A Study of Multi-Objective Optimisation Algorithms

Peiqi Gao[a]
Mathematics and Statistics, Xidian University, Xi`an, Shaanxi Province, 710071, China

Keywords: Multi-Objective Optimisation, AHP Combined with EWM, NSGA-II, PSO.

Abstract: In production activities, the problem often involves the optimisation of multiple objectives, and the traditional single-objective problem-solving methods are unable to deal with optimisation problems with multiple objectives. Traditional single-objective optimization methods usually focus on the optimal solution of one objective function, while multi-objective optimization problems need to consider multiple objective functions at the same time. This paper offers a comprehensive summary of the approaches to multi-objective optimization problems and proposes recommendations for future development. Firstly, the development history of multi-objective optimisation algorithms is reviewed, and then the related concepts of multi-objective problems, such as pareto optimal solution set, are briefly explained. In this paper, multi-objective optimisation algorithms are broadly classified into three categories: multi-objective weighting methods, multi-objective population genetic algorithms, and multi-objective individual evolutionary algorithms. The advantages and disadvantages of the three main types of methods are analysed by practical examples of the methods, and suggestions for subsequent improvements are given based on limitations.

1 INTRODUCTION

In engineering and scientific contexts, it's common to encounter optimisation problems where the goal is to achieve optimality within a specific domain. These are referred to as multi-objective optimisation problems when multiple objectives are involved. For example, when optimising the purchase of an item, the decision maker usually has to balance price and quality.

The solution to multi-objective optimisation problems is very commonly used in production activities, and researchers are constantly proposing new ideas to deal with them. From 1896, Pareto proposed the optimal solution of pareto to the beginning of the 20th century, multi-objective optimisation was introduced into finance and other fields, marking the gradual formation of the theoretical basis of multi-objective optimisation; During the mid-to-late 20th century, scientists and researchers primarily approached multi-objective optimization problems by converting them into single-objective problems, such as: the objective function weighting method (Xiao, 2011). During the transition from the 20th to the 21st century, researchers applied population genetic algorithms in the field of computational intelligence to multi-objective optimisation. Later individual algorithms, such as Particle Swarm Optimization Algorithm (PSO) were applied to multi-objective optimisation. Since then, new methods have been proposed, such as Multi-objective Evolutionary Algorithm based on Decomposition (MOEA/D), but they are not considered in this paper for the time being.

When dealing with multi-objective optimisation problems, the main problem is how to make multiple objectives reach the optimal solution. In the problem, when one objective is optimized, the performance of the other objectives usually decreases. This is because a multi-objective optimization problem involves a contradiction between the various sub-objectives. Therefore, a compromise must be made to achieve the best possible outcome. In other words, there are solutions that cannot be compared in terms of their advantages and disadvantages. Consequently, solutions to multi-objective optimization problems are not unique. Instead, there exists a set of optimal solutions, which is referred to as the Pareto optimal solution set.

[a] https://orcid.org/0009-0008-7480-8746

Gao, P.
A Study of Multi-Objective Optimisation Algorithms.
DOI: 10.5220/0013680600004670

In *Proceedings of the 2nd International Conference on Data Science and Engineering (ICDSE 2025)*, pages 175-181
ISBN: 978-989-758-765-8

This paper classifies multi-objective optimisation, which is often used in scientific experiments, into three categories, multi-objective weighting method (which uses weighting to convert a multi-objective problem into a single-objective problem.), multi-objective population genetic algorithm (which introduces multi-objective optimisation into the framework of genetic algorithms), and multi-objective individual evolutionary algorithms (which applies an initial solution to generate a subsequent solution). The corresponding computational principles are analysed as well as specific applications in scientific experiments. Ultimately and give directions for improvement.

