Filtering of Polytopic-Type Uncertain State-Delayed Noisy Systems

Eli Gershon
Holon Institute of Technology, HIT, Holon, Israel
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Abstract: The problem of H_∞ state estimation is considered for uncertain polytopic retarded linear discrete-time stochastic systems. We first bring the solution of the estimation problem for the nominal case based on a previously developed BRL for state-delayed stochastic systems. We then extend our solution to the robust uncertain polytopic case where a vertex-dependent approach is applied. The latter is achieved via the application of a modified version of the Finsler lemma. The use of this lemma enable us to derive a solution which is less conservative comparing to the "quadratic" solution where a single Lyapunov function is applied over all the uncertain polytope. The solution obtained for the robust case is composed of a set of LMIs based on only two tuning parameters. The theory presented is demonstrated by a numerical example.

1 INTRODUCTION

We address the problem of state-estimation of linear stochastic state-multiplicative systems which are state-delayed and may contain large uncertainties. In our study we make use of a general-type filter and we first bring the solution of the estimation problem for nominal systems via a single LMI condition based on a Bounded Real Lemma (BRL) for these systems (Gershon and Shaked, 2013),(Fridman, 2014). Based on the latter solution, we solve the estimation problem for polytopic-type uncertain systems where we apply a special version of the Finsler lemma. This lemma enable us to assign a different Lyapunov function to each vertex of the uncertain polytope, thus greatly reducing the over-design which is inherent to the quadratic solution, which, in turn, is based on assigning the same Lyapunov function over all the uncertain polytope.

The field of control and estimation of stochastic state-multiplicative noisy systems has been greatly developed since the onset of the H_∞ control theory in the early 80‘s (see (Gershon and Shaked, 2013), (Gershon et al., 2005) and the references therein). Within a span of more than four decades, many approaches to the study of the various stochastic control and filtering problems, including those that ensure a worst case performance bound in the H_∞ sense, have been derived for both: delay-free systems (Gershon et al., 2005), (Dragan and Stoica, 1998),(Dragan and Morozan, 1997a), (Dragan and Morozan, 1997b), (Dragan et al., 1992), (Hinriechsen and Pritchard, 1998), (Bouhtouri et al., 1999) and state-delayed, linear, stochastic systems (Gershon et al., 2007), (Gershon and Shaked, 2011), (Verriest and Florchinger, 1995), (Mao, 1996), (Xu et al., 2005), (Chen et al., 2005), (Gao and Chen, 2007), (Yue et al., 2009).

Delay-free systems with parameter uncertainties that are modeled as white noise processes in a linear setting have been treated in (Gershon and Shaked, 2013), (Gershon et al., 2005), (Dragan and Stoica, 1998) for both the continuous-time and the discrete-time cases. Such models of uncertainty are encountered in many areas of applications such as: nuclear fission and heat transfer, population models and immunology. In control theory, such models are encountered in gain scheduling when the scheduling parameters are corrupted with measurement noise.

The study of both continuous-time and discrete-time retarded systems has been developed extensively since the emergence of optimal control theory. In the field of control and estimation theory, the input-output approach has been applied, in the last two decades, to both continuous and discrete-time retarded systems. The input-output approach is based on the representation of the system's delay action by delay-free linear operators which allows one to replace the underlying system with one that possesses a norm-bounded uncertainty, and therefore may be treated by the theory of norm bounded uncertain, non-retarded systems with state-multiplicative

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noise (Gershon et al., 2005). Based on the latter approach, the stability and BRL issues of stochastic state-multiplicative systems were obtained (Gershon and Shaked, 2013), followed by the solution of various control problems that include, among others, state-feedback control, filtering and static and dynamic output-feedback control. We note that the robust solution of both the state-feedback control and filtering problems for uncertain systems in (Gershon and Shaked, 2013), rely on assigning a single Lyapunov function over all the uncertainty polytope (the so called "quadratic" solution) thus leading to the most conservative solution type (see (Gershon and Shaked, 2013) and the references therein). Obviously, the latter handicap can be relaxed by resorting to vertex-dependent solution which is the merit of the present work.

Similarly to the systems treated in (Gershon and Shaked, 2013), in the present theory, our systems may encounter a time-varying delay where the uncertain stochastic parameters multiply both the delayed and the non delayed states in the dynamics state-space model of the systems as well as the non-delayed states in the system measurement. We treat both the nominal case and the uncertain case where the system matrices encounter polytopic type uncertainties. In the latter case, we apply the Finsler lemma which leads to a less conservative solution compared to the simple "quadratic" solution where the same decision variables are assigned to all the vertices of the polytope.

The paper is organized as follows: Following the problem formulation in Section 2, we bring a preliminary result in Section 3. Based on the latter result, we bring the solution of the improved robust vertex-dependant H_∞ filter in Section 4. In Section 5 an example is given that demonstrates the applicability and tractability of the various solution methods derived in this work.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space and $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices. For a symmetric $P\in\mathcal{R}^{n\times n}$, $P>0$ means that it is positive definite. We denote expectation by $\mathcal{E}\{\cdot\}$ and we provide all spaces $\mathcal{R}^k$, $k\geq 1$ with the usual inner product $<\cdot,\cdot>$ and with the standard Euclidean norm $||\cdot||$. By $||f(t)||_R^2$ we denote the product of $f^T(t)Rf(t)$. We denote by $L^2(\Omega,\mathcal{R}^k)$ the space of square-integrable $\mathcal{R}^k-$ valued functions on the probability space $(\Omega,\mathcal{F},\mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subsets of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_t)_{t>0}$ we denote an increasing family of σ-algebras $\mathcal{F}_t\subset\mathcal{F}$. We also denote by $\tilde{L}^2([0,\infty);\mathcal{R}^k)$ the space of nonanticipative stochastic processes $f(\cdot)=(f(t))_{t\in[0,\infty)}$ in $\mathcal{R}^k$ with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$ satisfying

