_{12}^{-1}\mathbf{D}_{20}^{-T}\mathbf{K}_{20} &= \mathbf{K}_{30}\mathbf{D}_{10}^{-1}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\mathbf{K}_{10}
\end{aligned}
\tag{42}
$$

Then, using the first and third relations, the symmetry of the stiffness matrices $\mathbf{K}_{ij}$ as well as commutativity of the above matrix products, and applying transposition, one can get expressions

$$
\begin{aligned}
\mathbf{K}_{10}\mathbf{D}_{20}^{-1}\mathbf{K}_{12}^{-1}\mathbf{D}_{20}^{-T}\cdot\mathbf{K}_{20} &= \mathbf{K}_{30}\mathbf{D}_{30}^{-1}\mathbf{K}_{23}^{-1}\mathbf{D}_{30}^{-T}\cdot\mathbf{K}_{20} \\
\mathbf{K}_{20}\mathbf{D}_{20}^{-1}\mathbf{K}_{12}^{-1}\mathbf{D}_{20}^{-T}\cdot\mathbf{K}_{10} &= \mathbf{K}_{30}\mathbf{D}_{10}^{-1}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\cdot\mathbf{K}_{10}
\end{aligned}
\tag{43}
$$

allowing the derivation of relations between $\mathbf{K}_{10}, \mathbf{K}_{20}, \mathbf{K}_{30}$ as

$$
\begin{aligned}
\mathbf{K}_{10} &= \mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{20}\mathbf{D}_{30}^{-1}\mathbf{K}_{23}^{-1}\mathbf{D}_{30}^{-T}\cdot\mathbf{K}_{30} \\
\mathbf{K}_{20} &= \mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{20}\mathbf{D}_{10}^{-1}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\cdot\mathbf{K}_{30}
\end{aligned}
\tag{44}
$$

Which using properties (10) can be further simplified down to

$$
\begin{aligned}
\mathbf{K}_{10} &= \mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{23}\mathbf{K}_{23}^{-1}\mathbf{D}_{30}^{-T}\cdot\mathbf{K}_{30} \\
\mathbf{K}_{20} &= \mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{21}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\cdot\mathbf{K}_{30}
\end{aligned}
\tag{45}
$$

Substituting these relations into the third relation of the original system (41) and

$$
\begin{aligned}
&\mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{21}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\cdot\mathbf{K}_{30}\mathbf{D}_{30}^{-1}\mathbf{K}_{23}^{-1}\mathbf{D}_{30}^{-T}\mathbf{K}_{30} = \\
&\quad = \mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{23}\mathbf{K}_{23}^{-1}\mathbf{D}_{30}^{-T}\cdot\mathbf{K}_{30} + \\
&\quad + \mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{21}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\cdot\mathbf{K}_{30}+\mathbf{K}_{30}
\end{aligned}
\tag{46}
$$

After executing relevant simplifications, one can obtain the desired solution for the stiffness matrix $\mathbf{K}_{30}$ in the form

$$
\begin{aligned}
\mathbf{K}_{30} &= \mathbf{D}_{10}^{T}\mathbf{K}_{31}\mathbf{D}_{10}+\mathbf{D}_{30}^{T}\mathbf{K}_{23}\mathbf{D}_{30} + \\
&+ \mathbf{D}_{30}^{T}\mathbf{K}_{23}\mathbf{D}_{30}\cdot\mathbf{D}_{20}^{-1}\mathbf{K}_{12}^{-1}\mathbf{D}_{20}^{-T}\cdot\mathbf{D}_{10}^{T}\mathbf{K}_{31}\mathbf{D}_{10}
\end{aligned}
\tag{47}
$$

Which can also be presented as

$$
\mathbf{K}_{30} = {}^{0}\mathbf{K}_{31} + {}^{0}\mathbf{K}_{23} + {}^{0}\mathbf{K}_{23}\cdot{}^{0}\mathbf{K}_{12}^{-1}\cdot{}^{0}\mathbf{K}_{31} \tag{48}
$$

which operates with the modified stiffness matrices of Δ-structures ${}^{0}\mathbf{K}_{12}, {}^{0}\mathbf{K}_{23}, {}^{0}\mathbf{K}_{31}$ obtained from the original once $\mathbf{K}_{12}, \mathbf{K}_{23}, \mathbf{K}_{31}$ by shifting the reference point to node #0 in accordance with

$$\begin{aligned} {}^{0}\mathbf{K}_{12} &= \mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{20} \\ {}^{0}\mathbf{K}_{23} &= \mathbf{D}_{30}^{T}\mathbf{K}_{23}\mathbf{D}_{30} \\ {}^{0}\mathbf{K}_{31} &= \mathbf{D}_{10}^{T}\mathbf{K}_{31}\mathbf{D}_{10} \end{aligned} \tag{49}$$

