Verifying Positivity of Piecewise Quadratic Lyapunov Functions*

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Abstract: Continuous, piecewise quadratic (CPQ) Lyapunov functions are frequently used to assert stability for switched, cone-wise linear systems. It is advantageous to construct such Lyapunov functions in two steps: first a function is parameterized that is decreasing along all system trajectories, then it is verified whether this function is positive definite. Usually these steps have been performed using linear matrix inequalities (LMIs), but recently a linear programming (LP) approach for the first step has been suggested. In this paper we present a new algorithm to verify the positivity of CPQ Lyapunov function candidates, parameterized either with LMIs or LP. Further, we prove that the algorithm is non-conservative and will always be able to either assert positive definiteness of a CPQ Lyapunov function candidate or find a point where it is negative.

1 INTRODUCTION

Switched, cone-wise linear systems have received much interest in the control engineering community, in particular since the seminal works of Mikael Johansson and Anders Rantzer (Johansson and Rantzer, 1998; Johansson, 1999). For these systems, the stability of the origin has been, inter alia, asserted by using continuous and piecewise affine (CPA) Lyapunov functions, see e.g. (Andersen et al., 2023b; Andersen et al., 2023a), or by using continuous and piecewise quadratic (CPQ) Lyapunov functions as in the works of Johansson and Rantzer. Such systems have, for example, been successfully used to study hybrid integrator-gain systems (HIGS), see e.g. (van den Eijnden et al., 2020; Deenen et al., 2021; van den Eijnden et al., 2022).

Recently an algorithm different to the usual linear matrix inequality (LMI) approach for the computation of CPQ Lyapunov functions was presented, where linear programming (LP) is used instead of semi-definite optimization to parameterize CPQ Lyapunov functions, see (Palacios Roman et al., 2024; Andersen et al., 2024). Just as in the LMI approach, it is advantageous to construct the CPQ Lyapunov function in two steps; see Section 4.8 in (Johansson, 1999). First, a CPQ real-valued function V from the state-space is parameterized, that is decreasing along all system trajectories; this function is referred to as Lyapunov function candidate. In a second step it is verified whether V is positive definite or not. If V is positive definite, i.e. $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, then V is a Lyapunov function for the system and the origin is asymptotically stable. If there exists an $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $V(\mathbf{x}) < 0$, then the origin is unstable. Both of these properties follow from the fact that V is decreasing along all system trajectories. In more detail: If V is positive definite, then all solution trajectories must approach the minimum at the origin. If there is an $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $V(\mathbf{x}) < 0$, then $V(a\mathbf{x}) < 0$ for all $a > 0$, and solutions starting at a point $a\mathbf{x}$ must approach infinity, where the values of V are lower. It follows that it is impossible that V takes on negative values if the origin is asymptotically stable. Further, the third possibility, i.e. that $V(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ and there is an $\mathbf{y} \in \mathbb{R}^n$ different to the origin such that $V(\mathbf{y}) = 0$, is impossible; see Theorem 1.

Hence, dividing the algorithm into two steps is not only computationally more efficient, but asymptotic stability of the origin will be asserted if it is asymptotically stable, and as an added bonus, instability can be asserted for instable systems. The main contribution of this paper is the presentation of a new algorithm to verify the positivity of CPQ Lyapunov function candidates parameterized with the method from (Palacios Roman et al., 2024; Andersen et al.,

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2024) or by LMIs. We prove in Theorem 11 that this new method is non-conservative. Note that our new algorithm is computationally more efficient than the LMI approach from (Johansson, 1999), discussed in Section 2.4, which additionally introduces some conservatism; see e.g. (Scherer, 2006) for various LMI relaxation methods used in control theory. Further, our approach can be used to verify positivity for more general functions than piecewise quadratic. Comparison with more recent LMI approaches presented in (Kruszewski et al., 2009; Sala and Arino, 2007; Gonzaleza et al., 2017), which are sufficient and asymptotically necessary, will be the subject of a subsequent publication.

2 CPQ LYAPUNOV FUNCTIONS

In this paper we consider switched, cone-wise linear systems and CPQ Lyapunov functions that are positively homogeneous of order two. For this it is advantageous to first consider triangulations of a neighborhood of the origin of a specific type and then extend these triangulations to a conical subdivision of the whole state-space $\mathbb{R}^n$. Hence, we first define triangulations suitable for our application, before we discuss conical subdivisions, our class of systems, and CPQ Lyapunov functions.

2.1 Triangulations

A triangulation $\mathcal{T}$ of a set $\mathcal{D}_{\mathcal{T}} \subset \mathbb{R}^n$ is a set of n-simplices $\mathcal{T} := \{\mathfrak{S}_\nu \colon \nu \in I\}$, such that $\mathcal{D}_{\mathcal{T}} = \bigcup_{\nu \in I} \mathfrak{S}_\nu$; I is an index set. Recall that an n-simplex $\mathfrak{S}_\nu$ is defined as

$$\begin{aligned}\mathfrak{S}_\nu &:= \mathrm{co}\{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\}\\ &= \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i^\nu, \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1\right\},\end{aligned}$$

where the vectors $\mathbf{x}_i^\nu \in \mathbb{R}^n$ are called the vertices of $\mathfrak{S}_\nu$ and are assumed to be affinely independent, i.e. the vectors $\mathbf{x}_i^\nu - \mathbf{x}_0^\nu$, $i = 1, 2, \ldots, n$ are linearly independent. For our purposes, we additionally require that the triangulation $\mathcal{T}$ is shape-regular, i.e., every two different simplices $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$ either intersect in a common lower-dimensional face or do not intersect at all. This means that if $\mathfrak{S}_\nu \cap \mathfrak{S}_\mu \neq \emptyset$, then $\mathfrak{S}_\nu \cap \mathfrak{S}_\mu$ is a k-simplex, $0 \leq k < n$, whose vertices are the vertices common to $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$.

For our specific application we further demand that $\mathbf{x}_0^\nu = \mathbf{0}$ for all $\mathfrak{S}_\nu \in \mathcal{T}$ and that $\mathcal{D}_{\mathcal{T}}$ is a neighborhood of the origin $\mathbf{0} \in \mathbb{R}^n$.

An efficient implementation of a triangulation that satisfies these requirements is the triangular fan $\mathcal{T}_{K,\mathrm{fan}}^{\mathrm{std}}$, $K \in \mathbb{N} := \{1, 2, \ldots\}$, discussed in (Hafstein, 2019) where a formula for the vertices $\mathbf{x}_i^\nu$ is given. In the following we write $\mathcal{T}_K$ for $\mathcal{T}_{K,\mathrm{fan}}^{\mathrm{std}}$. The vertex set of the triangulation $\mathcal{T}_K$, i.e. the set of all vertices of all simplices, is

$$\{\mathbf{0}\} \cup \{\mathbf{z} \in \mathbb{Z}^n : \ \|\mathbf{z}\|_\infty = K\},$$

where the scaling parameter $K \in \mathbb{N}$ determines the fineness of the triangulation around zero. The number of simplices in the triangulation $\mathcal{T}_K$ is given by the formula $2^n \cdot K^{n-1} \cdot n!$.