2 MATHEMATICAL DESCRIPTION

The multi-objective problem can be described in the following standard form

$$\begin{cases} \min\ y = \left(f_1\ (x), f_2(x), \ldots, f_m(x)\right) \\ s.t.\ g_i(x) \leqslant 0 \quad i = 1, 2 \ldots, q \\ \qquad h_j(x) = 0 \quad j = 1, 2, \ldots p \end{cases} \tag{1}$$

There are m objective functions, and the variables involved in decision-making are n-dimensional. The vector x, which includes x_1 through x_n, belongs to X, an n-dimensional space for making decisions. The variable representing objectives, y, which includes y_1 through y_m, belongs to Y, the m-dimensional space of objectives. Furthermore, there are q constraints where $g_i(x)$ is less than or equal to zero $(i = 1, 2, \ldots, q)$ and p constraints are established where h_j equals zero$(j = 1, 2, \ldots, p)$.

Definition 1 (Feasible Solution) A solution is said to be feasible if and only if it satisfies the equality constraints $h_j = 0(j = 1, 2, \ldots p)$ and the inequality constraints $g_i \leqslant 0(i = 1, 2, \ldots q)$ in . The set of all feasible solutions, denoted as X_f, and $X_f \in X$

Definition 2 (pareto optimal solution) Assume that $x_1, x_2 \in X_f$, x_1 is pareto dominated x_2 (dominated as $x_1 \prec x_2$,) if $\forall i = 1, 2, \ldots m\ f_i(x_1) \leqslant fi(x_2)$ and $\exists\, j = 1, 2 \ldots m\ f_j(x_1) < f_j(x_2)$. A feasible solution x^* is a pareto optimal solution (or non-dominated solution) if there is no $\dot{x} \in X_f\ \ \dot{x} \prec x^*$. and the set of all pareto optimal solutions is the set of pareto optimal solutions P.

3 MULTI-OBJECTIVE WEIGHTING METHOD

3.1 Overview of the Weighting Method

The core idea of the weighting method is to convert a multi-objective optimization problem into a single-objective one by assigning weights to the various objective functions. Let's denote the weights as $w = (w_1, w_2, \ldots w_m)$ and the objective functions as $y = \left(f_1(x), f_2(x), \ldots, f_m(x)\right) = (y_1, y_2, \ldots y_m)$. By calculating $F = w \times y = w_1 \cdot y_1 + w_2 \cdot y_2 + \ldots + w_m \cdot y_m$, the multi-objective optimization problem is effectively transformed into finding the feasible solution that maximizes F, thereby identifying the optimal solution.

The core of the weighting method lies in the assignment of weights. There are three main approaches to assigning weights: the subjective weighting method, the objective weighting method, and the integrated subjective and objective weighting method. Among the commonly used subjective weighting techniques, the Analytic Hierarchy Process (AHP) and the Grey Analytic Hierarchy Process (GAHP) are often utilized. However, these methods are rather rough, and personal subjective factors have a significant impact on the solution (Guo, 2008). When it comes to problems requiring higher precision, the results may not be consistent with the actual situation.

Objective weighting techniques encompass methods such as the Entropy Weight Method (EWM) and Principal Component Analysis (PCA), among others. This type of method is calculated based on the data of the program (Wang, 2011). The results are relatively objective, avoiding the influence of the evaluator's subjective factors on the weight of the indicators. Nevertheless, the Entropy Weight Method (EWM) also has its drawbacks. The weights of the indicators obtained through it only indicate the relative intensity of the competition among the indicators, rather than the true significance of the metrics. Moreover, the determination of its weight relies entirely on the relationship between the objective data. When the objective data is more special, the weight will differ from the actual situation.

The subjective-objective comprehensive weighting method usually combines the two approaches mentioned above, such as the Analytic Hierarchy Process-Entropy Weight Method (AHP-EWM)(Fei, 2009). However, in the process of seeking the integrated weights with this method, the combination is merely a simple synthesis of the results of the indicators after using each of the subjective and objective methods to find the weights of all the indicators.

3.2 Analytic Hierarchy Process-Entropy Weight Method

Among all weighting methods, the most commonly used in scientific research is (AHP-EWM) in the subjective-objective integrated weighting method.