Let us now consider relations (1) and (2) in the system (42), and using the symmetry of the stiffness matrices $\mathbf{K}_{ij}$ as well as the commutativity of the above matrix products, and applying transposition, one can get the following expressions

$$\begin{aligned} \mathbf{K}_{10}\mathbf{D}_{20}^{-1}\mathbf{K}_{12}^{-1}\mathbf{D}_{20}^{-T}\cdot\mathbf{K}_{20} &= \mathbf{K}_{30}\mathbf{D}_{30}^{-1}\mathbf{K}_{23}^{-1}\mathbf{D}_{30}^{-T}\cdot\mathbf{K}_{20} \\ \mathbf{K}_{20}\mathbf{D}_{30}^{-1}\mathbf{K}_{23}^{-1}\mathbf{D}_{30}^{-T}\cdot\mathbf{K}_{30} &= \mathbf{K}_{10}\mathbf{D}_{10}^{-1}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\cdot\mathbf{K}_{30} \end{aligned} \tag{50}$$

allowing the derivation of relations between $\mathbf{K}_{10}, \mathbf{K}_{20}, \mathbf{K}_{30}$ as

$$\begin{aligned} \mathbf{K}_{30} &= \mathbf{D}_{30}^{T}\mathbf{K}_{23}\mathbf{D}_{30}\mathbf{D}_{20}^{-1}\mathbf{K}_{12}^{-1}\mathbf{D}_{20}^{-T}\cdot\mathbf{K}_{10} \\ \mathbf{K}_{20} &= \mathbf{D}_{30}^{T}\mathbf{K}_{23}\mathbf{D}_{30}\mathbf{D}_{10}^{-1}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\cdot\mathbf{K}_{10} \end{aligned} \tag{51}$$

Which using properties (10) can be further simplified down to

$$\begin{aligned} \mathbf{K}_{30} &= \mathbf{D}_{30}^{T}\mathbf{K}_{23}\mathbf{D}_{32}\mathbf{K}_{12}^{-1}\mathbf{D}_{20}^{-T}\cdot\mathbf{K}_{10} \\ \mathbf{K}_{20} &= \mathbf{D}_{30}^{T}\mathbf{K}_{23}\mathbf{D}_{31}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\cdot\mathbf{K}_{10} \end{aligned} \tag{52}$$

Substituting these relations into the third relation of the original system (41) and executing relevant simplifications, one can obtain the desired solution for the stiffness matrix $\mathbf{K}_{30}$ in the form

$$\begin{aligned} \mathbf{K}_{10} = {} & \mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{20} + \mathbf{D}_{10}^{T}\mathbf{K}_{31}\mathbf{D}_{10} + \\ & + \mathbf{D}_{10}^{T}\mathbf{K}_{31}\mathbf{D}_{10}\cdot\mathbf{D}_{30}^{-1}\mathbf{K}_{23}^{-1}\mathbf{D}_{30}^{-T}\cdot\mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{20} \end{aligned} \tag{53}$$

Which can also be presented as

$$\mathbf{K}_{10} = {}^{0}\mathbf{K}_{12} + {}^{0}\mathbf{K}_{31} + {}^{0}\mathbf{K}_{31}\cdot{}^{0}\mathbf{K}_{23}^{-1}\cdot{}^{0}\mathbf{K}_{12} \tag{54}$$

In a similar way, the expressions can also be derived for $\mathbf{K}_{20}$

$$\begin{aligned} \mathbf{K}_{20} = {} & \mathbf{D}_{30}^{T}\mathbf{K}_{23}\mathbf{D}_{30} + \mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{20} + \\ & + \mathbf{D}_{20}^{T}\mathbf{K}_{12}\mathbf{D}_{20}\cdot\mathbf{D}_{10}^{-1}\mathbf{K}_{31}^{-1}\mathbf{D}_{10}^{-T}\cdot\mathbf{D}_{30}^{T}\mathbf{K}_{23}\mathbf{D}_{30} \end{aligned} \tag{55}$$

