Energy Optimal Control of Collision Avoidance from Two Inertial Objects

Pengfei Zhang, Hao Zhang, Feng Song and Yiqun Zhang*
Beijing Institute of Electronic System Engineering, Beijing, 100854, China

Keywords: Optimal Control, Differential Games, Two vs One, Collision Avoidance.

Abstract: Energy optimal control of collision avoidance is significant in aerospace and robotics. In this paper, we are to investigate the energy optimal control problem of avoiding collision from two inertial objects, in a differential game manner. In particular, we develop the optimal control structure by applying the already published payoff augmentation method, and the result shows that the control of the evader is zero in some time periods. For verification, a simulation case is constructed and the results are consistent with the theoretical analysis.

1 INTRODUCTION

The study of energy optimal control is significant in collision avoidance problems. The problem can be solved by regarding it as a pursuit evasion game (Başar and Zaccour, 2018; Friedman, 2013), in which the obstructive objects are assumed to be intelligent. Pursuit evasion games are widely used in aerospace, collision avoidance, robotics (Isaacs and Philip, 1966; Exarchos et al., 2015; Exarchos et al., 2016; Kumkov et al., 2014). In a pursuit evasion game, the pursuer tries to approach the evader, while the evader avoids collision with the pursuer. In many cases, there might be more than one obstructive objects, and the evader has to design its optimal control taking all the obstructive objects into consideration. The problem becomes the well known two pursuers one evader game when two obstructive objects exist, which is to be investigated in this paper.

Two pursuers and one evader game is more difficult than one pursuer and one evader game (Ganebny et al., 2012; Garcia et al., 2017; Exarchos et al., 2016), for the payoff function has a more complex form. When considering the energy optimal control problem, the energy term should be included in the payoff function. Extensive studies have researched the two pursuers and one evader differential games, among which the linear dynamic model and simple motion model of the players are mostly considered (Le Ménec, 2011; Garcia et al., 2017; Makkapati et al., 2018; Ganebny et al., 2012; Hagedorn and Breakwell, 1976; Ho et al., 1965; Kumkov et al., 2014; Pachter et al., 2020; Pachter et al., 2019; Sun et al., 2017). For the linear dynamic model game, the zero effort miss approach is widely adopted, which brings remarkable convenience. For the simple motion games (Garcia et al., 2017; Makkapati et al., 2018; Pachter et al., 2020; Pachter et al., 2019), it has shown the most of the optimal controls are moving straight. When facing the inertial model two pursuers one evader games, Zhang et.al has analysed the time optimal problem by proposing a payoff augmentation method and an open loop Stackelberg approach. This paper will analyse the energy optimal problem by the method proposed by the reference (Zhang et al., 2022).

In this paper, we study the energy optimal control of avoiding collision from two inertial objects, in a differential game manner. It is assumed that our flying vehicle (the evader) is hit by the obstructive objects when their distance is less than l. We adopt the payoff augmentation method introduced by reference (Zhang et al., 2022) to obtain the optimal control form. As a result, it will be shown that the energy optimal control problem has a different result with the time optimal problem. The major difference is that the evader adopts zero control in some time periods, which is consistent with intuition.

Zhang, P., Zhang, H., Song, F. and Zhang, Y.
Energy Optimal Control of Collision Avoidance from Two Inertial Objects.
DOI: 10.5220/0012016200003612
In *Proceedings of the 3rd International Symposium on Automation, Information and Computing (ISAIC 2022)*, pages 681-685
ISBN: 978-989-758-622-4; ISSN: 2975-9463

2 PROBLEM STATEMENT

2.1 State Equation

Three players move on a planar. One is the evader, the rest two are the pursuers P1 and P2. The evader survives should the distance of the evader and the closest pursuer is bigger or equal than l. Th game operates in the time interval $[0,T]$. The three players have inertial dynamic models controlled by acceleration vectors, written as

$$\begin{cases} \dot{x}^{p1} = v_x^{p1} \\ \dot{y}^{p1} = v_y^{p1} \\ \dot{v}_x^{p1} = a_1 \cos u_1 \\ \dot{v}_y^{p1} = a_1 \sin u_1 \end{cases}, \begin{cases} \dot{x}^{p2} = v_x^{p2} \\ \dot{y}^{p2} = v_y^{p2} \\ \dot{v}_x^{p2} = a_2 \cos u_2 \\ \dot{v}_y^{p2} = a_2 \sin u_2 \end{cases}, \begin{cases} \dot{x}^{e} = v_x^{e} \\ \dot{y}^{e} = v_y^{e} \\ \dot{v}_x^{e} = a_3 \cos u_3 \\ \dot{v}_y^{e} = a_3 \sin u_3 \end{cases} \tag{1}$$

where the supscripts pi and e are used for the i-th pursuer and the evader respectively. x, y are the positions, v_x, v_y are the velocities. $a_1 \in [0,a_{1m}]$, $a_2 \in [0,a_{2m}]$ and $a_3 \in [0,a_{3m}]$ are the magnitudes of the acceleration vectors. u_1, u_2 and u_3 represent the headings of the acceleration vectors of P1, P2 and E.

