Algebraic Subset of N-Dimensional Vector Space on Affine Scheme

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Abstract: In this paper, we study the connection between the most basic Hodge theory in compact complex manifold and the affine scheme in algebraic geometry. By introducing the definitions of algebraic subset and affine scheme, the Hodge operator on-dimensional affine space is defined, the ringed space of algebraic subset defined on affine space is constructed, and the proof the Nullstellensatz theorem is obtained.

1 INTRODUCTION

As we known, Hodge theory in compact complex manifold is a very important theory in algebraic manifold. It has become an important research topic in algebraic geometry whether similar theories can be established in the theory of algebraic variety, that is, the famous Cheeger-Goresky-MacPherson conjecture (Goresky M, 1980), (Goresky M, 1983). This is still a very difficult job that is to solve at present.

2 ALGEBRAIC SUBSETS ON HERMITIAN VECTOR SPACE

The polynomial ring $k[x_1,\cdots,x_n]$ defined on n -dimensional vector space is a commutative ring. By using the definition of algebraic subset, the Hodge operator defined on n -dimensional affine space is obtained, which is equivalent to Hermitian exterior algebra. What is more important is to construct a ringed space on this basis, establish the relationship between the algebraic subset of the vector space and the affine scheme, and further prove the Nullstellensatz theorem satisfied by the algebraic subset defined in the vector space.

2.1 Hermitian Exterior Algebra and Hodge Operator on Algebraic Subset

Definition 2.1.1: Let $f \in k[x_1,\cdots,x_n]$ be a polynomial with coeffcients in a field k . The function defined by f is called a polynomial function on n -dimensional vector space k^n over k , with values in k .

If k is infinite, then all polynomials on n -dimensional vector space k^n over k can form a commutative ring $k[x_1,\cdots,x_n]$. Thus distinct polynomials define distinct polynomial functions by the "Definition 2.1.1([David Eisenbud, 2008])". Since no polynomial function other than 0 can vanish identically on k^n, then k^n is usually called affine n -space over k , written $A^n(k)$ or A^n . Hence we have the following definition:

Definition 2.1.2: Given a subset $I \subset k[x_1,\cdots,x_n]$, then we define a corresponding algebraic subset of k^n to be

$$Z(I) = \left\{ (a_1,\cdots,a_n) \in k^n \middle| f(a_1,\cdots,a_n) = 0 \quad for \quad all \quad f \in I \right\}. \tag{1}$$

The "Definition 2.1.2" gives an isomorphism between algebraic subsets and subsets of a affine space. Consider $k[x_1,\cdots,x_n]$ as a polynomial ring, if $I \subset k[x_1,\cdots,x_n]$ is an ideal, then we have an isomorphism:

$$Z(I) \to I .$$

Similarly, we also have another isomorphism about a subset of an affine space. Given any set $X \subset k^n$, we can define

$$I(X) = \left\{ g \in k[x_1,\cdots,x_n] \middle| g(b_1,\cdots,b_n) = 0 \quad for \quad all \quad (b_1,\cdots,b_n) \in X \right\}. \tag{2}$$

It is clear that $I(X)$ is an ideal and $I(X) \to X$ is isomorphic.

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Let V be a real finite-dimensional vector space of dimension m which is equipped with an inner product $\langle,\rangle$, a Euclidean vector space. Actually, V is an affine m-space by the previous discussion. Namely, if $\{e_1,\cdots,e_m\}$ is an orthonormal basis for V, then we have

$$I\left(\{e_i\}\right)\rightarrow\{e_i\}, i=1,2,\cdots,n$$

are isomorphic by the (2). Thus for the exterior algebra $\wedge^s V$, we can obtain the corresponding orthonormal basis

$$\left\{Z\left(\prod_{1\le p\le s} I\left(\left\{e_{i_p}\right\}\right)\right):1\le i_1\le i_2\le\cdots\le i_s\le m\right\}, \quad (3)$$

moreover the Hodge operator can be defined. The Hodge operator is a mapping

$$H^*:\left\{Z\left(\prod_{1\le p\le s} I\left(\left\{e_{i_p}\right\}\right)\right):1\le i_1\le i_2\le\cdots\le i_s\le m\right\}\rightarrow\left\{Z\left(\prod_{1\le q\le m-s} I\left(\left\{e_{j_q}\right\}\right)\right):1\le j_1\le j_2\le\cdots\le j_{m-s}\le m\right\}$$

by the (3).

In order to better define the form of the Hodge operator H^*, we need to prove the proposition introduced below. It will be beneficial to establish a relationship with the scheme theory.