Or in the form

$$\mathbf{K}_{20} = {}^{0}\mathbf{K}_{23} + {}^{0}\mathbf{K}_{12} + {}^{0}\mathbf{K}_{12}\cdot{}^{0}\mathbf{K}_{31}^{-1}\cdot{}^{0}\mathbf{K}_{23} \tag{56}$$

Hence, the final solution has the following presentation

$$\begin{aligned} \mathbf{K}_{10} &= {}^{0}\mathbf{K}_{12} + {}^{0}\mathbf{K}_{31} + {}^{0}\mathbf{K}_{31}\cdot{}^{0}\mathbf{K}_{23}^{-1}\cdot{}^{0}\mathbf{K}_{12} \\ \mathbf{K}_{20} &= {}^{0}\mathbf{K}_{23} + {}^{0}\mathbf{K}_{12} + {}^{0}\mathbf{K}_{12}\cdot{}^{0}\mathbf{K}_{31}^{-1}\cdot{}^{0}\mathbf{K}_{23} \\ \mathbf{K}_{30} &= {}^{0}\mathbf{K}_{31} + {}^{0}\mathbf{K}_{23} + {}^{0}\mathbf{K}_{23}\cdot{}^{0}\mathbf{K}_{12}^{-1}\cdot{}^{0}\mathbf{K}_{31} \end{aligned} \tag{57}$$

Also, after relevant matrix transformations and inversion of eq. (57), the desired solutions can be presented with respect to the compliance

$$\begin{aligned} \mathbf{K}_{10}^{-1} &= {}^{0}\mathbf{K}_{12}^{-1}\left({}^{0}\mathbf{K}_{12}^{-1} + {}^{0}\mathbf{K}_{23}^{-1} + {}^{0}\mathbf{K}_{31}^{-1}\right)^{-1}{}^{0}\mathbf{K}_{31}^{-1} \\ \mathbf{K}_{20}^{-1} &= {}^{0}\mathbf{K}_{23}^{-1}\left({}^{0}\mathbf{K}_{12}^{-1} + {}^{0}\mathbf{K}_{23}^{-1} + {}^{0}\mathbf{K}_{31}^{-1}\right)^{-1}{}^{0}\mathbf{K}_{12}^{-1} \\ \mathbf{K}_{30}^{-1} &= {}^{0}\mathbf{K}_{31}^{-1}\left({}^{0}\mathbf{K}_{12}^{-1} + {}^{0}\mathbf{K}_{23}^{-1} + {}^{0}\mathbf{K}_{31}^{-1}\right)^{-1}{}^{0}\mathbf{K}_{23}^{-1} \end{aligned} \tag{58}$$

Thus, the obtained expressions (37), (38), (57) and (58) allow the transformation of the Y-type elastic structure into the equivalent Δ-type one and vice versa. They are similar to the scalar expressions from electrical engineering.

$$\begin{aligned} R_{10} &= \frac{R_{12}R_{31}}{R_{12}+R_{23}+R_{31}} \\ R_{20} &= \frac{R_{12}R_{23}}{R_{12}+R_{23}+R_{31}} \\ R_{30} &= \frac{R_{23}R_{31}}{R_{12}+R_{23}+R_{31}} \end{aligned} \tag{59}$$

However, the expressions for stiffness transformations are based on the 6×6 matrix operations and include additional components $\mathbf{D}_{ij}$ that take into account the geometry of the relevant mechanical structure, although they can be excluded if all stiffness matrices $\mathbf{K}_{ij}$ are presented with respect to the node #0, i.e. in the form ${}^{0}\mathbf{K}_{ij}$ defined by eq. (49). It is worth mentioning that for $\Delta \rightarrow \mathrm{Y}$ transformations, the location of the node #0 can be assigned arbitrarily. Besides, it should be noted that because of the symmetry of the matrices $\mathbf{K}_{ij}$ and ${}^{0}\mathbf{K}_{ij}$ leading to commutativity of some matrix products, one can obtain slightly different expressions for equivalent stiffness/compliance matrices, which are equal up to a transposition.