$$||f(\cdot)||^2_{\tilde{L}_2}=\mathcal{E}\{\int_0^\infty ||f(t)||^2dt\}=$$
$$\int_0^\infty \mathcal{E}\{||f(t)||^2\}dt<\infty,$$

and by $\mathcal{C}o\{\bar{\Omega}_1,\bar{\Omega}_2,...,\bar{\Omega}_N\}$ the convex hull of the matrices $\bar{\Omega}_i$, $i=1,...,N$. Stochastic differential equations will be interpreted to be of *Itô* type (Hinriechsen and Pritchard, 1998).

2 PROBLEM FORMULATION

We consider the following nominal linear stochastic retarded system:

$$\begin{array}{l} dx=[A_0x(t)+A_1x(t-\tau(t))+B_1w(t)]dt+\\ Hx(t-\tau(t))d\nu(t)+Gx(t)d\beta(t),\quad x(\theta)=0,\quad \theta\leq 0,\\ y(t)=C_2x(t)+D_{21}n(t)+Fx(t)d\zeta(t),\\ z(t)=Cx(t),\end{array}\tag{1a-c}$$

where $x(t)\in R^n$ is the state vector, $w(t)\in\tilde{L}^2_{\mathcal{F}_t}([0,\infty);\mathcal{R}^q)$ is an exogenous disturbance, $y(t)\in R^m$ is the measurement vector, $n(t)\in\tilde{L}^2_{\mathcal{F}_t}([0,\infty);\mathcal{R}^q)$ is an additive measurement noise, $z(t)\in R^r$ is the objective vector, A_0, A_1, B_1, C, C_2, D_{21} and F, G, H are time-invariant matrices of the appropriate dimension. $\tau(t)$ is an unknown time-delay which satisfies:

$$0\leq\tau(t)\leq h,\quad \dot{\tau}(t)\leq d<1.\tag{2a,b}$$

The zero-mean real scalar Wiener process $\beta(t)$, $\nu(t)$, $\zeta(t)$ satisfy:

$$\mathcal{E}\{\zeta(t)\zeta(s)\}=min(t,s),\ \ \mathcal{E}\{\beta(t)\beta(s)\}=min(t,s),$$
$$\mathcal{E}\{\nu(t)\nu(s)\}=min(t,s),$$
$$\mathcal{E}\{\beta(t)\nu(s)\}=0,\ \mathcal{E}\{\nu(t)\zeta(t)\}=0,\ \mathcal{E}\{\beta(t)\zeta(s)\}=0$$

In the uncertain case, we assume that the system matrices in (1a-c), lie within the following polytope:

$$\bar{\Omega}\stackrel{\Delta}{=}\begin{bmatrix} A & A_1 & B_1 & C & C_2 & D_{21} & F & G & H\end{bmatrix},\tag{3}$$

which is described by the vertices:

$$\bar{\Omega}=\mathcal{C}o\{\bar{\Omega}_1,\bar{\Omega}_2,...,\bar{\Omega}_N\},\tag{4}$$

where $\bar{\Omega}_i\stackrel{\Delta}{=}$

$$\begin{bmatrix} A^{(i)} & A_1^{(i)} & B_1^{(i)} & C^{(i)} & C_2^{(i)} & D_{21}^{(i)} & F^{(i)} & G^{(i)} & H^{(i)}\end{bmatrix}\tag{5}$$

and where N is the number of vertices. In other words:

$$\bar{\Omega} = \sum_{i=1}^{N} \bar{\Omega}_i f_i, \quad \sum_{i=1}^{N} f_i = 1, \quad f_i \geq 0. \qquad (6)$$

We treat the following problem:

i) Robust H_∞ Filtering:
We consider the system of (1a-c) where the system matrices lie within the polytope $\bar{\Omega}$ of (3). We consider an estimator of the following general form:

$$\begin{array}{l} d\hat{x}(t) = A_c\hat{x}(t)dt + B_c y(t), \\ \hat{z}(t) = C_c\hat{x}(t). \end{array} \qquad (7a,b)$$

We denote

$$e(t) = x(t) - \hat{x}(t) \quad \text{and} \quad \bar{z}(t) = z(t) - \hat{z}(t), \qquad (8)$$

and consider the following cost function:

$$J_F \stackrel{\Delta}{=} ||\bar{z}(t)||^2_{\tilde{L}_2} - \gamma^2[||w(t)||^2_{\tilde{L}_2} + ||n(t)||^2_{\tilde{L}_2}]. \qquad (9)$$

Given $\gamma > 0$, we seek an estimate $C_c\hat{x}(t)$ of $Cx(t)$ over the infinite time horizon $[0,\infty)$ such that J_F given by (9) is negative for all nonzero $w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^q)$, $n(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$.

3 PRELIMINARY RESULT

In this section we bring the solution of the continuous-time robust quadratic filtering problem of uncertain polytopic systems with state-multiplicative noise. The result in the sequel was obtained in (Gershon and Shaked, 2013), Chapter 2, Theorem 2.12).

