Weakly Mixing and Topological Conjugation of Discrete Dynamical System

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Abstract: In this paper, by using Urysohn lemma to construct Metric space, introduce topological discrete dynamical system, and study its weakly mixing and simple ergodicity. Moreover, by using topological conjugation relation, the corresponding communicative condition that between measure preserving systems are given.

1 INTRODUCTION

Dynamical system describes the evolution of a point in geometric space over time. The weakly mixing, arbitrary order mixing and ergodic theory are the core problems in the study of dynamic system. Especially Rohlin's problem is an important issue that urgently needs to be solved in this field. For example, Host proved mixing of all orders and pairwise independent joining of systems with singular spectrum (Host B, 1991), Kalikow proved twofold mixing implies arbitrary mixing for rank one transformation (Host B, 1984), and Ryzhikov proved twofold mixing implies arbitrary mixing for finite rank (Ryzhikov, V. V, 1991). This article lays the theoretical foundation for attempting to study Rohlin's problem.

2 METRIC SPACE AND WEAKLY MIXING OF MEASURE PRESERVING SYSTEM

First, the metric topological space is obtained by constructing a given topological space and using Urysohn lemma. Thus topological discrete dynamical system is introduced. The weakly mixing of measure preserving system is proved first, and several commutative relationships of measure preserving system are given by commutative diagrams. Considering topological conjugation, the corresponding communication conditions between measure preserving systems are given.

2.1 Introduction of Measurable Space and Probability Space

Definition 2.1.1: Select the subsets of X_0 to establish a subset family χ_0, then X_0 must be the maximum element in χ_0, we have

(a) $X_0 \in \chi_0$;

(b) $\forall A \subset X_0$, because χ_0 is consist of the subsets of X_0, so $A \in \chi_0$, then $A^{X_0} \subset X_0$ derives $A^{X_0} \in \chi_0$ in the same way;

(c) If $B = \bigcup_{n=1}^{\infty} A_n$ where $A_n \in \chi_0$, so $A_n \subset X_0, n = 1, 2, \cdots$ and $\bigcup_{n=1}^{\infty} A_n \subset X_0$, $B \subset X_0$, we know $B \in \chi_0$ on the basis of the condition (b).

So χ_0 is a $\sigma-$ ring of X_0, the X_0 is a measurable space. Let the measure of X_0 be μ, $T: X_0 \to X_0$ is also measurable. According to the definition of the probability space, let $\mu: \chi_0 \to [0,1]$ and $\mu(X_0) = 1$. We have a probability space (X_0, χ_0, μ). According to the "Definition 2.1.1([Paul R., 1974])" and the definition of measurable space, it be satisfied that

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$$\forall A \in \chi_0, \mu(T^{-1}A) = \mu(A),$$

We have measuring system (X_0, χ_0, μ, T). We need to construct and show X have the relationship with X_0 is a compact metric space in next step.

2.2 Construction of Compact Metric Space and Introduction of Discrete Dynamical Systems

Definition 2.1.2: Let every single point set in X is closed set. If there are two open sets contain any point x in X and a closed set B exclude x respectively, then X is a regular space (Munkres J R, 2004).

Definition 2.1.3: The topology $\mathcal{T}$ is subset family of X, it satisfied with these conditions:

(a) $\varnothing$ and X in $\mathcal{T}$;

(b)The union of element of any subfamily of $\mathcal{T}$ in $\mathcal{T}$;

(c)The intersection of element of any finite subfamily of $\mathcal{T}$ in $\mathcal{T}$;

Then X is a topological space, $\mathcal{T}$ is topology of X (Munkres J R, 2004).

The probability space (X_0, χ_0, μ) be satisfied with the definition of measurable space. It is easy to know χ_0 is the subfamily of X_0 and X_0 is the maximum element in χ_0. It is natural that $X_0 \in \chi_0$ and $\varnothing \subset X_0 \in \chi_0$.

Now selecting any subfamily of X_0 is $\chi = \{A_i\}_{i\in I}$, where $A_i \subset X_0$. According to the condition (c) of the measurable space, we have $\bigcup_{i\in I} A_i \subset X_0$, and then $\bigcup_{i\in I} A_i \in \chi_0$. Selecting any finite subfamily of χ_0 is $\chi' = \{B_1, B_2, \cdots B_m\}$, where $B_k \subset X_0, k = 1, \cdots, m$. So $\bigcap_{k=1}^{m} B_k \subset B_k \subset X_0$, and then $\bigcap_{k=1}^{m} B_k \in \chi_0$. After simple verification, we can see χ_0 is a topology of X_0 and (X_0, χ_0) is a topological space.