3 APPLICATION EXAMPLE

To demonstrate the value of the proposed technique, let us apply it to the stiffness analysis of the Gough-Stewart manipulator with a non-rigid mobile platform (see 0). Legs' stiffness modelling (0) does not create any problems due to their strictly serial kinematics (Klimchik *et al.*, 2012). However, due to the platform's elasticity, the entire mechanism cannot be presented as a serial-parallel structure, as is typically considered in relevant works. In fact, the platform contains multiple elastic cross-linkages that make it impossible to handle within the frame of the conventional VJM approach. However, using the developed $\Delta \rightarrow Y$ transformation, one can obtain an equivalent pure serial-parallel topology suitable for stiffness modelling employing the VJM approach.

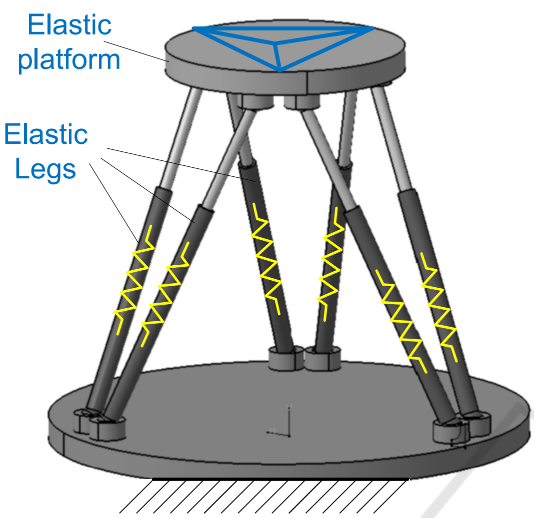

Figure 7: Gough-Stewart platform with 3-3 connection.

The considered elastic platform consists of six mutually connected elastic beams forming the frame, as shown in 0. For each beam, the 6×6 the stiffness matrix is computed using the following expression

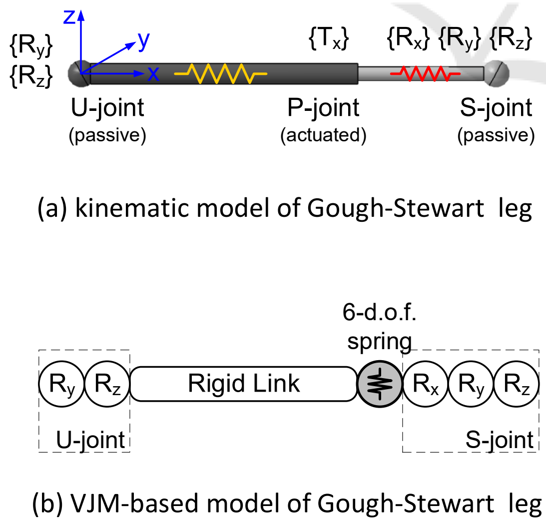

Figure 8: VJM-based stiffness models for Gough-Stewart's leg.

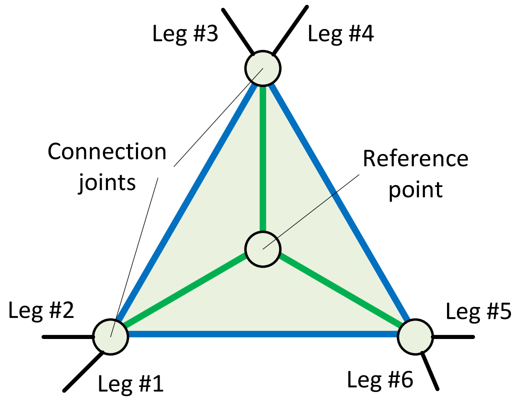

Figure 9: Gough-Stewart's Δ+Y structure of mobile platform.