Denoting $\xi^T(t) \stackrel{\Delta}{=} [x(t)^T \quad \hat{x}(t)^T]$, $\bar{w}^T(t) \stackrel{\Delta}{=} [w(t)^T \quad n(t)^T]$ we obtain the following augmented system:

$$\begin{array}{l} d\xi(t) = [\tilde{A}_0\xi(t) + \tilde{B}\bar{w}(t)]dt + \tilde{A}_1\xi(t-\tau(t))dt \\ +\tilde{H}\xi(t-\tau(t))d\nu(t) + \tilde{G}\xi(t)d\beta(t) + \tilde{F}\xi(t)d\zeta(t), \\ \xi(\theta) = 0, \; over[-h\; 0], \\ \tilde{z}(t) = \tilde{C}\xi(t), \end{array} \qquad (10)$$

where

$$\tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ B_cC_2 & A_c \end{bmatrix}, \tilde{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_cD_{21} \end{bmatrix},$$

$$\tilde{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \tilde{H} = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{G} = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}, \tilde{F} = \begin{bmatrix} 0 & 0 \\ B_cF & 0 \end{bmatrix}, \tilde{C} = [C_1 \; -C_c]. \qquad (11)$$

where the system matrices lie within the polytope $\bar{\Omega}$ of (3). We obtain the following lemma:

Lemma 1. (Gershon and Shaked, 2013) Consider the system of (1a-c) where the system matrices lie within the polytope $\bar{\Omega}$ of (3). For a prescribed scalar $\gamma > 0$ and tuning scalar ε_f, there exists a filter of the structure (7) that achieves (9) for all nonzero $w \in \tilde{L}^2([0,\infty);\mathcal{R}^q)$, $n \in \tilde{L}^2([0,\infty);\mathcal{R}^q)$, if there exist matrices $\bar{X} > 0$, $Y > 0$, K_0, U, $\bar{R}_1$, $\tilde{M}$, Z, that satisfy (13).

$$\Gamma_1^i = \left[\begin{array}{ccccc} \tilde{\Upsilon}_{i,11} & \tilde{\Upsilon}_{i,12} & \tilde{M} & \tilde{\Upsilon}_{i,14} & \tilde{\Upsilon}_{i,15} \\ * & -\bar{R}_1 & 0 & 0 & \tilde{\Upsilon}_{i,25} \\ * & * & -\varepsilon_f\bar{X}_Y & 0 & -\varepsilon_f h\tilde{M}^T \\ * & * & * & -\gamma^2 I_{2q} & \tilde{\Upsilon}_{i,45} \\ * & * & * & * & -\varepsilon_f\bar{X}_Y \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array} \right. \qquad (12)$$

$$\left. \begin{array}{cccc} \tilde{\Upsilon}_{i,16} & \tilde{\Upsilon}_{i,17} & \tilde{\Upsilon}_{i,18} & 0 \\ 0 & 0 & 0 & \tilde{\Upsilon}_{i,29} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I_r & 0 & 0 & 0 \\ ** & -\bar{X}_Y & 0 & 0 \\ * & * & -\bar{X}_Y & 0 \\ * & * & * & -\bar{X}_Y \end{array} \right] < 0, \qquad (13)$$

$\forall i, \; i = 1,2,....,N$, where

$$\tilde{\Upsilon}_{i,11} = \begin{bmatrix} \bar{X}A_0^i + A_0^{i,T}\bar{X} & \tilde{\Upsilon}_{i,11a} \\ YA_0^i + UC_2^i + K_0 + A_0^{i,T}\bar{X} & \tilde{\Upsilon}_{i,11b} \end{bmatrix},$$

$$+\tilde{M} + \tilde{M}^T + \tfrac{1}{1-d}\bar{R}_1,$$

$$\tilde{\Upsilon}_{i,11a} = \bar{X}A_0^i + A_0^{i,T}Y + C_2^{i,T}U^T + K_0^T,$$

$$\tilde{\Upsilon}_{i,11b} = YA_0^i + UC_2^i + A_0^{i,T}Y + C_2^{i,T}U^T$$

$$\tilde{\Upsilon}_{i,12} = \begin{bmatrix} \bar{X}A_1^i & \bar{X}A_1^i \\ YA_1^i & YA_1^i \end{bmatrix} - \tilde{M},$$

$$\tilde{\Upsilon}_{i,14} = \begin{bmatrix} \bar{X}B_1^i & 0 \\ YB_1^i & UD_{21}^i \end{bmatrix},$$

$$\tilde{\Upsilon}_{i,15} = \varepsilon_f h[\begin{bmatrix} A_0^{i,T}\bar{X} & A_0^{i,T}Y + C_2^{i,T}U^T + K_0^T \\ A_0^{i,T}\bar{X} & C_2^{i,T}U^T \end{bmatrix} + \tilde{M}^T],$$

$$\tilde{\Upsilon}_{i,17} = \begin{bmatrix} G^{i,T}\bar{X} & G^{i,T}Y \\ G^{i,T}\bar{X} & G^{i,T}Y \end{bmatrix}, \ \tilde{\Upsilon}_{i,18} = \begin{bmatrix} 0 & F^{i,T}U^T \\ 0 & F^{i,T}U^T \end{bmatrix}$$

$$\tilde{\Upsilon}_{i,25} = \varepsilon_f h \begin{bmatrix} A_1^{i,T}\bar{X} & A_1^{i,T}Y \\ A_1^{i,T}\bar{X} & A_1^{i,T}Y \end{bmatrix} + \varepsilon_f h \tilde{M}^T,$$