Proposition 2.1.3: The intersection of collection of algebraic subsets is algebraic subset, i.e.

$$\bigcap_i Z(J_i)=Z\left(\bigcup_i J_i\right). \quad (4)$$

Furthermore, if we define $\prod_{i=1}^{n} J_i$ to be the set consisting of all products of one function from each J_i, then we have

$$\bigcup_{i=1}^{n} Z(J_i)=Z\left(\prod_{i=1}^{n} J_i\right). \quad (5)$$

Proof. By the "Definition 2.1.1", $\forall$ $(\alpha_1,\cdots,\alpha_r)\in\bigcap_i Z(J_i)$ we have $f_i(\alpha_1,\cdots,\alpha_r)=0$ for all $f_i\in J_i$, this implies that $\forall$ $f\in\bigcup_i J_i$ such that $f(\alpha_1,\cdots,\alpha_r)=0$. Then

$$(\alpha_1,\cdots,\alpha_r)\in Z\left(\bigcup_i J_i\right)\Rightarrow\bigcap_i Z(J_i)\subset Z\left(\bigcup_i J_i\right); \quad (6)$$

Conversely, $\forall$ $(\beta_1,\cdots,\beta_r)\in Z\left(\bigcup_i J_i\right)$, it satisfies $f(\beta_1,\cdots,\beta_r)=0$ for any $f\in\bigcup_i J_i$. Hence $\forall$ $f_i\in J_i$ we have $f_i(\beta_1,\cdots,\beta_r)=0$ such that $(\beta_1,\cdots,\beta_r)\in Z(J_i)$. Thus

$$(\beta_1,\cdots,\beta_r)\in\bigcap_i Z(J_i)\Rightarrow\bigcap_i Z(J_i)\supset Z\left(\bigcup_i J_i\right). \quad (7)$$

Then (4) right by (6) and (7).

Consider $\forall(\lambda_1,\cdots,\lambda_r)\in\bigcup_{i=1}^{n} Z(J_i)$, then for all $f_i\in J_i$, $i=1,2,\cdots,n$, we have

$$f_i(\lambda_1,\cdots,\lambda_r)=0.$$

Since $\forall f\in\prod_{i=1}^{n} J_i$, $f:k^r\rightarrow k^n$ defined, then $f(\lambda_1,\cdots,\lambda_r)=(0,\cdots,0)$. Thus

$$(\lambda_1,\cdots,\lambda_r)\in Z\left(\prod_{i=1}^{n} J_i\right)\Rightarrow\bigcup_{i=1}^{n} Z(J_i)\subset Z\left(\prod_{i=1}^{n} J_i\right); \quad (8)$$

Conversely, $\forall(\mu_1,\cdots,\mu_r)\in Z\left(\prod_{i=1}^{n} J_i\right)$ and $g\in\prod_{i=1}^{n} J_i$, there exists the corresponding $g_i\in J_i$ such that

$$g(\mu_1,\cdots,\mu_r)=\left(g_1(\mu_1,\cdots,\mu_r),\cdots,g_n(\mu_1,\cdots,\mu_r)\right)=(0,\cdots,0),$$

where $i=1,2,\cdots,n$. Then we have

$$g_i(\mu_1,\cdots,\mu_r)=0\Rightarrow(\mu_1,\cdots,\mu_r)\in\bigcup_{i=1}^{n} Z(J_i)\Rightarrow\bigcup_{i=1}^{n} Z(J_i)\supset Z\left(\prod_{i=1}^{n} J_i\right) \quad (9)$$

for all $g_i\in J_i$, where $i=1,2,\cdots,n$. Then (5) right by (8) and (9).

Q.E.D

According to the "Proposition 2.1.3", we can let

$$\Omega_i^s=\bigcup_{1\le p\le s} Z\left(I\left(\left\{e_{i_p}\right\}\right)\right)=Z\left(\prod_{1\le p\le s} I\left(\left\{e_{i_p}\right\}\right)\right). \quad (10)$$

Hence the Hodge operator become

$$H^*:\Omega_i^s\rightarrow\Omega_j^{m-s}$$

by the (10), and $\Omega_i^s\otimes\Omega_j^{m-s}=\Omega^m$ is an orthonormal basis for V. If we find that $a,b\in\wedge^s V$, then we have

$$a\wedge H^*b=\left(\bigotimes_{|I|=s} a_I\Omega_I^s\right)\wedge\left(\bigotimes_{|J|=s} b_J\Omega_J^{m-s}\right)=\langle a,b\rangle\otimes\Omega^m. \quad (11)$$

Therefore, after the above discussion, a representation combining Hermitian exterior algebra and Hodge operator with algebraic subsets theory has been obtained. Moreover, the operation method between one element and another element transformed by Hodge operator H^* in exterior algebra $\wedge^s V$ is given through formulas (11).

2.2 Affine Scheme of Algebraic Subset and Nullstellensatz Theory

Now we need to construct a ringed space based on the affine spaces discussed in the previous section, and introduce the scheme theory to study related issues.