For computations it is usually better to map the vertices $\mathcal{T}_K$ with the mapping $\mathbf{F} \colon \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{F}(\mathbf{x}) = \frac{\|\mathbf{x}\|_\infty}{\|\mathbf{x}\|_2}\mathbf{x}$. The resulting triangulation is denoted $\mathcal{T}_K^{\mathbf{F}}$ and the set $\mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}}$ is approximately spherically symmetric, see Figures 1 and 2.

For proving theorems, it is often convenient to scale down all the vertices of $\mathcal{T}_K$ by the factor K^{-1}. The resulting triangulation is denoted $K^{-1}\mathcal{T}_K$ and all its non-zero vertices are at the boundary of the unit hypercube $[-1, 1]^n$ and for any two different non-zero vertices $\mathbf{x}, \mathbf{y}$ of a simplex $\mathfrak{S}_\nu \in K^{-1}\mathcal{T}_K$ we have $\|\mathbf{x} - \mathbf{y}\|_\infty = K^{-1}$.

For a fixed $K \in \mathbb{N}$, all the triangulations $\mathcal{T}_K$, $\mathcal{T}_K^{\mathbf{F}}$, and $K^{-1}\mathcal{T}_K$ give the same conical subdivision of the state-space $\mathbb{R}^n$ discussed in the next section.

2.2 Conical Subdivision of the State-Space

Given a triangulation $\mathcal{T}$ as in the last section, one can define a corresponding conical subdivision of the state-space through

$$\mathfrak{C}_\nu := \{\mathbf{x} \in \mathbb{R}^n \colon c\mathbf{x} \in \mathfrak{S}_\nu \text{ for some } c > 0\}$$

for every $\mathfrak{S}_\nu \in \mathcal{T}$. Since $\mathfrak{S}_\nu = \mathrm{co}\{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\}$ it is easy to see that

$$\mathfrak{C}_\nu = \mathrm{cone}\{\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu\} := \left\{\sum_{i=1}^{n} \lambda_i \mathbf{x}_i^\nu \colon \lambda_i \geq 0\right\}.$$

Hence, every $\mathbf{x} \in \mathfrak{C}_\nu$ has a unique set of numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ such that $\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i^\nu$ because the $\mathbf{x}_i^\nu$ are linearly independent; recall that $\mathbf{x}_0^\nu = \mathbf{0}$. Since $\mathcal{T}$ is a triangulation of a neighborhood of the origin $\mathcal{D}_{\mathcal{T}}$, the set-theoretic union of all $\mathfrak{C}_\nu$ is equal to $\mathbb{R}^n$.

2.3 Switched Linear Systems and CPQ Lyapunov Functions

We consider systems of the form

$$\dot{\mathbf{x}}(t) = A_{s(t)}\mathbf{x}(t), \ \ A_j \in \mathbb{R}^{n\times n} \ \text{ for } j \in \{1, 2, \ldots, M\},$$

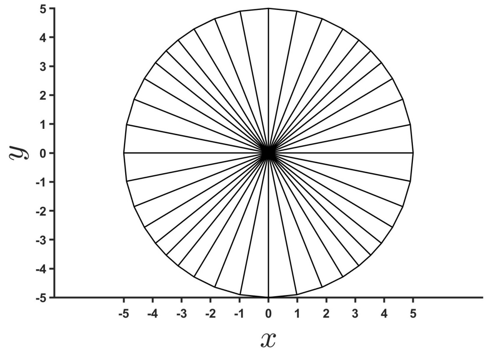

Figure 1: The triangulation $\mathcal{T}_K^{\mathbf{F}}$ in two dimensions and with $K = 5$.

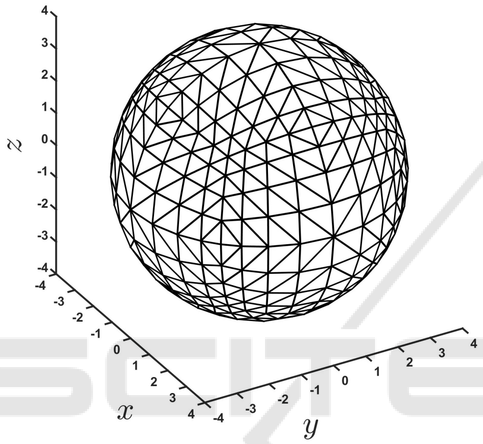

Figure 2: The triangulation $\mathcal{T}_K^{\mathbf{F}}$ in three dimensions and with $K = 4$. Note that every simplex in $\mathcal{T}_K^{\mathbf{F}}$ is a tetrahedron with zero as a vertex, together with three other vertexes at the boundary of a sphere centered at the origin and with radius K.

where $s\colon [0,\infty) \to \{1,2,\ldots,M\}$, $M \in \mathbb{N}$, is a right-continuous, piecewise constant function, called switching signal, with a only a finite number of discontinuity points, called switching times, on any finite time interval. The switching signal can either be arbitrary, in which case the systems is said to be arbitrary switched, or one can introduce restrictions in the form $\sigma(t) = j$ only if $\mathbf{x}(t) \in \mathcal{F}_j$, where the $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_M \subset \mathbb{R}^n$ are closed simplicial cones with the apex at the origin, similar to the $\mathfrak{C}_\nu$s above, fulfilling that the $\bigcup_{j=1}^{M} \mathcal{F}_j = \mathbb{R}^n$ and the intersection of the interiors $\mathcal{F}_j^\circ$ and $\mathcal{F}_k^\circ$ of two different cones $\mathcal{F}_j$ and $\mathcal{F}_k$ is empty. In the latter case the system is said to have state-dependent switching; see (Palacios Roman et al., 2024) for more details.

The solutions to arbitrary switched systems are understood in the sense of Carathéodory, see e.g. (Walter, 1998), and the solutions to systems with state-dependent switching are understood in the sense of Filippov, see (Filippov, 1988) or e.g. (Camlibel and Pang, 2006) for cone-wise linear systems, which takes care of sliding modes. For both arbitrary switched systems and systems with state-dependent switching, the origin is said to be globally uniformly exponentially stable (GUES) if there exist constants $c \geq 1$, $\lambda > 0$ such that all solutions fulfill

$$\|\mathbf{x}(t)\|_2 \leq ce^{-\lambda t}\|\mathbf{x}(0)\|_2 \text{ for all } t \geq 0.$$

In both cases the GUES of the origin can be asserted with the existence of a CPQ Lyapunov function fulfilling: Let $\{\mathfrak{C}_\nu\}_{\nu\in I}$ be a conical subdivision of the state-space as in Section 2.2 and assume $V\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that for, w.l.o.g. symmetric, matrices $P_\nu \in \mathbb{R}^{n\times n}$ we have

$$V(\mathbf{x}) = \mathbf{x}^T P_\nu \mathbf{x} \quad \text{if } \mathbf{x} \in \mathfrak{C}_\nu. \tag{1}$$

Assume that the matrices $P_\nu \in \mathbb{R}^{n\times n}$ fulfill for some constants $c_1, c_2, c_3 > 0$, that for all $\nu \in I$ and all $j \in \{1,2,\ldots,M\}$ we have

$$c_1\|\mathbf{x}\|_2^2 \leq \mathbf{x}^T P_\nu \mathbf{x} \leq c_2\|\mathbf{x}\|_2^2,\ \forall \mathbf{x} \in \mathfrak{C}_\nu, \tag{2}$$

$$\mathbf{x}^T(A_j^T P_\nu + P_\nu A_j)\mathbf{x} \leq -c_2\|\mathbf{x}\|_2^2,\ \forall \mathbf{x} \in \mathfrak{C}_\nu \cap \mathcal{F}_j, \tag{3}$$

where in (3) we set $\mathcal{F}_j := \mathbb{R}^n$ for all j in the case of arbitrary switched systems.