$$\tilde{\Upsilon}_{i,29} = \begin{bmatrix} H^{i,T}\bar{X} & H^{i,T}Y \\ H^{i,T}\bar{X} & H^{i,T}Y \end{bmatrix}, \quad \tilde{\Upsilon}_{i,45} = \varepsilon_f h \tilde{\Upsilon}_{i,14}^T, \tag{14}$$

and $\tilde{\Upsilon}_{i,16} = \begin{bmatrix} C^{i,T} - Z^T \\ -C^{i,T} \end{bmatrix}$. In the latter case the filter parameters can be extracted as follows:

$$A_c = N^{-T}K_0XM^{-1}, \ B_c = N^{-T}U, \ C_c = Z\bar{X}M^{-1}. \tag{15}$$

Proof: see (Gershon and Shaked, 2013), Section 2.6.1.

4 ROBUST h_∞ FILTERING

The solution of the filtering problem for uncertain systems has already appeared in (Gershon and Shaked, 2013) for the most conservative case where the same Lyapunov function is assigned all over the uncertainty polytope. In this section we show that relying on Lemma 1, one can manipulate the resulting LMI condition of the latter lemma in such a way that a vertex-dependent solution is obtained for a minimal number of tuning parameters. Eventually, the quadratic solution can be derived from our new vertex-dependent solution as a special case. We consider the result of Lemma 1 and we denote $E = \begin{bmatrix} I_n & 0 \\ -I_n & I_n \end{bmatrix}$ and $J_E = diag\{E,\ E,\ E,\ \hat{I},\ E,\ I_r,\ E,\ E,\ E\}$. We then consider the inequality of $\hat{J}_1^i < 0$ where,

$$\hat{J}_1^i \triangleq J_E \Gamma_1^i J_E^T = \begin{bmatrix} \hat{\Upsilon}_{i,11} & \hat{\Upsilon}_{i,12} & \hat{M} & \hat{\Upsilon}_{i,14} & \hat{\Upsilon}_{i,15} & \hat{\Upsilon}_{i,16} & \hat{\Upsilon}_{i,17} & \hat{\Upsilon}_{i,18} & 0 \\ * & -\hat{R}_1 & 0 & 0 & \hat{\Upsilon}_{i,25} & 0 & 0 & 0 & \hat{\Upsilon}_{i,29} \\ * & * & \hat{\Upsilon}_{i,33} & 0 & \hat{\Upsilon}_{i,35} & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2\hat{I} & \hat{\Upsilon}_{i,45} & 0 & 0 & 0 & 0 \\ * & * & * & * & \hat{\Upsilon}_{i,33} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -I_r & 0 & 0 & 0 \\ * & * & * & * & * & * & \hat{\Upsilon}_{i,33} & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\Upsilon}_{i,33} & 0 \\ * & * & * & * & * & * & * & * & \hat{\Upsilon}_{i,33} \end{bmatrix}, \tag{16}$$

$\hat{R}_1 = E\bar{R}_1E^T$, $\hat{M} = E\tilde{M}E^T$, $\hat{I} = \begin{bmatrix} I_q & 0 \\ -I_q & I_q \end{bmatrix}$,
$\hat{\Upsilon}_{i,33} = -\varepsilon_f diag\{\bar{X}^i, \Delta^i\}$, $\hat{\Upsilon}_{i,35} = -\varepsilon_f h\hat{M}^T$,

$$\hat{\Upsilon}_{i,11} = \begin{bmatrix} \bar{X}^iA_0^i & -\Delta^iA_c^T \\ \Delta^iA_0^i + B_cC_2^i & \Delta^iA_c^T \end{bmatrix} + \hat{M}$$
$$+ (\begin{bmatrix} \bar{X}^iA_0^i & -\Delta^iA_c^T \\ \Delta^iA_0^i + B_cC_2^i & \Delta^iA_c^T \end{bmatrix} + \hat{M})^T + \frac{1}{(1-d)}\hat{R}_1$$

$$\hat{\Upsilon}_{i,12} = \begin{bmatrix} \bar{X}^iA_1^i & 0 \\ \Delta^iA_1^i & 0 \end{bmatrix} - \hat{M},$$

$$\hat{\Upsilon}_{i,14} = \begin{bmatrix} \bar{X}^iB_1^i & -\bar{X}^iB_1^i \\ \Delta^iB_1^i & -\Delta^iB_1^i + B_cD_{21}^i \end{bmatrix},$$

$$\hat{\Upsilon}_{i,15} = \varepsilon_f h \begin{bmatrix} A_0^{i,T}\bar{X}^i & A_0^{i,T}\Delta^i + C_2^{i,T}B_c^T - \Delta^iA_c^T \\ 0 & -A_0^{i,T}\Delta^i - \Delta^iA_c^T \end{bmatrix}$$
$$+\varepsilon_f h\hat{M}^T,$$

$$\hat{\Upsilon}_{i,16} = \begin{bmatrix} C^{i,T} + \Delta^iC_c^T \\ -2C^{i,T} - \Delta^iC_c^T \end{bmatrix}, \ \hat{\Upsilon}_{i,17} = \begin{bmatrix} G^{i,T}\bar{X}^i & G^{i,T}\Delta^i \\ 0 & 0 \end{bmatrix}$$

$$\hat{\Upsilon}_{i,18} = \begin{bmatrix} 0 & F^{i,T}B_c^T \\ 0 & 0 \end{bmatrix}, \ \hat{\Upsilon}_{i,25} = \varepsilon_f h \begin{bmatrix} A_1^{i,T}\bar{X}^i & A_1^{i,T}\Delta^i \\ 0 & 0 \end{bmatrix}$$

$$+\varepsilon_f h\hat{M}^T, \ \hat{\Upsilon}_{i,29} = \begin{bmatrix} H^{i,T}\bar{X}^i & H^{i,T}\Delta^i \\ 0 & 0 \end{bmatrix},$$

$$\text{and} \ \ \hat{\Upsilon}_{i,45} = \varepsilon_f h\hat{\Upsilon}_{i,14}^T. \tag{17}$$