Choose denumerable closed sets with single point of X_0: $\{p_1\}, \{p_2\}, \{p_3\} \cdots$.Let

$$p_1 \in U_1, p_2 \in U_2, p_3 \in U_3, \cdots,$$

where $U_1, U_2, U_3, \cdots$ are the neighbourhoods of the corresponding points and $\bigcap_{i=1}^{\infty} U_i = \varnothing$. $\{p_m\}$ is a single point closed set. Let the other be $\{p_1\}, \cdots, \{p_{m-1}\}, \{p_{m+1}\}, \cdots$, and

$$E = \{p_1\} \bigcup \cdots \bigcup \{p_{m-1}\} \bigcup \{p_{m+1}\} \bigcup \cdots.$$

Because $\{p_1\}, \cdots, \{p_{m-1}\}, \{p_{m+1}\}, \cdots$ are all closed sets. According to the theory of topology as we know, E is a closed set. Let

$$U = U_1 \bigcup U_2 \bigcup \cdots \bigcup U_{m-1} \bigcup U_{m+1} \bigcup \cdots.$$

Because $p_k \in U_k, k = 1, 2, \cdots$. It is obvious that $\{p_k\} \subset U_k, k = 1, 2, \cdots$. By using the principle of inclusion relation between sets, $E \subset U$. Due to $p_i \neq p_j, i \neq j$, we have $p_m \notin U$ and $p_m \in \{p_m\} \subset U_m$. It is known $\bigcap_{i=1}^{\infty} U_i = \varnothing$, so $U_m \bigcap U = \varnothing$. It is obvious that

$$\{p_k\} \subset X_0, k = 1, 2, \cdots,$$

from $p_1, p_2, p_3, \cdots \in X_0$ and property of neighbourhood.

Then

$$\{p_m\} \subset U_m \subset X_0 \text{ and}$$

$$U = U_1 \bigcup \cdots \bigcup U_{m-1} \bigcup U_{m+1} \bigcup \cdots \subset X_0.$$

Let $X = U_m \bigcup U \subset X_0$. By the "Definition 2.1.1", X_0 is a regular space.

According to the previous discussion, it is known that $X \subset X_0$ and X_0 is a topological space. By Using inherited principle of topological space, X is also a regular space. The topology $\mathcal{T}$ of X is determined at the same time.

Theorem 2.1.4: Let X is a topological space. ℓ is a subfamily of open sets of X. For any open set U of X and any point $x \in U$, an element C of ℓ existed, Such that $x \in C \subset U$. Then ℓ is a basis on the topology of X (Munkres J R, 2004).

It is known that the topological space X is a measurable space. We need to prove that any open set U of X be corresponding to denumerable open subsets $C_1, C_2, \cdots$, such that $U = \bigcup_{i=1}^{\infty} C_i$ and $\bigcap_{i=1}^{\infty} C_i = \varnothing$.

It is obvious that open set $U \subset X$ be equal to union of disjoint subsets, I. e. $U = \bigcup_{i=1}^{\infty} D_i$ and $\bigcap_{i=1}^{\infty} D_i = \varnothing$. If D_l is a closed set, the others are open sets. There be an open set C_l such that $D_l \subset C_l$. It is evident that C_l could have intersection with some disjoint open sets, i. e.

$$C_l \cap D_{i_k} \neq \varnothing, k = 1, 2, \cdots.$$

According to the theory of topology as we know, the finite intersection of open sets is the open set. We can have denumerable disjoint open sets $C_1, C_2, \cdots$ by rearranging $D_{i_k} - (C_l \cap D_{i_k}), k = 1, 2, \cdots$ and the rest of open sets, such that $U = \bigcup_{i=1}^{\infty} C_i$. If non-open sets $D_{j_1}, D_{j_2}, \cdots$ exists, we can find open sets $C_{j_1}, C_{j_2}, \cdots$ have intersection with some disjoint open sets by same way. I. e.

$$C_{j_1} \cap D_{n_k^{(1)}} \neq \varnothing, C_{j_2} \cap D_{n_k^{(2)}} \neq \varnothing, \cdots, k = 1, 2, \cdots.$$

By the same reason,

$$C_{j_1} \cap D_{n_k^{(1)}}, C_{j_2} \cap D_{n_k^{(2)}}, \cdots, k = 1, 2, \cdots$$

are open sets. We can have denumerable disjoint open sets $C_1, C_2, \cdots$ by rearranging

$$D_{n_k^{(1)}} - (C_{j_1} \cap D_{n_k^{(1)}}), D_{n_k^{(2)}} - (C_{j_2} \cap D_{n_k^{(2)}}), \cdots, k = 1, 2, \cdots$$

and the rest of open sets, such that $U = \bigcup_{i=1}^{\infty} C_i$. So this problem has been proved.