3.2.1 Calculation of Analytic Hierarchy Process weights

Assume that the judgement matrix $A=(a_{ij})_{m\times m}=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix}$ a_{ij} is $\dfrac{Importance\ of\ indicator\ i}{Importance\ of\ indicator\ j}$

1) Consistency test. Test whether the whole judgement matrix meets $a_{ik}=a_{ij}\times a_{jk}$, by judging the size of the consistency ratio coefficient $CR=\dfrac{CI}{RI}$. If $CR<0.1$, the consistency of the judgement matrix isn't acceptable, adjustments are needed to ensure it meets the required consistency. The consistency coefficient $CI=\dfrac{\lambda_{\max}-m}{m-1}$, $\lambda_{\max}$ is the largest eigenvalue of the judgement matrix. RI is determined by n , and the specific function value can be obtained through the relevant table.

2) Calculate the weights by performing column normalisation followed by arithmetic mean to the weights, $\theta_i=\dfrac{1}{m}\sum_{j=1}^{m}\dfrac{a_{ij}}{\sum_{k=1}^{m}a_{kj}}$

3.2.2 Calculation of Entropy Weighting Method Weights

Let there be a total of m data points, each data point corresponds to n objective function values, the corresponding objective function value of the data points constitutes an evaluation matrix $X=(x_{ij})_{m\times n}=\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{vmatrix}$ The operation procedure is as follows

Pre-processing of the above matrix such as data regularisation, normalisation etc.

Calculate the entropy measure of the jth parameter e_j , $e_j=-\dfrac{1}{\ln m}\sum_{i=1}^{m}a_{ij}lna_{ij}$, where $\ln m$ denotes the logarithm of m

Calculate the entropy value corresponding to the entropy weights $\beta_j=\dfrac{1-e_j}{n-\sum_{i=1}^{n}e_i}$ $j=1,2,...,m$ to get the entropy weights. $\beta=(\beta_1,\beta_2,...,\beta_m)$

3.2.3 The Final Weights

Multiply the weights of the hierarchical analysis and entropy weighting methods and then normalise by multiplication to get the final weights $w=(w_1,w_2,...,w_m)$

3.3 Example of the Use of Hierarchical Analysis-Entropy Weight Method

In practical applications, the hierarchical analysis-entropy weight method is mainly used in the case of limited decision-making methods, and Wang Huibin and others applied this method to photovoltaic power generation projects. The three main indicators in photovoltaic power generation are scale, cost, and benefit (Wang, 2022). The scale is determined by the installed capacity, the number of hours of power generation, and the amount of power generated. Cost is determined by the averageised cost of electricity, the investment per unit of electricity, and the time to start earning. Benefits are determined by the internal rate of return (IRR), which indicates profitability over the entire operating period, and the net present value (NPV), which measures net income per unit of operating time. In this context, all indicators for scale and benefit are maximizing. Indicators that are maximized have larger corresponding values, while those that are minimized have smaller values.

All indicators are normalised (all indicators are converted into very large ones, which can be inverted) and then dimensionless. Use Hierarchical Analysis-Entropy Weighting to get the priority of each scenario. The prioritisation results were compared with the results obtained using only Hierarchical Analysis and only Entropy Weights. The results obtained were found to be superior.

4 MULTI-OBJECTIVE POPULATION GENETIC ALGORITHM

4.1 Overview of Multi-Objective Population Genetic Algorithms

Over the past 30 years, genetic algorithms have developed rapidly. Scientists have applied genetic algorithms to multi-objective optimisation problems (Ma, 2007). This has been accompanied by the introduction of concepts related to the Pareto optimal solution set. As a result, the success rate of multi-objective genetic algorithms has been further guaranteed in terms of their computational results.

The Genetic Algorithm (GA) is a method for finding optimal solutions, inspired by the biological principle of "survival of the fittest." In this context, the fitness value indicates the quality of a solution, which is reflected by the final function value. Additionally, GA involves encoding and decoding operations, where feasible solutions are converted into strings of characters, such as numbers or letters.

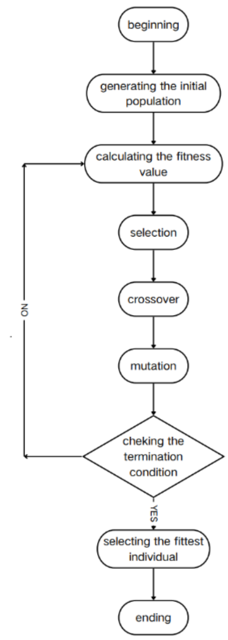

Figure 1: The flow of the genetic framework.(Picture credit : Original)

The fundamental processes include selection, recombination, and mutation, in figure 1.