Remark 1: As given in (Gershon and Shaked, 2013), Section 2.6.1, since $YX + N^TM = I$ we chose $N = I$ and obtained that $K_0 = -A_c\Delta$ where $\Delta = Y - \bar{X}$. We also have then that $U = B_c$ and that $M\bar{X} = -\Delta$. We also denote:

$$\hat{\Delta}^i \triangleq diag\{\Delta^i,\ \Delta^i,\ 0_{5n+2q},\ \Delta^i,\ 0_{n+r},\ \Delta^i,\ 0_{3n},\ \Delta^i\},$$

$$\hat{X}^i \triangleq diag\{\bar{X}^i,\ 0_{5n+2q},\ \bar{X}^i,\ 0_{n+r},\ \bar{X}^i,\ 0_{3n},\ \bar{X}^i,\ 0_n\},$$

$$\Phi_\Delta^i = \left[\begin{array}{cccc} \begin{bmatrix} 0 & -A_c^T \\ A_0^i & A_c^T \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ A_1^i & 0 \end{bmatrix} & 0_{2n} & \begin{bmatrix} 0 & 0 \\ B_1^i & -B_1^i \end{bmatrix} \\ 0_{2n} & 0_{2n} & 0_{2n} & 0_{2n,2q} \\ 0_{2n} & 0_{2n} & 0_{2n} & 0_{2n,2q} \\ 0_{2q,2n} & 0_{2q,2n} & 0_{2q,2n} & 0_{2q} \\ \varepsilon_f h \begin{bmatrix} 0 & 0 \\ A_0^i & -A_0^i \end{bmatrix} & \varepsilon_f h \begin{bmatrix} 0 & 0 \\ A_1^i & 0 \end{bmatrix} & 0_{2n} & \varepsilon_f h \begin{bmatrix} 0 & 0 \\ B_1^i & -B_1^i \end{bmatrix} \\ 0_{r,2n} & 0_{r,2n} & 0_{r,2n} & 0_{r,2q} \\ \begin{bmatrix} 0 & 0 \\ G^i & 0 \end{bmatrix} & 0_{2n} & 0_{2n} & 0_{2n,2q} \\ 0_{2n} & 0_{2n} & 0_{2n} & 0_{2n,2q} \\ 0_{2n} & \begin{bmatrix} 0 & 0 \\ H^i & 0 \end{bmatrix} & 0_{2n} & 0_{2n,2q} \end{array} \right.$$

$$\left.\begin{matrix} \varepsilon_f h \begin{bmatrix} 0 & -A_c^T \\ 0 & -A_c^T \end{bmatrix} & \begin{bmatrix} C_c^T \\ -C_c^T \end{bmatrix} & 0_{2n,6n} \\ 0_{2n} & 0_{2n,r} & 0_{2n,6n} \\ 0_{2n} & 0_{2n,r} & 0_{2n,6n} \\ 0_{2q,2n} & 0_{2q,r} & 0_{2q,6n} \\ 0_{2n} & 0_{2n,r} & 0_{2n,6n} \\ 0_{r,2n} & 0_{r} & 0_{r,6n} \\ 0_{2n} & 0_{2n,r} & 0_{2n,6n} \\ 0_{2n} & 0_{2n,r} & 0_{2n,6n} \\ 0_{2n} & 0_{2n,r} & 0_{2n,6n} \end{matrix}\right]$$

and $\Phi_X^i =$

$$\begin{bmatrix} \Phi_{11}^i & \Phi_{12}^i & 0_{2n} & \Phi_{14}^i & 0_{2n} & 0_{2n,6n+r} \\ 0_{2n} & 0_{2n} & 0_{2n} & 0_{2n,2q} & 0_{2n} & 0_{2n,6n+r} \\ 0_{2n} & 0_{2n} & 0_{2n} & 0_{2n,2q} & 0_{2n} & 0_{2n,6n+r} \\ 0_{2q,2n} & 0_{2q,2n} & 0_{2q,2n} & 0_{2q,2q} & 0_{2q,2n} & 0_{2q,6n+r} \\ \Phi_{11}^i & \Phi_{12}^i & 0_{2n} & \Phi_{14}^i & 0_{2n} & 0_{2n,6n+r} \\ 0_{r,2n} & 0_{r,2n} & 0_{r,2n} & 0_{r,2q} & 0_{r,2n} & 0_{r,6n+r} \\ \Phi_{71}^i & 0_{2n} & 0_{2n} & 0_{2n,2q} & 0_{2n} & 0_{2n,6n+r} \\ 0_{2n} & 0_{2n} & 0_{2n} & 0_{2n,2q} & 0_{2n} & 0_{2n,6n+r} \\ 0_{2n} & \begin{bmatrix} H^i & 0 \\ 0 & 0 \end{bmatrix} & 0_{2n} & 0_{2n,2q} & 0_{2n} & 0_{2n,6n+r} \end{bmatrix} \quad \text{(18a-d)}$$

where

$$\Phi_{11}^i = \begin{bmatrix} A_0^i & 0 \\ 0 & 0 \end{bmatrix}, \quad \Phi_{12}^i = \begin{bmatrix} A_1^i & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Phi_{14}^i = \begin{bmatrix} B_1^i & -B_1^i \\ 0 & 0 \end{bmatrix}, \quad \Phi_{71}^i = \begin{bmatrix} G^i & 0 \\ 0 & 0 \end{bmatrix},$$

where 0_m and $0_{m,r}$ are the $m \times m$ and $m \times r$ matrices of zeros, respectively.