It is necessary to construct the denumerable open set family ℓ:

Situation i: If X has the only one open set U, aforementioned $C_1, C_2, \cdots$ can be regarded as the elements of ℓ. I. e. $\ell = \{C_1, C_2, \cdots\}$ is a denumerable family;

Situation ii: If X has two open sets U, V, by the previous mentioned method about V to get $V = \bigcup_{i=1}^{\infty} D_i$, where D_i is in accordance with C_i, $i = 1, 2, \cdots$. I. e. $\ell = \{C_1, D_1, C_2, D_2, \cdots\}$ is a denumerable family by the arrangement principle of denumerable sets;

Situation iii: If X has the denumerable open sets $U_1, U_2, \cdots$, let $U_1 = \bigcup_{i=1}^{\infty} C_i^{(1)}, U_2 = \bigcup_{i=1}^{\infty} C_i^{(2)}, \cdots$, where $C_i^{(j)}, i, j = 1, 2, \cdots$ has the same definition with situation i and situation ii.

Make the following arrangement (I P. Natanson, 2016):

$$\ell = \begin{Bmatrix} C_1^{(1)}, & C_2^{(1)}, & C_3^{(1)}, & \cdots \\ C_1^{(2)}, & C_2^{(2)}, & C_3^{(2)}, & \cdots \\ C_1^{(3)}, & C_2^{(3)}, & C_3^{(3)}, & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{Bmatrix}.$$

Put the elements with the same superscript plus subscript together to correspond to the elements of N and rearrange them. We have $\ell = \{C_1^{(1)}, C_2^{(1)}, C_1^{(2)}, C_3^{(1)}, C_2^{(2)}, C_1^{(3)}, C_4^{(1)}, \cdots\} \sim N$. It is obvious that ℓ is denumerable.

The above is the ideal situation, that is to say, denumerable open subsets of X satisfy $\bigcap_{i=1}^{\infty} U_i = \varnothing$. If $\bigcap_{i=1}^{\infty} U_i \neq \varnothing$, we can take all of open sets as a whole. And then the detail process is similar to the situation i. If some terms $U_{i_1}, U_{i_2}, \cdots, U_{i_n}$ have intersection, we can think of these terms as a whole and rearrange them. The result of the rearrangement will not change. In conclusion, ℓ is a denumerable family.

Assume that $\ell = \{A_1, A_2, A_3, \cdots\}$, where A_i always be contained in some open sets of X. For any open set V of X, we have $V = \bigcup_{i=1}^{\infty} A_{k_i}$. According to the previous discussion, we have $\bigcap_{i=1}^{\infty} A_{k_i} = \varnothing$, where $A_{k_i} \subset V, i = 1, 2, \cdots$. So for any point $x \in V$, we always find some A_{k_m} such that $x \in A_{k_m} \subset V$. Because $A_{k_m} \in \ell$ is known, so ℓ is denumerable basis of topology $\mathcal{T}$ of X by introduction of theorem 1. Therefore regular space X has the denumerable basis.

Theorem 2.1.5: (Urysohn) Every regular space X with the denumerable basis is a metric space (Munkres J R, 2004).

By the "Theorem 2.1.5", we have the condition (I): Topological space X is a metric space.

According to above discussion (topology space X has the denumerable base) and the second axiom of countability, we have X is a second denumerable space. Because the second denumerable space is the separable Lindelof space by topological theory as we know. So we have condition (II): X is a compact space. According to condition (I) and condition (II), topological space X is a compact metric space. Considering the situation in question, so let

$$.\rho(x,y)=d_X(x,y)=\max|x-y|,\forall x,y\in X$$

Because X is a compact metric space, we can definite the self-action $T:X\to X$ on X .The topology discrete dynamic system is introduced：

$$\{\cdots,T^{-n},\cdots,T^{-1},T^0,T^1,\cdots T^n,\cdots\}.$$

(Note: Let $C^0(X)$ be the collection of all consecutive self-mappings of X , then $T\in C^0(X)$. (X,T) represents compact system which be generated by continuous self-mapping T of compact metric space X .)