Selection is used to choose better individuals from the group. It increases the probability of these better individuals being selected. This can ensure the convergence of the algorithm. Common selection methods include the roulette wheel selection method (RWS). The individual's fitness value is directly converted into the probability of being selected.

Crossover, where information is exchanged for multiple parent individuals selected to produce new child individuals. The diversity of solutions can be ensured and local convergence due to too fast convergence can be avoided to some extent. Common crossover methods include single-point crossover, etc.

mutation, one of the selected parent individuals is manipulated to produce a new child individual. It also ensures the diversity of solutions and to some extent avoids local convergence due to too fast convergence. However, the difference with crossover is that crossover requires at least two parent individuals, while mutation can be achieved with only one parent individual; at the same time, the probability of crossover occurring is greater than the probability of mutation occurring. A common method of mutation is single point mutation.

4.2 Non-Dominated Sorting Genetic Algorithm II (NSGA-II)

The common multi-objective genetic algorithms are Vector Evaluated Genetic Algorithm (VEGA), Non-dominated Sorting Genetic Algorithm (NSGA), and Strength Pareto Evolutionary Algorithm (SPEA)(Xu,2007). All of these methods use the framework of genetic algorithms and also apply pareto dominance relationships, which finally have significant results in ensuring diversity. However, although all of these methods employ the framework of genetic algorithms, the specifics of each step are different. In the following section, the commonly used multi-objective genetic algorithm is described in detail: Non-dominated Sorting Genetic Algorithm II (NSGA-II)(Gao, 2006)

Perform a fast non-inferiority stratification for the population P . That is, according to the pareto dominance relationship, individuals that do not have dominance relationships with each other are in the same non-inferiority stratum, where individuals that are not dominated by any individual are in the first stratum, those that are only dominated by individuals in the first stratum are in the second stratum, and so on until all individuals of the population have been stratified. Individuals in the same stratum are also a group of individuals, with the first stratum being the first group and the second stratum being the second group. According to the result of non-inferiority stratification, the population P can be divided into ρ non-inferiority groups, i.e $P=\bigcup_{j=1}^{\rho} P_j$.,the first group P_1 is the optimal non-inferiority group of the population.

Crowding degree calculation, Crowding degree can indicate the density of individuals in the space. Assume that the jth non-inferiority group consists of m individuals. After the jth non-inferiority group is desorted for the mth objective function value, an infinite distance is assigned between the first and last points, and the congestion degree distance for the ith individual is calculated as $L_i^{(m)}=\frac{f_m(x_{i+1})-f_m(x_{i-1})}{f_m^{\max}-f_m^{\min}}$ $f_m^{\max}, f_m^{\min}$ denoting the maximum and minimum values, respectively

The dominance relation can be further extended by the crowding degree calculation in step 2). That is, $when\ x_1 \in P_j, x_2 \in P_j\ and\ L_1^{(m)} > L_2^{(m)}\ then\ x_1 \succ x_2$, i.e., it means that when two solutions belong to the same non-inferior group, the solution with greater crowding degree is preferred.

Add the elite strategy in selecting the best individual. That is, the selection range of individuals is increased to the concatenation of parents and children, expanding the search range and making the algorithm less likely to fall into local optimal solutions.

4.3 Application of NSGA-II Algorithm

Wang Xi employed the NSGA-II algorithm to tackle the issue of connecting wind farms to the grid and expanding the grid, with the aim of reducing costs, minimizing grid expansion, and lowering pollutant emissions (Wang, 2011). The final objective is reached by adjusting the independent variables wind farm access location, access capacity, and the scheme of grid expansion.

By comparing the other algorithms, it is concluded that the NSGA-II ensures the stability of the algorithm results, i.e., it works well for most of the cases. At the same time, it does not need a priori knowledge to get the weights, and the calculation process is more intelligent.