$$\mathbf{K}_{beam} = \frac{E}{L^3}\left[\begin{array}{ccc|ccc} AL^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12I_z & 0 & 0 & 0 & -6I_zL \\ 0 & 0 & 12I_y & 0 & 6I_yL & 0 \\ \hline 0 & 0 & 0 & K_{44} & 0 & 0 \\ 0 & 0 & 6I_yL & 0 & 4I_yL^2 & 0 \\ 0 & -6I_zL & 0 & 0 & 0 & 4I_zL^2 \end{array}\right] \quad (60)$$

where $K_{44} = JL^2(1+\upsilon)/2$, Young's modulus E and Poisson's ratio coefficient υ describe beam's elastic properties, its geometry is described by length L and cross-section area A, the variables I_y, I_z, and J are the cross-section quadratic and polar moments of inertia. For the considered example, it is assumed that actuated legs are connected to the elastic platform at the corners of the equilateral triangle with the edge length a, while the reference point is located at the triangle's centre. For such an arrangement, the lengths of the links (1,2), (2,3) and (3,1) are equal to the triangle parameter a and the lengths of the links (1,0), (2,0) and (3,0) are $b = a/\sqrt{3}$. The remaining parameters included in the matrix $\mathbf{K}_{beam}$ are computed as $A = \pi \cdot d^2/4$, $I_y = I_z = \pi \cdot d^4/64$, $J = \pi \cdot d^4/32$, where d is the link diameter that is assumed to have a circular cross-section.

The original platform consisted of six mutually connected elastic elements: three beams of length a and three beams of length b (0a). After applying the developed $\Delta \rightarrow Y$ transformation, the original model is converted into an equivalent double-Y-structure composed of six elements of length b each (0b). Then this double-Y-structure was transformed into a classical Y-structure that can be easily handled by the conventional VJM approach (0c).

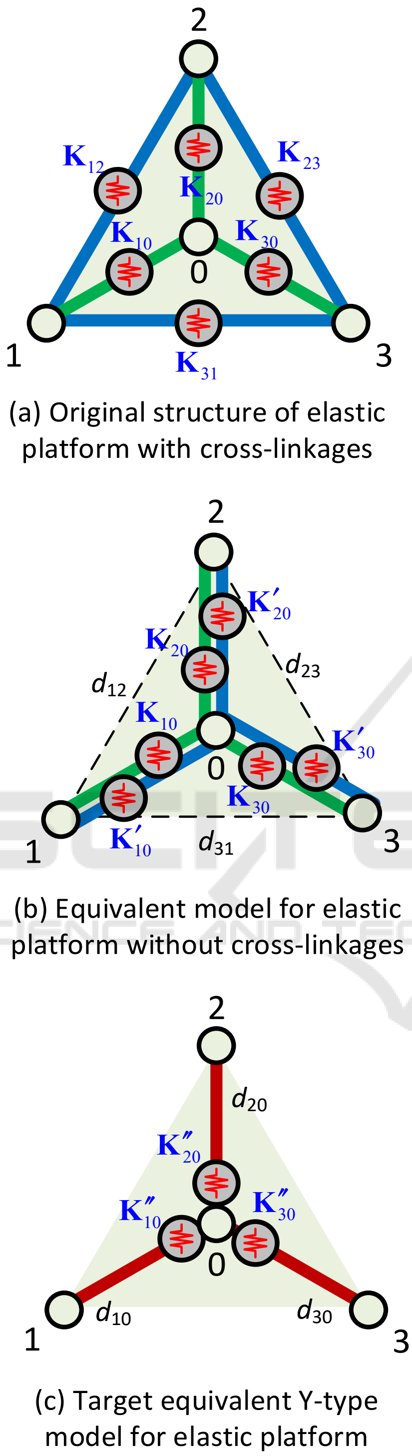

Figure 10: VJM-based stiffness models for the Δ-structure and their parameters.

Using the developed $\Delta \rightarrow \mathrm{Y}$ transformation, we can compute $\mathbf{K}'_{10}, \mathbf{K}'_{20}, \mathbf{K}'_{30}$ as follows

$$\begin{aligned}
\mathbf{K}'_{10} &= {}^{0}\mathbf{K}_{12} + {}^{0}\mathbf{K}_{31} + {}^{0}\mathbf{K}_{31} \cdot {}^{0}\mathbf{K}_{23}^{-1} \cdot {}^{0}\mathbf{K}_{12} \\
\mathbf{K}'_{20} &= {}^{0}\mathbf{K}_{23} + {}^{0}\mathbf{K}_{12} + {}^{0}\mathbf{K}_{12} \cdot {}^{0}\mathbf{K}_{31}^{-1} \cdot {}^{0}\mathbf{K}_{23} \\
\mathbf{K}'_{30} &= {}^{0}\mathbf{K}_{31} + {}^{0}\mathbf{K}_{23} + {}^{0}\mathbf{K}_{23} \cdot {}^{0}\mathbf{K}_{12}^{-1} \cdot {}^{0}\mathbf{K}_{31}
\end{aligned} \tag{61}$$