We emphasize: if the conditions (2) and (3) are fulfilled for the arbitrary switched system or the system with state-dependent switching, the function V is called a CPQ Lyapunov function for the system and the origin is GUES; see e.g. Theorems 3 and 5 in (Palacios Roman et al., 2024).

2.4 Computing CPQ Lyapunov Functions

Both the LMI method from (Johansson and Rantzer, 1998; Johansson, 1999) and the LP method from (Palacios Roman et al., 2024; Andersen et al., 2024) parameterize a CPQ Lyapunov function candidate of the form (1) fulfilling the conditions (3). The conditions (2) are then verified a posteriori for the candidate using the LMI:

For every $\nu \in I$ find a symmetric matrix $U_\nu \in \mathbb{R}^{n\times n}$ with entries $[U_\nu]_{ij} \geq 0$ such that

$$P_\nu - (X_\nu^{-1})^T U_\nu X_\nu^{-1} \succeq 0, \tag{4}$$

where $\succeq 0$ means that the matrix on the left-hand-side is symmetric and positive definite and

$$X_\nu := \begin{bmatrix} \mathbf{x}_1^\nu & \mathbf{x}_2^\nu & \ldots & \mathbf{x}_n^\nu \end{bmatrix} \in \mathbb{R}^{n\times n}$$

has the non-zero vertices of $\mathfrak{S}_\nu$ as its columns. Since for every $\mathbf{x} \in \mathfrak{C}_\nu$ there are unique $\boldsymbol{\lambda} =$

$(\lambda_1,\lambda_2,\ldots,\lambda_n)^T$, $\lambda_i \geq 0$, such that $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{x}_i^\nu$, the condition (4) implies for every such $\mathbf{x} \in \mathfrak{C}_\nu$ that

$$\begin{aligned} 0 &\leq \mathbf{x}^T(P_\nu - (X_\nu^{-1})^T U_\nu X_\nu^{-1})\mathbf{x} \qquad (5)\\ &= \mathbf{x}^T P_\nu \mathbf{x} - \boldsymbol{\lambda}^T U \boldsymbol{\lambda} \\ &\leq \mathbf{x}^T P_\nu \mathbf{x}. \end{aligned}$$

The fact that V fulfills the constraints 2 now easily follows from the fact that V is continuous, homogeneous of order two, and that $V(\mathbf{x}) = 0$ for $\mathbf{x} \neq \mathbf{0}$ is impossible if $V(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ by the next theorem; see also Section 3.2.

Theorem 1. *Assume V is of the form* (1) *and fulfills the conditions* (3). *If $V(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$, then $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.*

Proof. Let V fulfill the assumptions of the theorem and assume $V(\boldsymbol{\xi}) = 0$, $\boldsymbol{\xi} \neq \mathbf{0}$, and consider a solution $\mathbf{x}(t)$ starting at $\mathbf{x}(0) = \boldsymbol{\xi}$ for an arbitrary switched system. Let $h > 0$ be so small that $\mathbf{x}(t) \neq \mathbf{0}$ for all $0 \leq t \leq h$, which is possible because $\mathbf{x}(t)$ is continuous, and so small that no switching occurs on the interval $[0,h]$. Then for appropriate $\nu \in I$ and $j \in \{1,2,\ldots,M\}$ we have

$$\begin{aligned} V(\mathbf{x}(h)) &= V(\mathbf{x}(h)) - V(\mathbf{x}(0)) = \int_0^h \frac{d}{dt} V(\mathbf{x}(t))dt \\ &= \int_0^h \mathbf{x}(t)^T (A_j^T P_\nu + P_\nu A_j)\mathbf{x}(t)dt \\ &\leq -c_3 \int_0^h \|\mathbf{x}(t)\|_2^2 dt < 0 \qquad (6) \end{aligned}$$

which contradicts $V(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$.

For systems with state-dependent switching this follows similarly, but by the Fundamental Theorem of Calculus for Lebesgue Integrals, see e.g. Chapter III, Section 10 in (Walter, 1998). For Filippov solutions we have

$$\dot{\mathbf{x}}(t) = \sum_{j=1}^M \lambda_j(t) A_j \mathbf{x}(t), \quad \sum_{j=1}^M \lambda_j(t) = 1, \ \text{a.e.},$$

for some non-negative functions λ_j and since a.e.

$$\begin{aligned} \frac{d}{dt} V(\mathbf{x}(t)) &= \dot{\mathbf{x}}(t) P_\nu \mathbf{x}(t) + \mathbf{x}(t)^T P_\nu \dot{\mathbf{x}}(t) \\ &= \sum_{j=1}^M \lambda_j(t) \mathbf{x}(t)^T (A_j^T P_\nu + P_\nu A_j)\mathbf{x}(t) \\ &\leq -c_3 \|\mathbf{x}(t)\|_2^2 \end{aligned}$$

we can conclude, similarly as in (6), that $V(\mathbf{x}(t)) < 0$ for small enough $t > 0$, in contradiction to $V(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$.

□

Although the LMIs conditions (4) are sufficient to assert the conditions (2) for a CPQ Lyapunov function, they are not necessary. In the following section we propose a different method that is both sufficient and necessary, and, as an added bonus, computationally less demanding.

Remark 2. Note that the condition $\mathbf{x}^T P_\nu \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathfrak{C}_\nu$ in (5) is nothing else than the condition of copositivity of the matrix $X_\nu^T P_\nu X_\nu$, i.e.

$$\boldsymbol{\lambda}^T X_\nu^T P_\nu X_\nu \boldsymbol{\lambda} \geq 0 \ \text{ for all } \ \boldsymbol{\lambda} \in \mathbb{R}_+^n := [0,\infty)^n.$$

This problem of deciding whether a matrix is copositive is known to be co-NP-complete; for more details on copositive matrices see e.g. (Ikramov and Savel-eva, 2000).