We also define:

$$\hat{J}_0^i = \hat{J}_1^i - \hat{\Delta}^i \Phi_\Delta^i - \Phi_\Delta^{i,T} \hat{\Delta}^i - \hat{X}^i \Phi_X^i - \Phi_X^{i,T} \hat{X}^i \qquad (19)$$

and build below the following requirement by applying the Finsler lemma

$$\hat{J}_0^i + \hat{\Delta}^i \Phi_\Delta^i + \Phi_\Delta^{T,i} \hat{\Delta}^i + \hat{X}^i \Phi_X^i + \Phi_X^{T,i} \hat{X}^i < 0 \qquad (20)$$

Denoting:

$$\begin{aligned} \hat{S}_1 &= diag\{S_1,\, S_1,\, 0_{5n+2q},\, S_1,\, 0_{n+r},\, S_1,\, 0_{3n},\, S_1\}, \\ \hat{S}_2 &= diag\{S_2,\, 0_{5n+2q},\, S_2,\, 0_{n+r},\, S_2,\, 0_{3n},\, S_2,\, 0_n\}, \\ \hat{H}_1 &= diag\{\varepsilon_1 S_1,\, S_1,\, I_{5n+2q},\, H_1,\, 0_{n+r},\, H_1,\, 0_{3n},\, H_1\}, \\ \hat{H}_2 &= diag\{H_2,\, I_{5n+2q},\, H_2,\, I_{n+r},\, H_2,\, I_{3n},\, H_2,\, 0_n\}, \end{aligned}$$

where ε_1 is a positive tuning parameter. We thus obtain the following theorem:

Theorem 1: Consider the system of (1a-c) where the system matrices lie within the polytope $\bar{\Omega}$ of (3). For a prescribed scalar $\gamma > 0$ and tuning positive scalars ε_f, ε_1, there exists a filter of the structure (7) that achieves (9) for all nonzero $w \in \tilde{L}^2([0,\infty);\mathcal{R}^q)$, $n \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$, if there exist a matrix B_c and positive definite matrices: Δ^i, $\hat{X}^i$, $i = 1,...,N$, H_1, H_2, S_1 and S_2 that, for $i = 1,...,N$, solve the following set of inequalities:

$$\begin{bmatrix} \bar{\Psi}^i & \bar{\Omega}_1^i & \bar{\Omega}_2^i \\ * & -2diag\{\tilde{H}_1, \tilde{H}_2\} \end{bmatrix} < 0, \quad i = 1,...,N \qquad (21)$$

where:

$$\bar{\Psi}^i \triangleq \begin{bmatrix} \hat{J}_0^i + \begin{bmatrix} \hat{S}_1 & \hat{S}_2 \end{bmatrix} \begin{bmatrix} \Phi_\Delta^i \\ \Phi_X^i \end{bmatrix} + \begin{bmatrix} \Phi_\Delta^{i,T} & \Phi_X^{i,T} \end{bmatrix} \begin{bmatrix} \hat{S}_1 \\ \hat{S}_2 \end{bmatrix} \end{bmatrix} =$$

$$\begin{bmatrix} \Psi_{i,11} & \Psi_{i,12} & \hat{M} & \Psi_{i,14} & \Psi_{i,15} & \Psi_{i,16} & \Psi_{i,17} & \Psi_{i,18} & 0 \\ * & -\hat{R}_1 & 0 & 0 & \Psi_{i,25} & 0 & 0 & 0 & \Psi_{i,29} \\ * & * & X_{\Delta^i}^i & 0 & -\varepsilon_f h \hat{M}^T & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 \hat{I} & \Psi_{i,45} & 0 & 0 & 0 & 0 \\ * & * & * & * & X_{\Delta^i}^i & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -I_r & 0 & 0 & 0 \\ * & * & * & * & * & * & X_{\Delta^i}^i & 0 & 0 \\ * & * & * & * & * & * & * & X_{\Delta^i}^i & 0 \\ * & * & * & * & * & * & * & * & X_{\Delta^i}^i \end{bmatrix} \qquad (22)$$

with: $\hat{I} = \begin{bmatrix} I_q & 0 \\ -I_q & I_q \end{bmatrix}$, $X_{\Delta^i}^i = -\varepsilon_f diag\{\bar{X}^i, \Delta^i\}$,

$$\Psi_{i,11} = \begin{bmatrix} S_2 A_0^i & -\hat{A}_F^T \\ S_1 A_0^i + B_c C_2^i & \hat{A}_F^T \end{bmatrix} + \hat{M}$$

$$+ [\begin{bmatrix} S_2 A_0^i & -\hat{A}_F^T \\ S_1 A_0^i + B_c C_2^i & \hat{A}_F^T \end{bmatrix} + \hat{M}]^T + \frac{1}{(1-d)} \hat{R}_1$$