5 MULTI-OBJECTIVE INDIVIDUAL EVOLUTIONARY ALGORITHMS

5.1 Brief Description of Multi-Objective Individual Evolutionary Algorithms

In the last fifteen years, population-based algorithms such as genetic algorithms have not developed significantly, but some optimisation algorithms based on individual search mechanisms have developed rapidly, such as Particle Swarm Optimization (PSO) (Zhang, 2004), Ant Colony Optimization (ACO) (Duan, 2004). These algorithms are different from the mechanism of eliminating individuals in genetic algorithms, but update and iterate each individual in the population based on the optimal solution of the current individual and population. Therefore, this type of algorithm has more interaction of information between individuals, and has memory and consistency for the update route. At the same time, the update is more flexible and less likely to fall into local optimal solutions.

5.2 Multi-Objective Particle Swarm Algorithm

Among the individual-based search mechanism optimisation methods, the particle swarm algorithm is more frequently utilized. In this study, we will first introduce the particle swarm algorithm, and then describe how to apply the particle swarm algorithm to multi-objective optimisation problems.

5.2.1 The Particle Swarm Algorithm

The basic particle swarm algorithm is mainly for single-objective optimisation problems, let D be the size of the parameter space, NP be the size of the population, and the particle is , $x_i = (x_{i1}, x_{i2}, \ldots, x_{iD})$ $i = 1, 2, \ldots, NP$ .The corresponding velocity $v_i = (v_{i1}, v_{i2}, \ldots, v_{iD})$ $i = 1, 2, \ldots, NP$ The best position for this particle so far is $p_i = (p_{i1}, p_{i2}, \ldots, p_{iD})$, and the best position for the whole population is. $p_g = (p_{g1}, p_{g2}, \ldots, p_{gD})$ According to the iterative formula

$$v_{id}^{t+1} = wv_{id}^t + c_1 r_1 (p_{id}^t - x_{id}^t) + c_2 r_2 (p_{gd}^t - x_{id}^t) \quad (1)$$

$$x_{id}^{t+1} = x_{id}^t + v_{id}^{t+1} \quad d = 1, 2, \ldots, D \quad i = 1, 2, \ldots, NP \quad (2)$$

where w is the inertia index and is a number generated by a random process in$(0, 1)$. reflecting the dependence on the previous speed

c_1, c_2 is the acceleration weight, a number generated by a random process in$(0, 2)$

r_1 is the stochastic number in $[0, 1]$ is called the cognitive factor. Reflects dependence on one's own experience

r_2 is the random number in $[0, 1]$ is the social factor. Reflects the degree of dependence on group experience

The coordinates of the particles in the population are updated iteratively until the requirements of the question are met (either the number of iterations required or the accuracy required). By observing the iteration formula (1), we can find that the velocity iteration formula is roughly divided into three parts, the first part wv_{id}^t indicates that it receives the influence of the last velocity vector, i.e., it is influenced by its own velocity at the last moment, which is also known as the memory term. The second part $c_1 r_1 (p_{id}^t - x_{id}^t)$ Indicates that the velocity of the particle receives the influence of the vector from the current position to the particle's optimal point, i.e., it indicates that it receives the influence of the particle's optimal point, which is also known as the self-cognition term. The third part $c_2 r_2 (p_{gd}^t - x_{id}^t)$ indicates that the velocity receives the influence of the vector from the current individual position to the group optimal position, which can reflect the influence of receiving the group optimal point, also known as the group cognitive term. Also All are random numbers, which can increase the diversity of the solution.

5.2.2 Applications of Multi - Objective PSO

For the multi-objective particle swarm algorithm, since there is no single optimal solution, only the optimal solution set exists, i.e., The ultimate objective is to achieve the best possible solutions. Based on the pareto dominance relation, the population P of N is divided into two populations, one is the non-dominated subset A , the other is the dominated subset B , and the corresponding numbers of individuals are n_1, n_2 $(n_1 + n_2 = N)$. Each update of the individual coordinates of the particle population is only for the dominant subset B . Determine the dominance relationship between the updated subset $\dot{B}$ and the individuals in the non-dominated subset$A$, if there is an individual in the subset$A$ that is dominated by an individual in the dominated subset$\dot{B}$, then replace the corresponding individual. The update is iterated until the termination condition (accuracy or number of iterations) is met.