3 VERIFICATION OF POSITIVITY

We start by discussing in general how the positivity of a function defined on a simplex can be verified. The following lemma, proved as Lemma 4.16 in (Marinósson, 2002) using Taylor-expansions, is fundamental for our approach:

Lemma 3. *On an m-simplex $\mathfrak{S} := \text{co}\{\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_m\} \subset \mathbb{R}^n$, $m \leq n$, we have for a function $g \in C^2(U)$, $U \subset \mathbb{R}^n$ open neighborhood of $\mathfrak{S}$, and every $\mathbf{x} = \sum_{i=0}^m \lambda_i \mathbf{x}_i \in \mathfrak{S}$ and any $d \in \{0,1,\ldots,n\}$, that*

$$\left| g\Big(\sum_{i=0}^m \lambda_i \mathbf{x}_i\Big) - \sum_{i=0}^m \lambda_i g(\mathbf{x}_i) \right| \leq \sum_{i=0}^m \lambda_i E_i, \qquad (7)$$

where

$$E_i \geq \sum_{r,s=1}^n \frac{B_{rs}}{2} |[\mathbf{x}_i - \mathbf{x}_d]_r| (|[\mathbf{x} - \mathbf{x}_d]_s| + |[\mathbf{x}_i - \mathbf{x}_d]_s|), \quad (8)$$

$[\mathbf{y}]_i$ is the ith component of the vector $\mathbf{y}$ and

$$B_{rs} := \max_{\mathbf{x} \in \mathfrak{S}} \left| \frac{\partial^2 g}{\partial x_r \partial x_s}(\mathbf{x}) \right|. \qquad \square$$

This lemma can be used to rigorously verify computationally whether $g(\mathbf{x}) \geq 0$ on an m-simplex $\mathfrak{S} := \text{co}\{\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_m\} \subset \mathbb{R}^n$, $m \leq n$.

Test for Positivity 4. *The test consist of the three following steps:*

1. *If $g(\mathbf{x}_i) < 0$ for some $i \in \{0,1,\ldots,m\}$ then clearly $g(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in \mathfrak{S}$ does not hold true.*
2. *If*

$$g(\mathbf{x}_i) - E_i \geq 0 \ \text{ for } i = 0,1,\ldots,m, \qquad (9)$$

then, because

$$\begin{aligned} g(\mathbf{x}) &= g(\mathbf{x}) - \sum_{i=0}^{m} \lambda_i g(\mathbf{x}_i) + \sum_{i=0}^{m} \lambda_i g(\mathbf{x}_i) \\ &\geq \sum_{i=0}^{m} \lambda_i g(\mathbf{x}_i) - |g(\mathbf{x}) - \sum_{i=0}^{m} \lambda_i g(\mathbf{x}_i)| \\ &\geq \sum_{i=0}^{m} \lambda_i (g(\mathbf{x}_i) - E_i) \geq 0, \end{aligned}$$

we have $g(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in \mathfrak{S}$.

3. *If neither of the criteria above hold true, i.e. if $g(\mathbf{x}_i) \geq 0$ for all $i = 0, 1, \ldots, m$ but there is an $i \in \{0, 1, \ldots, m\}$ such that $g(\mathbf{x}_i) - E_i < 0$, then the test is inconclusive.*

In the inconclusive case one can subdivide the simplex $\mathfrak{S}$ into smaller m-simplices and verify whether $g(\mathbf{x}) \geq 0$ on these smaller simplices or not with the same method. If we can guaranty that the bounds E_is approach zero as the simplices get smaller and smaller, this will indeed give us an algorithm that asserts computationally that $g(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathfrak{S}$ if $\min_{\mathbf{x}\in\mathfrak{S}} g(\mathbf{x}) > 0$. Further, if the vertices of the ever further subdivided simplices build a dense set in $\mathfrak{S}$, the method will also deliver a point $\mathbf{x} \in \mathfrak{S}$ with $g(\mathbf{x}) < 0$ if $\min_{\mathbf{x}\in\mathfrak{S}} g(\mathbf{x}) < 0$. Both these conditions are easy to guaranty if the diameter of the simplices converges to zero as we keep on subdividing the simplices; we prove this in Corollary 8 and Lemma 10.

3.1 Some Notes on the Bounds E_i

For the term $|[\mathbf{x} - \mathbf{x}_d]_s|$ in (8) we can use $\max_{j\in\{0,1,\ldots,m\}} |[\mathbf{x}_j - \mathbf{x}_d]_s|$ as an upper bound, which is independent of $\mathbf{x}$, because $\mathbf{x} = \sum_{j=0}^{m} \lambda_j \mathbf{x}_j$ and

$$\begin{aligned} \left| \left[\sum_{j=0}^{m} \lambda_j \mathbf{x}_j - \mathbf{x}_d \right]_s \right| &= \left| \sum_{j=0}^{m} \lambda_j \left[\mathbf{x}_j - \mathbf{x}_d\right]_s \right| \qquad (10) \\ &\leq \sum_{j=0}^{m} \lambda_j \left|[\mathbf{x}_j - \mathbf{x}_d]_s\right| \\ &\leq \max_{j\in\{0,1,\ldots,m\}} |[\mathbf{x}_j - \mathbf{x}_d]_s|. \end{aligned}$$

Hence, $\max_{j\in\{0,1,\ldots,m\}} |[\mathbf{x}_j - \mathbf{x}_d]_s|$ can be substituted for $|[\mathbf{x} - \mathbf{x}_d]_s|$ in the formula on the right-hand-side of (8).

A less conservative bound for right-hand-side of (8), shown analogously, is given by

$$\begin{aligned} &\sum_{r,s=1}^{n} \frac{B_{rs}}{2} |[\mathbf{x}_i - \mathbf{x}_d]_r| (|[\mathbf{x} - \mathbf{x}_d]_s| + |[\mathbf{x}_i - \mathbf{x}_d]_s|) \qquad (11) \\ &\leq \max_{j\in\{0,1,\ldots,m\}} \sum_{r,s=1}^{n} \frac{B_{rs}}{2} |[\mathbf{x}_i - \mathbf{x}_d]_r| \times \\ &\qquad (|[\mathbf{x}_j - \mathbf{x}_d]_s| + |[\mathbf{x}_i - \mathbf{x}_d]_s|). \end{aligned}$$

However, using these tighter bound is computationally somewhat more involved.

Further, one can choose the $d \in \{0, 1, \ldots, m\}$ in formula (8) freely, or try different ones and select the best according to some criteria, but note that one must use the same d for all the E_is for (7) to hold true. A rather straight-forward a priori choice is to select d such that

$$g(\mathbf{x}_d) \leq g(\mathbf{x}_i) \quad \text{for } i = 0, 1, \ldots, m,$$

because we can set $E_d = 0$ and this automatically delivers that (9) holds true for $i = d$ if $g(\mathbf{x}_d) \geq 0$.