$$\Psi_{i,12} = \begin{bmatrix} S_2 A_1^i & 0 \\ S_1 A_1^i & 0 \end{bmatrix} - \hat{M},$$

$$\Psi_{i,14} = \begin{bmatrix} S_2 B_1^i & -S_2 B_1^i \\ S_1 B_1^i & -S_1 B_1^i + B_c D_{21}^i \end{bmatrix},$$

$$\Psi_{i,15} = \varepsilon_f h \begin{bmatrix} A_0^{i,T} S_2 & A_0^{i,T} S_1 + C_2^{i,T} B_c^T - \hat{A}_F^T \\ 0 & -A_0^{i,T} S_1 - \hat{A}_F^T \end{bmatrix} + \varepsilon_f h \hat{M}^T$$

$$\Psi_{i,16} = \begin{bmatrix} C^{i,T} + \hat{C}_F^T \\ -2C^{i,T} - \hat{C}_F^T \end{bmatrix}, \Psi_{i,17} = \begin{bmatrix} G^{i,T} S_2 & G^{i,T} S_1 \\ 0 & 0 \end{bmatrix},$$

$$\Upsilon_{i,18} = \begin{bmatrix} 0 & F^{i,T} B_c^T \\ 0 & 0 \end{bmatrix}, \Psi_{i,25} = \varepsilon_f h \begin{bmatrix} A_1^{i,T} S_2 & A_1^{i,T} S_1 \\ 0 & 0 \end{bmatrix}$$

$$+\varepsilon_f h \hat{M}^T, \Psi_{i,29} = \begin{bmatrix} H^{i,T} S_2 & H^{i,T} S_1 \\ 0 & 0 \end{bmatrix},$$

$$\Psi_{i,45} = \varepsilon_f h \Psi_{i,14}^T, \qquad (23)$$

and where we denoted $\hat{A}_F = A_c S_1$ and $\hat{C}_F = C_c S_1$.

$$\bar{\Omega}_1^i =$$

$$\begin{bmatrix} \bar{\Omega}_{11}^i & \begin{bmatrix} A_0^{iT} H_1 \\ -A_0^{iT} H_1 \end{bmatrix} & \begin{bmatrix} G^{iT} H_1 \\ 0_n \end{bmatrix} & 0_{2n,n} \\ \bar{\Omega}_{21}^i & \begin{bmatrix} \varepsilon_f h A_1^{iT} H_1 \\ 0_n \end{bmatrix} & \begin{bmatrix} 0_n \\ 0_n \end{bmatrix} & \begin{bmatrix} H^{iT} H_1 \\ 0_n \end{bmatrix} \\ 0_{2n} & 0_{2n} & 0_{2n,n} & 0_n \\ \bar{\Omega}_{41}^i & h\varepsilon_f \begin{bmatrix} B_1^{iT} H_1 \\ -B_1^{iT} H_1 \end{bmatrix} & 0_{2n,n} & 0_{2n,n} \\ \bar{\Omega}_{51}^i & \begin{bmatrix} 0_n \\ \Delta^i - S_1 \end{bmatrix} & 0_{2n,n} & 0_{2n,n} \\ \bar{\Omega}_{61}^i & 0_{r,n} & 0_{r,n} & 0_{r,n} \\ 0_{2n} & 0_{2n,n} & \begin{bmatrix} 0_n \\ \Delta^i - S_1 \end{bmatrix} & 0_{2n,n} \\ 0_{4n,2n} & 0_{4n,n} & 0_{4n,n} & \begin{bmatrix} 0_{3n,n} \\ \Delta^i - S_1 \end{bmatrix} \end{bmatrix}$$

where

$$\bar{\Omega}_{11}^i = \begin{bmatrix} \Delta^i - S_1 & \varepsilon_1 A_0^{iT} S_1 \\ -\varepsilon_1 A_F & \Delta^i - S_1 + \varepsilon_1 A_F \end{bmatrix},$$
$$\bar{\Omega}_{21}^i = \begin{bmatrix} 0_n & \varepsilon_1 A_1^{iT} S_1 \\ 0_n & 0_n \end{bmatrix},$$
$$\bar{\Omega}_{41}^i = \begin{bmatrix} 0_{q,n} & \varepsilon_1 B_1^{iT} S_1 \\ 0_{q,n} & -\varepsilon_1 B_1^{iT} S_1 \end{bmatrix},$$
$$\bar{\Omega}_{51}^i = \begin{bmatrix} 0_n & 0_n \\ \varepsilon_1 A_F & -\varepsilon_1 A_F \end{bmatrix},$$
$$\bar{\Omega}_{61}^i = \begin{bmatrix} \varepsilon_1 C_F & 0 \end{bmatrix},$$

is the matrix that is obtained by deleting in the matrix $\hat{\Delta}^i - \hat{S}_1 + \Phi_\Delta^{iT} \hat{H}_1$ the $9n + 2q + r$ columns that are identically zero, and