2.3 Weakly Mixing and Commutativity of Measure Preserving System

Since (X_0,χ_0,μ) is a probability space by the section 2.1. Due to $X\subset X_0$, (X,χ,μ) is also a probability space. Because (X_0,χ_0,μ) is a measurable space, (X,χ,μ) is obviously a measurable space. Therefore the measure preserving system (X,χ,μ,T) can be used as the research object.

Considering that measure preserving system (X,χ,μ,T) is weakly mixing, we have these conditions:

1) T is topological weakly mixed;

2) measure preserving system (X,χ,μ,T) is ergodic;

3) $(X,\chi,\mu,T)\times(X,\chi,\mu,T)$ is ergodic.

Condition 2) can be launched: if self-mapping T is a measure-preserving map, μ is a invariant measure, μ is ergodic, for $B\in\chi$, we have $T^{-1}(B)=B$ $\Rightarrow\mu(B)=1$ or $\mu(B)=0$ 。

The first description of independence about $\forall A,B\in\mathcal{X}$ under the known conditions:

Because $\{n\}\subset J$ is sequence with density 1, and μ is ergodic. When $n=1$, we have B with $T^{-1}(B)=B$ or A with $T^{-1}(A)=A$. According to relevant definitions, it is known that $\mu(A\cap T^{-1}B)=\mu(A\cap B)$. So we have $\mu(B)=1$ or $\mu(B)=0$.

If $\mu(B)=1$, then

$$\mu(A\cap B)=\mu(A)=\mu(A)\mu(B);$$

If $\mu(B)=0$, then

$$\mu(A\cap B)=\mu(B)=\mu(A)\mu(B).$$

$\mu(A\cap B)=\mu(A)\mu(B)$ is right. Therefore the independence about any $A,B\in\chi$ is right under the known conditions. Let $\mu_A(B)=\dfrac{\mu(A\cap B)}{\mu(A)}$, so $\mu_A(B)=\mu(B)$.

We need to build the exchange relationship between measurable spaces:

Let Y is a interval of real number field R . Assuming that $\sigma-$ ring $\mathcal{A}$ is made of subsets of Y. According to the definition of measurable space, we can construct measurable space $(Y,\mathcal{A},\mathfrak{M})$, where measure $\mathfrak{M}$ is a mapping, i. e. $\mathfrak{M}:R\to[0,1]$. $\mathfrak{M}$ has the following properties (I P. Natanson, 2016):

1) $\mathfrak{M}(\varnothing)=0$;

2) $\forall i\neq j,A_i\cap A_j=\varnothing$, where $A_i\in\mathcal{A},i=1,2\cdots$;

3) $\mathfrak{M}(\bigcup_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\mathfrak{M}(A_n)$ under condition 1) and condition 2).

Let Y be satisfied with $\mathfrak{M}(Y)=1$, we have the probability space $(Y,\mathcal{A},\mathfrak{M})$. According to the "Definition 2.1.1", $(Y,\mathcal{A},\mathfrak{M})$ is a metric space. The continuous self-mapping $S:Y\to Y$ can be established. Because $\forall A\in\mathcal{A},S^{-1}(A)\in\mathcal{A}$, so $S^{-1}:Y\to Y$ is a continuous self-mapping. The topological discrete dynamical system is introduced by

$$\{\cdots,S^{-n},\cdots,S^{-1},S^0,S^1,\cdots S^n,\cdots\}.$$

Because $(Y,\mathcal{A},\mathfrak{M})$ is a measurable space. So $S:Y\to Y$ and $S^{-1}:Y\to Y$ are measurable by the definition of measurable space. Thus $\forall A\in\mathcal{A}$,

${S^{-1}}_A : A \to A$ is measurable. As a result of $S^{-1}A \subset A$, we have

$$\mathfrak{M}(S^{-1}A) \le \mathfrak{M}(A). \qquad (1)$$

Because ${S^{-1}}_A : A \to A$ is measurable, $S_A : A \to A$ is also measurable and $SA \subset A$. We have

$$A = S(S^{-1}A) \subset S^{-1}A \Rightarrow \mathfrak{M}(A) = \mathfrak{M}(S(S^{-1}A)) \le \mathfrak{M}(S^{-1}A). \qquad (2)$$

$\mathfrak{M}(S^{-1}A) = \mathfrak{M}(A)$ can be got from formula (1) and formula (2). So $S : Y \to Y$ is measure-preserving, $\mathfrak{M}$ is invariant measure of S. Therefore measure preserving system $(Y, \mathcal{A}, \mathfrak{M}, S)$ is ergodic, where $\mathfrak{M}$ is ergodic.