3.2 Positivity of CPQ Functions

For a CPQ Lyapunov functions candidate V as in (1), that fulfills the conditions (3), we want to assert whether the conditions (2) hold true or not. The conditions (2) hold true, if for every $\nu \in I$ we have that $V(\mathbf{x}) = \mathbf{x}^T P_\nu \mathbf{x} > 0$ for every $\mathbf{x} \in \mathfrak{C}_\nu \setminus \{\mathbf{0}\}$; just note that since V is continuous and positively homogenous of order two, i.e. $V(s\mathbf{x}) = s^2 V(\mathbf{x})$ for every $s > 0$, we have with

$$0 < c_1 := \min_{\|\mathbf{x}\|_2=1} V(\mathbf{x}) \quad \text{and} \quad c_2 := \max_{\|\mathbf{x}\|_2=1} V(\mathbf{x})$$

that

$$c_1 \|\mathbf{x}\|_2^2 \leq V(\mathbf{x}) \leq c_2 \|\mathbf{x}\|_2^2.$$

To verify that $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathfrak{C}_\nu \setminus \{\mathbf{0}\}$, it suffices to verify that $V(\overline{\mathbf{x}}) \geq 0$ for all $\overline{\mathbf{x}} \in \mathfrak{S}_\nu^* := \operatorname{co}\{\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu\}$, as for every $\mathbf{x} \in \mathfrak{C}_\nu$ there is a unique set of numbers $\lambda_i \geq 0$ such that $\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i^\nu$ and for $\mathbf{x} \neq \mathbf{0}$ we have $\overline{\lambda} := \sum_{i=1}^{n} \lambda_i > 0$. Hence, $\overline{\mathbf{x}} := \mathbf{x}/\overline{\lambda} \in \mathfrak{S}_\nu^*$ and $V(\mathbf{x}) = \overline{\lambda}^2 V(\overline{\mathbf{x}}) \geq 0$; by Theorem 1 this implies that indeed $V(\mathbf{x}) > 0$. Note that $\mathfrak{S}_\nu^*$ is the face of $\mathfrak{S}_\nu$ obtained by removing the vertex $\mathbf{x}_0^\nu = \mathbf{0}$.

One might be tempted to think that to verify the positivity of V on $\mathfrak{C}_\nu \setminus \{\mathbf{0}\}$ it might be enough to check the positivity at the non-zero vertices of $\mathfrak{S}_\nu$, or maybe all vertices and all midpoints between vertices $(\mathbf{x}_i^\nu + \mathbf{x}_j^\nu)/2$, $i, j = 0, 1, \ldots, n$, because the values of $V(\mathbf{x}) = \mathbf{x}^T P_\nu \mathbf{x}$ at these points completely determine V, see e.g. Theorem 2.8 in (Giesl et al., 2025). However, as the next remark shows, this is not the case.

Remark 5. Consider the quadratic function

$$P(x, y) = \left(y - \frac{3}{4}x\right)^2 - \frac{1}{128}xy$$

on the triangle/simplex $\operatorname{co}\{(0,0)^T, (1,0)^T, (1,1)^T\}$. At the vertices we have $P(0,0) = 0$, $P(0,1) = 9/16 > 0$, and $P(1,1) = 7/128 > 0$ and at the midpoints between the vertices we have $P(1/2, 0) = 9/64 > 0$, $P(1/2, 1/2) = 7/512 > 0$, and $P(1, 1/2) = 15/256 >$

0. However, by construction $P(x,3x/4) = -3x/512 < 0$ for all $0 < x \leq 1$.

Similarly, one can also check that on the triangle/simplex $\mathrm{co}\{(1/2,0)^T,(1,0)^T,(1,1)^T\}$ the function P is strictly larger than zero at all vertices and all midpoints between vertices, as $P(3/4,0) = 81/256 > 0$ and $P(3/4,1/2) = 1/1024 > 0$, although P is not positive over the triangle/simplex.

$\square$

We will use Lemma 3 to verify the positivity of V on $\mathfrak{S}_\nu^*$. The constants B_{rs} in the E_is are easy to get:

$$B_{rs} = \left|\frac{\partial^2}{\partial x_r \partial x_s}\mathbf{x}^T P_\nu \mathbf{x}\right| = 2|[P_\nu]_{rs}|.$$

For the rest of the terms in E_i we could, for example, use (10) or (11).

Another strategy could be use less tight bounds and use formulas for the E_i that can be computed more quickly. For example, with the triangulation $K^{-1}\mathcal{T}_K$ of $[-1,1]^n$ from Section 2.1, we have for every simplex $\mathfrak{S}_\nu$ and the face $\mathfrak{S}_\nu^*$ at the boundary of $[-1,1]^n$ that $\|\mathbf{x}_i^\nu\|_\infty = 1$ and $\|\mathbf{x}_i^\nu - \mathbf{x}_d^\nu\|_\infty \leq 1/K$ for all $i = 1,2,\ldots,n$, and $\|\mathbf{x}-\mathbf{x}_d\|_\infty \leq 1/K$ and we can set

$$E_i = \frac{2}{K^2}\sum_{r,s=1}^{n}|[P_\nu]_{rs}|.$$

Hence, if

$$\begin{aligned}(\mathbf{x}_i^\nu)^T P_\nu \mathbf{x}_i^\nu &- \frac{2}{K^2}\sum_{r,s=1}^{n}|[P_\nu]_{rs}| \qquad (12)\\ &= \sum_{r,s=1}^{n}\left([\mathbf{x}_i^\nu]_r[\mathbf{x}_i^\nu]_s[P_\nu]_{rs} - \frac{2}{K^2}|[P_\nu]_{rs}|\right) \geq 0\end{aligned}$$

for $i = 1,2,\ldots,n$, then $V(\mathbf{x}) = \mathbf{x}^T P_\nu \mathbf{x} > 0$ for all $\mathbf{x} \in \mathfrak{C}_\nu \setminus \{\mathbf{0}\}$. Note that we can actually skip one i in the test (12), as we can choose our d with $E_d = 0$ freely. Hence, for $\mathbf{x}_d^\nu$ such that $0 < V(\mathbf{x}_d^\nu) \leq V(\mathbf{x}_i^\nu)$ for all $i = 1,2,\ldots,n$ we don't need (12) to hold true for $i = d$.

As we discussed before, if the test is inconclusive, i.e.

$$\begin{aligned}&V(\mathbf{x}_i^\nu) > 0 \quad \text{ for all } i = 1,2,\ldots,n \text{ but,}\\ &V(\mathbf{x}_j^\nu) - E_j < 0 \quad \text{ for some } j \in \{1,2,\ldots,n\}\,,\end{aligned}$$

then we can subdivide the simplex $\mathfrak{S}_\nu^* := \mathrm{co}\{\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu\}$ into smaller simplices, such that the E_is are smaller, and try again. This can then be repeated for those sub-simplices where the test is inconclusive. Before we prove that such an algorithm always succeeds in Theorem 11, be discuss the subdivision of simplices in the next section.