And then, considering the parallel connection of $\mathbf{K}_{10}, \mathbf{K}_{20}, \mathbf{K}_{30}$ and $\mathbf{K}'_{10}, \mathbf{K}'_{20}, \mathbf{K}'_{30}$ get stiffness matrices $\mathbf{K}''_{10}, \mathbf{K}''_{20}, \mathbf{K}''_{30}$ as

$$\begin{aligned}
\mathbf{K}''_{10} &= \mathbf{K}_{10} + \mathbf{K}'_{10} \\
\mathbf{K}''_{20} &= \mathbf{K}_{20} + \mathbf{K}'_{20} \\
\mathbf{K}''_{30} &= \mathbf{K}_{30} + \mathbf{K}'_{30}
\end{aligned} \tag{62}$$

To obtain the stiffness model for the entire manipulator, one can consider pairs of legs connected in parallel and attached to the mobile platform, i.e. we can write

$$\begin{aligned}
\mathbf{K}_{10}^{(leg)} &= \mathbf{K}_{leg}^{1} + \mathbf{K}_{leg}^{2} \\
\mathbf{K}_{10}^{(leg)} &= \mathbf{K}_{leg}^{3} + \mathbf{K}_{leg}^{4} \\
\mathbf{K}_{10}^{(leg)} &= \mathbf{K}_{leg}^{5} + \mathbf{K}_{leg}^{6}
\end{aligned} \tag{63}$$

Where leg stiffness matrices $\mathbf{K}_{leg}^{i}$ can be computed as follows (see (Klimchik *et al.*, 2025) for details)

$$\mathbf{K}_{leg}^{i} = K_{11} \cdot \left[\begin{array}{c|c} \mathbf{u}_i \cdot \mathbf{u}_i^T & \mathbf{0}_{3\times 3} \\ \hline \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \end{array} \right] \tag{64}$$

where $K_{11} = EA/L$ is the leg stiffness on the compression along the main axis and $\mathbf{u}_i$ The unit direction vectors specify the orientation of the leg. To integrate the legs' stiffness in the stiffness model of the manipulator, we need to move $\mathbf{K}_i^{(leg)}$ to the zero node using the following transformations

$$\begin{aligned}
\mathbf{K}_{leg10}^{(0)} &= \mathbf{D}_{10} \mathbf{K}_{leg}^{10} \mathbf{D}_{10i}^{T} \\
\mathbf{K}_{leg20}^{(0)} &= \mathbf{D}_{20} \mathbf{K}_{leg}^{20} \mathbf{D}_{20}^{T} \\
\mathbf{K}_{leg30}^{(0)} &= \mathbf{D}_{30} \mathbf{K}_{leg}^{30} \mathbf{D}_{30}^{T}
\end{aligned} \tag{65}$$

Thus, the final Cartesian stiffness matrix for the Gough-Stewart Platform can be computed as

$$\begin{aligned}
\mathbf{K}_C = {} & \left(\mathbf{K}''_{10} + \mathbf{K}_{leg10}^{(0)} \right) + \left(\mathbf{K}''_{20} + \mathbf{K}_{leg20}^{(0)} \right) + \\
& + \left(\mathbf{K}''_{30} + \mathbf{K}_{leg30}^{(0)} \right)
\end{aligned} \tag{66}$$

Hence, this development expands the application scope of the VJM method for over-constrained parallel manipulators, where cross-linkages are widely used to improve stiffness properties.

4 CONCLUSIONS

This paper proposes a new stiffness model transformation technique for modelling the elastic behaviour of hybrid over-constrained robotic manipulators with multiple cross-linkages. This technique helps users address the critical limitation of the VJM method. It provides an analytical expression for equivalent transforming the cross-linkages into serial-parallel structures suitable for the VJM.

To derive the desired transformations, the specific MSA-based representation is employed, which uses a conventional VJM-type 6×6 virtual springs. This helps to derive analytical relations between the equivalent models. The main results were obtained for 3-node structures, but they can be further generalised for the n-node case. To demonstrate the efficiency of the developed technique, Gough-Stewart manipulator with elastic platforms and compliant legs was considered.

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