Let $\Phi : X \to Y$ be continuous mapping. Because $(X, \mathcal{X}, \mu)$ and $(Y, \mathcal{A}, \mathfrak{M},)$ are metric spaces. Assuming that the metric of X is ρ_1 and the metric of Y is ρ_2, then $\forall \varepsilon > 0, x, y \in X$, exists $\delta > 0$, such that

$$\rho_1(x, y) < \delta \Rightarrow \rho_2(\Phi(x), \Phi(y)) < \varepsilon.$$

In a similar way, $\forall \delta > 0, w, z \in Y$, exists $\varepsilon > 0$, such that

$$\rho_2(w, z) < \varepsilon \Rightarrow \rho_1(\Phi^{-1}(w), \Phi^{-1}(z)) < \delta.$$

$\Phi^{-1} : Y \to X$ is a continuous mapping, thus Φ is homeomorphism mapping. Considering the following conditions:

1) $(Y, \mathcal{A}, \mathfrak{M}, S)$ is measure preserving and ergodic;

2) Φ is homeomorphism mapping;

3) $\mu : X \to [0,1]$ and $\mathfrak{M} : Y \to [0,1]$ are finite measures;

4) (X, T) and (Y, S) are compact systems.

We have the following commutative diagrams:

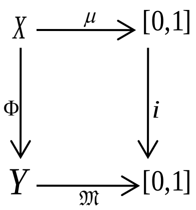

Figure 1. The commutative diagram of probability spaces.

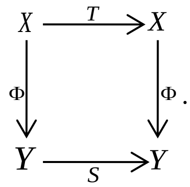

Figure 2. The commutative diagram of measure preserving systems.

We have $S\Phi = \Phi T$ by the figure 2, where Φ is topological conjugate from T to S. The figure 2 implies the figure 3:

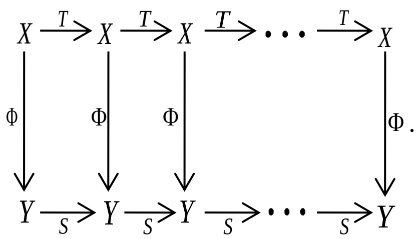

Figure 3. The finite order commutative diagram of measure preserving systems.

We have $S^n\Phi = \Phi T^n$ by figure 3, where Φ is topological conjugate from T^n to S^n. In a similar way, $\forall n > 0$, Φ^{-1} is topological conjugate from S to T, Φ^{-1} is topological conjugate from S^{-n} to T^{-n}. Therefore we have the following figure:

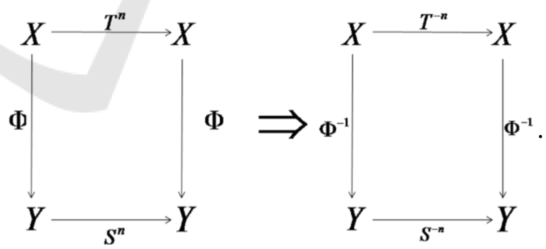

Figure 4: (a). The merge commutative diagram. (b). The inverse mapping commutative diagram.

The figure 1 implies $\mu = \mathfrak{M}\Phi$, the figure 4(a) and 4(b) imply that $\Phi T^{-n} = S^{-n}\Phi$, let it be called by corresponding commutative condition.

3 CONCLUSION

By constructing a given topological space and using Urysohn lemma, a metric topological space is obtained. On this basis, topological discrete dynamical system is introduced, weakly mixing of measure preserving system has been proved, and several commutative relationships of measure preserving systems are given. By using topological conjugation, the corresponding communicative condition that between measure preserving systems are given. In the future, more in-depth research is needed on the arbitrary mixing and ergodicity of discrete dynamical system. The application of mixing and complex ergodicity of discrete dynamical system in Markov chain is also an important research direction in the next step.

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