$$\bar{\Omega}_2^i = \begin{bmatrix} A_0^{iT} H_2 + \bar{X}^i - S_2 & A_0^{iT} H_2 & G^{iT} H_2 & 0_n \\ 0_n & 0_n & 0_n & 0_n \\ A_1^{iT} H_2 & h\varepsilon_f A_1^{iT} H_2 & 0_n & H_1^{iT} H_2 \\ 0_{3n,n} & 0_{3n,n} & 0_{3n,n} & 0_{3n,n} \\ B_1^{iT} H_2 & B_1^{iT} H_2 & 0_{q,n} & 0_{q,n} \\ -B_1^{iT} H_2 & -B_1^{iT} H_2 & 0_{q,n} & 0_{q,n} \\ 0_{2n,n} & \begin{bmatrix} \bar{X}^i - S_2 \\ 0_n \end{bmatrix} & 0_{2n,n} & 0_{2n,n} \\ 0_{r,n} & 0_{r,n} & 0_{r,n} & 0_{r,n} \\ 0_n & 0_n & \bar{X}^i - S_2 & 0_n \\ 0_{3n,n} & 0_{3n,n} & 0_{3n,n} & 0_{3n,n} \\ 0_n & 0_n & 0_n & \bar{X}_i - S_2 \\ 0_n & 0_n & 0_n & 0_n \end{bmatrix}$$

is the matrix that is obtained by deleting the $10n + r + 2q$ zero columns in $\hat{X}^i - \hat{S}_2 + \Phi_X^{iT} \hat{H}_2$.

$$\hat{H}_1 = diag\{\varepsilon_1 S_1, S_1, H_1, H_1, H_1\},$$

and

$$\hat{H}_2 = diag\{H_2, H_2, H_2, H_2\}.$$

Proof: The result of (21) is obtained by applying the Finsler lemma twice. We first apply the latter to: $\hat{J}_2^i + \hat{\Delta}^i \Phi_\Delta^i + \Phi^{i,T} \hat{\Delta}^i < 0$, where $\hat{J}_2^i = \hat{J}_0^i + \hat{X}^i \Phi_X^i + \Phi_X^{i,T} \hat{X}^i$
and obtain the requirement:

$$\begin{bmatrix} \hat{J}_2^i + \hat{S}_1 \Phi_\Delta^i + \Phi_\Delta^{i,T} \hat{S}_1 & \hat{\Delta}^i - \hat{S}_1 + \Phi_\Delta^{i,T} \hat{H}_1 \\ * & -2\hat{H}_1 \end{bmatrix} < 0.$$

To the (1,1) block of the latter inequality we apply again the Finsler lemma and readily obtain (21).

Remark 2: In the above, we consider $S_1 A_C^T \triangleq A_F^T$ and $S_1 C_C^T \triangleq C_F^T$ in the LMI of (21) to be decision variables.

5 EXAMPLE

We consider the system (1a,c) which is described by:

$$A_0 = \begin{bmatrix} -2 \pm a & 0 \\ 1 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix},$$

$G = H = \sqrt{0.1} I_2$, with

$$B_2 = 0,\ D_{12} = 0,\ C_2 = \begin{bmatrix} 1 & 2 \end{bmatrix},$$

$F = \begin{bmatrix} 0.03 & 0.02 \end{bmatrix}$, $D_{21} = 0.01$ and

$\bar{C}_2 = [0\ 0]$, $C_1 = \begin{bmatrix} -0.5 & 1 \end{bmatrix}$, $d = 0$, where $a = 0.9$.

We assume that the stochastic multiplicative noise processes in both the delayed and the non delayed states are uncorrelated. Considering the nominal case where a=0 and taking $h = 0.1$ and applying the results of Theorem 2.10 in (Gershon and Shaked, 2013) for the near minimum attenuation level of $\gamma = 0.136$ is obtained.
Considering the uncertain case and applying the various solution methods, the results of Table 1 are obtained. There, the quadratic H_∞ solution method was calculated by assigning a single Lyapunov function over the uncertainty polytope and is given in Lemma 1. We note that in the case where $h = 0.8$ the H_∞ attenuation level is $\gamma = 41.16$ whereas the quadratic approach of Lemma 1 fails to solve the problem. We note also the significant reduction in the H_∞ system norm, in the improved vertex-dependent case, compared to the quadratic solution.

The resulting filter parameters for the robust case are:

$$A_C = \begin{bmatrix} -25.8337 & 10.2329 \\ -20.8500 & 0.4604 \end{bmatrix}, B_C = \begin{bmatrix} -0.3239 \\ -2.5225 \end{bmatrix},$$

$$C_C = [24.6821\ -14.0506].$$

Table 1: The results of the various solution methods in Example 1. The nominal solution for the H_∞ case was obtained in (Gershon and Shaked, 2013), Chapter 2, Theorem 2.10. The "Robust Vertex-dependent" (RVd) result refers to the application of the Finsler lemma in the solution of the robust case [Theorem 1].

Solution Method	γ
Nominal	0.136, (Theorem 2.10.)
Quadratic	0.61, (Lem. 1)
RVd	0.48, $\varepsilon_1 = 0.001$, $\varepsilon_f = 5$,[Thm. 1]

6 CONCLUSIONS

In this paper the theory of robust linear H_∞ estimation of state-multiplicative noisy systems is developed and extended for state-delayed continuous-time uncertain systems with multiplicative noise, that is encountered in both the dynamic and the measurement matrices in the state space model of the system. Sufficient conditions are derived for the estimation problem of uncertain polytopic-type systems by applying a vertex-dependent Lyapunov function, based on the Finsler lemma. This approach enables us to apply a unique Lyapunov function to each vertex of the uncertain polytope. As shown in the example, the vertex-dependent approach performs better the the solution which is based on a single Lyapunov function over all the uncertain polytope. We note also that our solution depends only on two tuning parameters that can be readily determined.

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