4 SUBDIVISION OF SIMPLICES

To describe the subdivision we use, it is advantageous to use a little different notations. For a permutation $\sigma \in S_m$, i.e. a one-to-one $\sigma\colon \{1,2,\ldots,m\} \to \{1,2,\ldots,m\}$, and a number $a > 0$, define the m-simplex

$$\mathfrak{S}_\sigma^a := a\cdot \mathrm{co}\{\mathbf{x}_0^\sigma, \mathbf{x}_1^\sigma, \ldots, \mathbf{x}_m^\sigma\} \subset \mathbb{R}^m,$$

where, for $i = 0,1,\ldots,m$,

$$\mathbf{x}_i^\sigma := \sum_{j=1}^{i}\mathbf{e}_{\sigma(j)}.$$

Here $\mathbf{e}_i$ denotes the standard ith unit vector in $\mathbb{R}^m$ and recall that the empty sum is defined as zero, i.e. $\sum_{j=1}^{0}\mathbf{e}_{\sigma(j)} = \mathbf{0} \in \mathbb{R}^m$

Note that for a vector $\mathbf{x} \in \mathfrak{S}_\sigma^a$ we have

$$\begin{aligned}\mathbf{x} &= a\sum_{i=0}^{m}\lambda_i\mathbf{x}_i^\sigma = a\sum_{i=0}^{m}\lambda_i\sum_{j=1}^{i}\mathbf{e}_{\sigma(j)}\\ &= a\sum_{i=1}^{m}\left(\sum_{j=i}^{m}\lambda_j\right)\mathbf{e}_{\sigma(i)}\end{aligned}$$

and because $\lambda_j \geq 0$ for all j this means that the components of $\mathbf{x} = (x_1,x_2,\ldots,x_m)^T$, $x_{\sigma(i)} = \sum_{j=i}^{m}\lambda_j$, fulfill

$$a \geq x_{\sigma(1)} \geq x_{\sigma(2)} \geq \ldots \geq x_{\sigma(m)} \geq 0. \qquad (13)$$

Indeed, it is not difficult to see that

$$\begin{aligned}\lambda_m &:= \frac{1}{a}x_{\sigma(m)},\\ \lambda_{m-1} &:= \frac{1}{a}\left(x_{\sigma(m-1)} - x_{\sigma(m)}\right)\\ \lambda_{m-2} &:= \frac{1}{a}\left(x_{\sigma(m-2)} - x_{\sigma(m-1)}\right)\\ &\;\;\vdots\\ \lambda_1 &:= \frac{1}{a}\left(x_{\sigma(1)} - x_{\sigma(2)}\right)\\ \lambda_0 &= \frac{1}{a}\left(a - x_{\sigma(1)}\right)\end{aligned}$$

and that the simplex $\mathfrak{S}_\sigma^a$ is the set of those vectors $\mathbf{x} \in \mathbb{R}^m$ that fulfill (13).

Now consider the simplex $\mathfrak{S}_\sigma^1$ for a permutation $\sigma \in S_m$ and a vector $\mathbf{y} = (y_1,y_2,\ldots,y_m)^T$ with $y_i \in \{0,1\}$ for $i = 1,2,\ldots,m$. We want to find a permutation $\alpha \in S_m$ such that

$$\mathbf{y} + \mathfrak{S}_\sigma^1 \subset \mathfrak{S}_\alpha^2. \qquad (14)$$

To this end consider the matrix

$$\begin{bmatrix}\sigma(1) & \sigma(2) & \cdots & \sigma(m)\\ y_{\sigma(1)} & y_{\sigma(2)} & \cdot & y_{\sigma(m)}\end{bmatrix} \qquad (15)$$

and let

$$A_1 := \{i \in \{1,2,\ldots,m\} : y_i = 1\} = \{i_1, i_2, \ldots, i_k\},$$
$$A_0 := \{i \in \{1,2,\ldots,m\} : y_i = 0\} = \{i_{k+1}, i_{k+2}, \ldots, i_m\},$$

where $i_1 < i_2 < \ldots < i_k$ and $i_{k+1} < i_{k+2} < \ldots < i_m$.

Now rearrange the columns in the matrix in (15) such that

$$\begin{bmatrix} \sigma(i_1) & \cdots & \sigma(i_k) & \sigma(i_{k+1}) & \cdots & \sigma(i_m) \\ y_{\sigma(i_1)} & \cdot & y_{\sigma(i_k)} & y_{\sigma(i_{k+1})} & \cdots & y_{\sigma(i_m)} \end{bmatrix}$$
$$= \begin{bmatrix} \sigma(i_1) & \cdots & \sigma(i_k) & \sigma(i_{k+1}) & \cdots & \sigma(i_m) \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}$$

Now define $\alpha \in S_m$ through

$$\alpha(j) = \sigma(i_j), \quad j = 1,2,\ldots,m. \tag{16}$$

Since α is the composition of the two permutations σ and $j \mapsto i_j$ in S_m, it is clear that $\alpha \in S_m$. For $\mathbf{z} \in \mathbf{y} + \mathfrak{S}^1_\sigma$ we will show that

$$2 \geq z_{\alpha(1)} \geq z_{\alpha(2)} \geq \ldots \geq z_{\alpha(m)} \geq 0,$$

i.e. that $\mathbf{z} \in \mathfrak{S}^2_\alpha$.

Now for $\mathbf{z} = \mathbf{y} + \mathbf{x}$, $\mathbf{x} \in \mathfrak{S}^1_\sigma$, we have

$$\begin{aligned} z_{\alpha(1)} &= y_{\alpha(1)} + x_{\alpha(1)} = y_{\sigma(i_1)} + x_{\sigma(i_1)} \\ &\geq y_{\sigma(i_2)} + x_{\sigma(i_2)} = z_{\alpha(2)}, \end{aligned}$$

because if $k = 1$, i.e. $|A_1| = 1$, we have $y_{\sigma(i_1)} = 1$, $y_{\sigma(i_2)} = 0$, and $x_{\sigma(i_1)}, x_{\sigma(i_2)} \in [0,1]$, so

$$y_{\sigma(i_1)} + x_{\sigma(i_1)} \geq 1 \geq y_{\sigma(i_2)} + x_{\sigma(i_2)},$$

and if $k > 1$, then $y_{\sigma(i_1)} = y_{\sigma(i_2)} = 1$ and $x_{\sigma(i_1)} \geq x_{\sigma(i_2)}$ and again

$$y_{\sigma(i_1)} + x_{\sigma(i_1)} \geq y_{\sigma(i_2)} + x_{\sigma(i_2)}.$$

This argument can be repeated to show that

$$\begin{aligned} z_{\alpha(j)} &= y_{\alpha(j)} + x_{\alpha(j)} = y_{\sigma(i_j)} + x_{\sigma(i_j)} \\ &\geq y_{\sigma(i_{j+1})} + x_{\sigma(i_{j+1})} = z_{\alpha(j+1)} \end{aligned} \tag{17}$$

for $j = 1,2,\ldots,k$. For $j = k+1, k+2, \ldots, m$ the inequality (17) is equally clear by the construction of α, because $y_{\sigma(i_j)} = y_{\sigma(i_{j+1})} = 0$ and $x_{\sigma(i_j)} \geq x_{\sigma(i_{j+1})}$.