5.3 Application of Multi-Objective Particle Swarm Algorithm

Gu applied particle swarm optimisation （PSO）to a Combined Heat and Power (CHP)-based Microgrid system (Gu, 2012). A Combined Heat and Power (CHP)-based Microgrid system mainly refers to a system that can control the simultaneous production of electricity and heat. The final objective function is to minimise the operating cost, carbon monoxide emission and nitrogen oxide emission. Meanwhile, the constraints are: battery charging and discharging balance and surplus supply of heat energy. The particle swarm algorithm was applied to optimisation in two cases: a hospital and a school. The final calculation results were found to be reasonable. It was concluded that the particle swarm algorithm is widely used in the field of combined heat and power supply.

6 CONCLUSION

In this paper, the existence of multi-objective problems is firstly introduced widely, after which the basic concepts about multi-objective problems, such as feasible solution sets, are introduced. After that, the common methods for solving multi-objective problems are classified into three categories, describing their mathematical principles and applications in real life, and comparing the advantages and disadvantages between different methods. All three types of methods can be improved.

The main element involved in the weighting method is the determination of the weight vectors, a part that is difficult to improve if one wants to make innovations in the mathematical theory. In addition to the determination method can be improved, the weights can be made adaptive, that is, the weights are not fixed in the arithmetic process, and do not need human intervention to improve. Multi-objective population genetic algorithms are also relatively well-developed at the structural level of the algorithm. However, the determination of some parameters can utilize emerging computational methods in recent years, such as surrogate models and machine learning. Yet, the choice of specific methods still needs to be tailored to the specific application scenario. For the recently emerged multi-objective individual evolutionary algorithms, there are many innovations, such as the introduction of the farthest point from the point and the nearest point in the iterative formula to avoid the local optimal solution, as well as the introduction of other particles in the population to improve the iterative formula and so on.

REFERENCES

Duan, H., Wang, D., Zhu, J., et al., 2004. Progress in the theory and application research of ant colony algorithm. Control and Decision Making, (12), pp.1321-1326+1340.

Fei, Z., 2009. Research on entropy right-hierarchical analysis method and grey-hierarchical analysis method. Tianjin University.

Gao, Y., 2006. Research and application of non-dominated sorting genetic algorithm (NSGA). Zhejiang University.

Gu, W., Wu, Z., WANG, R., 2012. Multi-objective operation optimisation of cogeneration-type microgrid considering pollutant gas emission. Power System Automation, 36(14), pp.177-185.

Guo, J., Zhang, Z., Sun, Q., 2008. Research and application of hierarchical analysis. Chinese Journal of Safety Science, (05), pp.148-153.

Ma, Y., Yun, W., 2012. Research progress of genetic algorithm. Computer Application Research, 29(04), pp.1201-1206+1210.

Wang, Y., 2011. Comprehensive evaluation of professional journals of library and intelligence in mainland China - A comparative study based on entropy weighting, principal component analysis and simple linear weighting. Intelligence Science, 29(06), pp.943-947.

Wang, H., Hu, F., Liu, Y., 2022. Research on multi-objective decision analysis of photovoltaic power generation project. Hydropower Generation, 48(10), pp.99-103.

Wang, X., Zhang, P., 2011. Multi-objective grid planning for wind farms using NSGA-II hybrid intelligent algorithm. Chinese Journal of Electrical Engineering, 31(19), pp.17-24.

Xiao, X., Xiao, D., Lin, J., et al., 2011. Research overview of multi-objective optimisation problems. Computer Application Research, 28(03), pp.805-808+827.

Xu, L., 2007. Research and application of multi-objective optimisation problem based on genetic algorithm. Central South University.

Zhang, L., Zhou, C., Ma, M., et al., 2004. Solving multi-objective optimisation problems based on particle swarm algorithm. Computer Research and Development, (07), pp.1286-1291.