Consider polynomial commutative ring $k[x_1,\cdots,x_n]$ and n-dimensional vector space k^n, the topological space here is obtained by commutative ring from the "Definition 2.1.1", that is to say, the topological space is obtained by $Spec(k[x_1,\cdots,x_n])$, where the elements are Prime ideals in $k[x_1,\cdots,x_n]$. Thus we can construct such a ringed space $(Spec(k[x_1,\cdots,x_n]),\mathcal{O})$, where the sheaf of ring

$\mathcal{O}=\left\{\left(K\to\left\{K\to\bigcup_{k\in K}\left(Spec(k[x_1,\cdots,x_n])\right)_k\right\}\right):K\in Spec(k[x_1,\cdots,x_n])\right\}$,

and

$\forall(K,\varphi)\in\mathcal{O},K\in Spec(k[x_1,\cdots,x_n]),\varphi:K\to\bigcup_{k\in K}\left(Spec(k[x_1,\cdots,x_n])\right)_k$.

Definition 2.2.1: A ringed space $(X,\mathcal{O}_X)$ is an affine scheme, which means that it is isomorphic to a ringed space of the form $(SpecA,\tilde{A})$, where A is a ring, and then we call $\Gamma(X,\mathcal{O}_X)$ the ring of affine scheme.

For the ringed space formed by n-dimensional Vector space k^n, we can find $Y\subset k^n$, there exists a mapping

$$\Phi:I(Y)\to Spec(k[x_1,\cdots,x_n])$$

by setting $\forall h\in I(Y)$,

$$\Phi:h\to K_h.$$

It is clear that Φ is isomorphic. By the "Definition 2.2.1 (Alexander Grothendieck, 2018)", $(I(Y),\mathcal{O}_{I(Y)})$ is an affine scheme. Next, we can obtain the Nullstellensatz theorem satisfied by the base space of affine scheme $(I(Y),\mathcal{O}_{I(Y)})$.

Proposition 2.2.2: Let $I(O)$ be the base space of the affine probability form constructed above, and $O\subset k^n, I(O)\subset k[x_1,\cdots,x_n]$. Then

$$I(Z(I(O)))=rad(I(O)). \qquad (12)$$

Thus, the correspondences $I(O)\to Z(I(O))$ and $O\to I(O)$ induce a bijection between the collection of algebraic subsets of k^n and radical ideals of $k[x_1,\cdots,x_n]$.

Proof. Since k is an algebraically closed field, then we can obtain that the affine space k^n over k and defined polynomial function $f\in k[x_1,\cdots,x_n]$. Thus for any polynomial function

$\phi\in I(Z(I(O)))=\left\{f\in k[x_1,\cdots,x_n]\,\middle|\,f(a_1,\cdots,a_n)=0\ \ for\ \ all\ \ (a_1,\cdots,a_n)\in Z(I(O))\right\}$.

That is to say, $\phi(a_1,\cdots,a_n)=0$ for all

$(a_1,\cdots,a_n)\in Z(I(O))=\left\{(b_1,\cdots,b_n)\in k^n\,\middle|\,g(b_1,\cdots,b_n)=0\ \ for\ \ all\ \ g\in I(O)\right\}$.

By the property of that k is an algebraically closed field, then ϕ has no multiple roots, it implies that $\phi\in(\phi)=rad(I(O))$. Hence we have

$$I(Z(I(O)))\subset rad(I(O)). \qquad (13)$$

For any $\psi\in rad(I(O))$, there exists a m such that $\psi^m\in I(O)$. Thus $\forall(b_1,\cdots,b_n)\in Z(I(O))$, there exists $\psi^m(b_1,\cdots,b_n)=0$. Hence $\psi^m\in I(Z(I(O)))$ implies

$$I(Z(I(O)))\supset rad(I(O)). \qquad (14)$$

$I(Z(I(O)))=rad(I(O))$ right by the (12) and (13).

Consider the bijection $O\to I(O)$, then we have $I(Z(I(O)))\xrightarrow{\sim}Z(I(O))$ by the (12), and $Z(I(O))\xrightarrow{\sim}I(Z(I(O)))=rad(I(O))\xrightarrow{contain}I(O)$.

Consider the bijection $I(O)\to Z(I(O))$, we have $I(O)\xrightarrow{\sim}Z(I(O))\xrightarrow{contain}O$. Therefore, we can use Figure 1 below to describe the corresponding relationship mentioned above.

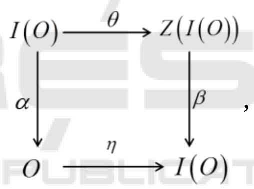

Figure 1: The commutative graph of algebraic subset and ideals.

where α,β are surjective by that $Z(I(O))\xrightarrow{contain}O$, and η,θ are isomorphic. Q.E.D

3 CONCLUSION

The expression of Hodge operator is given by defining affine space on n-dimensional vector space and introducing algebraic subset. The process of constructing prime spectrum based on polynomial ring is given, and on this basis, the ringed space satisfying the affine scheme condition is constructed. Finally, the Nullstellensatz theorem on the base space of affine scheme has been obtained. However, this paper only addresses some of the most basic issues in Hodge's theory and does not extend to the relevant theories of scheme. Therefore, it is necessary to further investigate the relationship between Hodge decomposition theory (Fox J, 1994) and Hodge's harmonic representation (Nagase M, 1988) and scheme in future.

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