This gives an algorithm to subdivide the simplices in $\mathfrak{S}^2_\alpha \subset [0,2]^m$, $\alpha \in S_m$, into 2^m simplices each, that are congruent to simplices in $\mathfrak{S}^1_\sigma \subset [0,1]^m$, $\sigma \in S_m$. However, every simplex $\mathfrak{S}^a_\sigma$ can be mapped one-to-one to a general m-simplex

$$\mathfrak{S} := \mathrm{co}\{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_m\} \subset \mathbb{R}^n, \ \ n \geq m,$$

i.e. the vectors $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathbb{R}^n$ are affinely independent, with the mapping

$$S(\mathbf{x}) = F\mathbf{x} + \mathbf{y}_0, \tag{18}$$

where the matrix $F \in \mathbb{R}^{n\times m}$ is defined by fixing its $\sigma(i)$th column as $\mathbf{F}_{\sigma(i)} = \frac{1}{a}[\mathbf{y}_i - \mathbf{y}_{i-1}]$, $i = 1,2,\ldots,m$, to see this just note that

$$\begin{aligned} S(a\mathbf{x}^\sigma_i) &= aF\sum_{j=1}^{i} \mathbf{e}_{\sigma(j)} + \mathbf{y}_0 = \sum_{j=1}^{i} \mathbf{F}_{\sigma(j)} + \mathbf{y}_0 \\ &= \sum_{j=1}^{i} [\mathbf{y}_j - \mathbf{y}_{j-1}] + \mathbf{y}_0 = \mathbf{y}_i, \end{aligned}$$

from which

$$S\left(a\sum_{i=0}^{m} \lambda_i \mathbf{x}^\sigma_i\right) = \sum_{i=0}^{m} \lambda_i \left(aF\mathbf{x}^\sigma_i + \mathbf{y}_0\right) = \sum_{i=0}^{m} \lambda_i \mathbf{y}_i$$

for $\sum_{i=0}^m \lambda_i = 1$ follows.

Hence, to subdivide $\mathfrak{S}$ into 2^m simplices, we can just subdivide

$$\mathfrak{S}^2_{\mathrm{id}} := \{\mathbf{x} \in \mathbb{R}^m : 2 \geq x_1 \geq x_2 \geq \ldots \geq x_m \geq 0\}$$

using the results above and then map the subdivision. This is indeed very simple. From (16) with $\alpha = \mathrm{id}$ it is clear that a necessary and sufficient condition is that $\sigma(i_j) = j$ for $j = 1,2,\ldots,m$. Further, $\mathbf{y} \in \mathbb{R}^m$ in (14) is given by $\mathbf{y} = \sum_{j=1}^k \mathbf{e}_{\sigma(i_j)} = \sum_{j=1}^k \mathbf{e}_j =: \mathbf{1}_k$.

This gives us a simple algorithm. For every $\mathbf{z} \in \{0,1\}^m$ do the following: Let $i_1 < i_2 < \ldots < i_k$ be the indices of $\mathbf{z}$ such that $z_{i_j} = 1$ and $i_{k+1} < i_{k+2} < \ldots < i_m$ be the indices where $z_{i_j} = 0$. Set $\sigma(i_j) = j$ for $j = 1,2,\ldots,m$, then $\mathbf{1}_k + \mathfrak{S}^1_\sigma \subset \mathfrak{S}^2_{\mathrm{id}}$. Since this gives us 2^m different simplices $\mathbf{1}_k + \mathfrak{S}^1_\sigma$, these are exactly the simplices that subdivide $\mathfrak{S}^2_{\mathrm{id}}$. For a graphical presentation of this approach see Figures 3 and 4

5 THE ALGORITHM

Using the results we have developed in the last section, we can now state an algorithm that combines Test for Positivity 4 with subdivision of simplices.

Test for Positivity 6. *Assume the function* $g: \mathfrak{S} \to \mathbb{R}$, $\mathfrak{S} := \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{R}^n$ *an* m*-simplex,* $m \leq n$*, fulfills the assumptions of Lemma 3. Then execute*:

1. *Perform Test for Positivity 4 on* $\mathfrak{S}$*.*
2. *If the test is inconclusive, then subdivide* $\mathfrak{S}$ *into* 2^m *sub-simplices,*

   $$\mathfrak{S}^k := \mathrm{co}\{\mathbf{x}^k_0, \mathbf{x}^k_1, \ldots, \mathbf{x}^k_m\}, \quad k = 1,2,3,\ldots,2^m,$$

   as described in Section 4 and go back to Step 1 with $\mathfrak{S} = \mathfrak{S}^k$ *for* $k = 1,2,3,\ldots,2^m$*.*
3. *If Step 2 finishes without having found a vertex* $\mathbf{y} \in \mathbb{R}^n$ *such that* $g(\mathbf{y}) < 0$ *in Step 1, then* $g(\mathbf{x}) \geq 0$ *for all* $\mathbf{x}$ *in the original simplex* $\mathfrak{S}$*.*

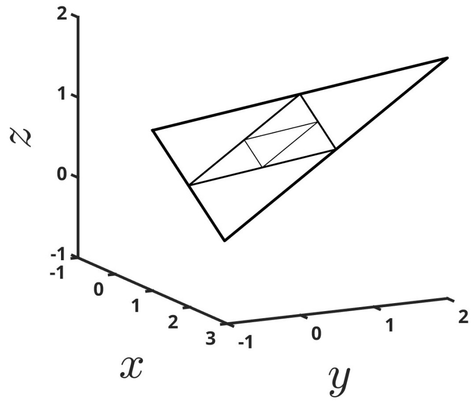

Figure 3: A triangle (2-simplex) in 3 dimensions subdivided into $2^2 = 4$ triangles, of which one is further subdivided into $2^2 = 4$ triangles.

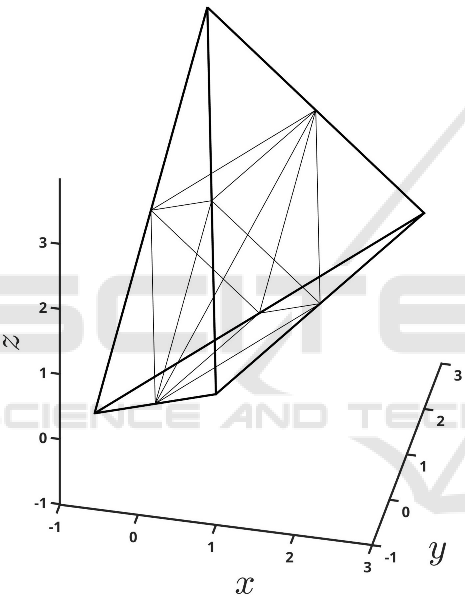

Figure 4: A tetrahedron (3-simplex) in 3 dimensions subdivided into $2^3 = 8$ tetrahedra.

Note that Test for Positivity 6 might end up in an infinite loop if g is non-negative but zero at some points in $\mathfrak{S}$. However, we will show that it always gives a definite answer for CPQ Lyapunov function candidates in the rest of this section.

Define the diameter of a simplex $\mathfrak{S} = \text{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{R}^n$ as

$$\text{diam}(\mathfrak{S}) := \max_{i,j=0,1,\ldots,m} \|\mathbf{x}_i - \mathbf{x}_j\|_2.$$

Lemma 7. *Let $\mathfrak{S} := \text{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{R}^n$ be an m-simplex and $\mathfrak{S}^k$, $k = 1, 2, 3, \ldots, 2^m$, be the simplices $\mathfrak{S}$ is subdivided into using the algorithm from Section 4. Let $S(\mathbf{x}) = F\mathbf{x} + \mathbf{x}_0$ be the mapping* (18) *that maps the vertices of $\mathfrak{S}^2_{\text{id}} \subset \mathbb{R}^m$ to the vertices of $\mathfrak{S}$. Then*

$$\text{diam}(\mathfrak{S}^k) \leq \frac{1}{2} \max_{\mathbf{y},\mathbf{z} \in \{0,2\}^m} \|F(\mathbf{y} - \mathbf{z})\|_2 \qquad (19)$$

for $k = 1, 2, 3, \ldots, 2^m$.

Proof. First note that all vertices of all simplices in $\mathfrak{S}^a_\sigma$, $\sigma \in S_m$, are vectors in the set $\{0, a\}^m$. Since $S(\mathbf{y}) - S(\mathbf{z}) = F(\mathbf{y} - \mathbf{z})$ and $\mathfrak{S} = F(\mathfrak{S}^2_{\text{id}})$ is subdivided into simplices of the form $F(\mathbf{1}_k + \mathfrak{S}^1_\sigma)$, $\sigma \in S_m$, which are congruent to the simplices in $F(\mathfrak{S}^1_\sigma)$, $\sigma \in S_m$, which in turn are congruent to the simplices in $F(\mathfrak{S}^2_\sigma)$, $\sigma \in S_m$, scaled down by a factor $1/2$, the estimate (19) follows. □

This lemma has an obvious corollary; just set $A := \max_{\mathbf{y},\mathbf{z} \in \{0,2\}^m} \|F(\mathbf{y} - \mathbf{z})\|_2$.

Corollary 8. *Let $\mathfrak{S} := \text{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{R}^n$ be an m-simplex. Then there is a constant $A > 0$ such that if $\mathfrak{S}$ is K times iteratively subdivided into simplices using the algorithm from Section 4, i.e. subdivided, then the simplices in the subdivision are subdivided, etc., then*

$$\text{diam}(\mathfrak{S}^k_K) \leq \frac{A}{2^K}$$

for every simplex $\mathfrak{S}^k_K$, $k = 1, 2, 3, \ldots, 2^{mK}$, in the Kth iteration. In particular

$$|[\mathbf{x} - \mathbf{y}]_r| \leq \frac{A}{2^K}$$

for every two vectors in $\mathfrak{S}^k_K$ and $r = 1, 2, \ldots, n$.

Another obvious corollary, and useful for our purposes, is the following.

Corollary 9. *If g in Lemma 3 fulfills $g(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathfrak{S}$ and if the E_is from Lemma 3 are scaled down in the obvious way in the iterations (jump from Step 2 to Step 1) in Test for Positivity 6, then the test will deliver the results $g(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathfrak{S}$ in a finite number of steps.*

Proof. This is indeed obvious from what we have shown. The only problem in the formulation is the inequality (8), as one could successively chose more and more conservative bounds E_i in the iterations. However, since the upper bounds B_{rs} cannot become larger when we go to smaller simplices, and the terms $|[\mathbf{x}_i - \mathbf{x}_j]_r|$ can be scaled down by a factor of $1/2$ in each iteration, this is unnecessary and makes no sense. Hence, we can let the E_i converge to zero uniformly over the iterations and then, at the latest in the iteration when all E_is are less than or equal to $\min_{\mathbf{x} \in \mathfrak{S}} g(\mathbf{x}) > 0$, Test for Positivity 4 in Step 1 of Test for Positivity 6 delivers that $g(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathfrak{S}$. □

Not only will Test for Positivity 6 deliver an affirmative answer if $g(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathfrak{S}$, but if there is a $\mathbf{y} \in \mathfrak{S}$ such that $g(\mathbf{y}) < 0$, then the test will also deliver the results that $g(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathfrak{S}$ is false, in a finite number of steps.

Lemma 10. *If for g in Lemma 3 there is a* $\mathbf{y} \in \mathfrak{S}$ *such that* $g(\mathbf{y}) < 0$*, then Test for Positivity 6 will deliver a point* $\mathbf{y}^* \in \mathfrak{S}$ *such that* $g(\mathbf{y}^*) < 0$ *in a finite number of steps.*

Proof. Since g is continuous there is an open neighborhood $U \subset \mathfrak{S}$ of $\mathbf{y}$ such that $g(\mathbf{x}) < 0$ for all $\mathbf{x} \in U$, and since S from (18) is continuous the set $S^{-1}(U)$ is open in $\mathfrak{S}^2_{\text{id}} \subset \mathbb{R}^m$ and there is an open ball $B \subset \mathfrak{S}^2_{\text{id}}$ where $g \circ S$ is negative. Now consider that the vertices of $\mathfrak{S}^2_{\text{id}}$ are the set $\{0,2\}^m \cap \mathfrak{S}^2_{\text{id}}$, the set of all vertices of all the simplices in the first subdivision of $\mathfrak{S}^2_{\text{id}}$ is $\{0,1,2\}^m \cap \mathfrak{S}^2_{\text{id}}$, the set of all vertices of all the simplices in the second iterative subdivision of $\mathfrak{S}^2_{\text{id}}$ is $\{0,1/2,1,3/2,2\}^m \cap \mathfrak{S}^2_{\text{id}}$, etc. Now, for a large enough $K \in \mathbb{N}$, there must exist an

$$\mathbf{x}^* \in \left\{0, \frac{1}{2^K}, \frac{2}{2^K}, \ldots, \frac{2 \cdot 2^K - 1}{2^K}, 2\right\}^m \cap B$$

and with $\mathbf{y}^* = S(\mathbf{x}^*)$ we have $0 > (g \circ S)(\mathbf{x}^*) = g(\mathbf{y}^*)$. □

Assume V is a CPQ Lyapunov function candidate, i.e. is of the form (1) and fulfills the conditions (3). In Theorem 1 we showed that if $V(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$, then $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$. Hence, for every $\nu \in I$, either $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathfrak{C}_\nu \setminus \{\mathbf{0}\}$ or there exists a $\mathbf{y} \in \mathfrak{C}_\nu$ such that $V(\mathbf{y}) < 0$. Combining Corollary 9 and Lemma 10 with these results delivers:

Theorem 11. *For a CPQ Lyapunov function candidate V of the form* (1) *that fulfills the conditions* (3)*, the Test for Positivity 6 is non-conservative when verifying the conditions* (2) *for a CPQ Lyapunov functions. That is, the test will give an affirmative answer in a finite number of steps, whether* (2) *holds true or not.*

6 CONCLUSIONS

For switched, cone-wise linear systems, either arbitrary switched or with state-dependent switching, we presented an algorithm to verify the positive definite conditions for CPQ Lyapunov function candidates parameterized using LMIs or LP. Further, we proved in Theorem 11 that the algorithm is non-conservative, in comparison to earlier approaches that do introduce some conservatism. In a subsequent publication we will describe an efficient implementation of our algorithm for n-dimensional system and demonstrate its applicability. Further, we will compare its numerical efficiency with the LMI approaches presented in (Kruszewski et al., 2009; Sala and Arino, 2007; Gonzaleza et al., 2017), which are sufficient and asymptotically necessary.

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