To obtain a reduced system, new variables are introduced

$$\begin{aligned} x_1 &= x^{p1} - x^e \\ y_1 &= y^{p1} - y^e \\ v_{1x} &= v_x^{p1} - v_x^e \\ v_{1y} &= v_y^{p1} - v_y^e \\ x_2 &= x^{p2} - x^e \\ y_2 &= y^{p2} - y^e \\ v_{2x} &= v_x^{p2} - v_x^e \\ v_{2y} &= v_y^{p2} - v_y^e \end{aligned} \tag{2}$$

where $x_1, y_1, v_{1x}, v_{1y}, x_2, y_2, v_{2x}, v_{2y}$ represent the relative positions and velocities.

Based on (2), (1) is rewritten as

$$\begin{aligned} \dot{x}_1 &= v_{1x} \\ \dot{y}_1 &= v_{1y} \\ \dot{v}_{1x} &= a_1 \cos u_1 - a_3 \cos u_3 \\ \dot{v}_{1y} &= a_1 \sin u_1 - a_3 \sin u_3 \\ \dot{x}_2 &= v_{2x} \\ \dot{y}_2 &= v_{2y} \\ \dot{v}_{2x} &= a_2 \cos u_2 - a_3 \cos u_3 \\ \dot{v}_{2y} &= a_2 \sin u_2 - a_3 \sin u_3 \end{aligned} \tag{3}$$

The payoff function is the evader's cost energy, under the state constraint that the distance of the evader and the pursuers are $\geq l$ in whole game period. In particular, the payoff function is written as

$$J = \int_0^T a_3 \mathrm{d}t \tag{4}$$

The evader wishes to minimize Eq. (4), while the pursuer has the opposite purpose. The state constraint is written as:

$$\begin{aligned} \min_{t\in[0,T]} \sqrt{x_1^2(t) + y_1^2(t)} \geq l \\ \min_{t\in[0,T]} \sqrt{x_2^2(t) + y_2^2(t)} \geq l \end{aligned} \tag{5}$$

where the evader has to make (5) hold while the pursuers wishes the opposite.

3 ANALYSIS OF OPTIMAL CONTROL

With the aid of the method proposed in reference (Zhang et al., 2022), the new payoff function is designed as

$$J = \int_0^T [a_3 + \frac{1}{k_1} e^{k_1^2(l^2 - x_1^2 - y_1^2)} + \frac{1}{k_2} e^{k_2^2(l^2 - x_2^2 - y_2^2)}] \mathrm{d}t \tag{6}$$

where k_1 approaches $+\infty$, k_2 approaches $+\infty$.

As what we have anticipated, the game is transformed to a new game of degree with payoff function (6).

Based on the state equation (3), the Hamiltonian function is written as

$$\begin{aligned} H = a_3 &+ \tfrac{1}{k_1} e^{k_1^2(l^2 - x_1^2 - y_1^2)} + \tfrac{1}{k_2} e^{k_2^2(l^2 - x_2^2 - y_2^2)} \\ &+ \lambda_1 v_{1x} + \lambda_2 v_{1y} + \lambda_3 (a_1 \cos u_1 - a_3 \cos u_3) \\ &+ \lambda_4 (a_1 \sin u_1 - a_3 \sin u_3) \\ &+ \lambda_5 v_{2x} + \lambda_6 v_{2y} + \lambda_7 (a_2 \cos u_2 - a_3 \cos u_3) \\ &+ \lambda_8 (a_2 \sin u_2 - a_3 \sin u_3) \end{aligned} \tag{7}$$

where $\lambda_i (i = 1,2,...,8)$ are co-states.

Based on (7), the co-state equation is derived as

$$\begin{aligned} \dot{\lambda}_1 &= 2k_1 x_1 e^{k_1^2(l^2 - x_1^2 - y_1^2)} \\ \dot{\lambda}_2 &= 2k_1 y_1 e^{k_1^2(l^2 - x_1^2 - y_1^2)} \\ \dot{\lambda}_3 &= -\lambda_1 \\ \dot{\lambda}_4 &= -\lambda_2 \\ \dot{\lambda}_5 &= 2k_2 x_2 e^{k_2^2(l^2 - x_2^2 - y_2^2)} \\ \dot{\lambda}_6 &= 2k_2 y_2 e^{k_2^2(l^2 - x_2^2 - y_2^2)} \\ \dot{\lambda}_7 &= -\lambda_5 \\ \dot{\lambda}_8 &= -\lambda_6 \end{aligned} \tag{8}$$

Since the pursuers maximizes H and the evader minimizes H in (7), the control equations are written as

$$\begin{aligned} & a_1 = a_{1m}, \cos u_1 = \tfrac{\lambda_3}{\sqrt{\lambda_3{}^2 + \lambda_4{}^2}}, \sin u_1 = \tfrac{\lambda_4}{\sqrt{\lambda_3{}^2 + \lambda_4{}^2}} \\ & a_2 = a_{2m}, \cos u_2 = \tfrac{\lambda_7}{\sqrt{\lambda_7{}^2 + \lambda_8{}^2}}, \sin u_2 = \tfrac{\lambda_8}{\sqrt{\lambda_7{}^2 + \lambda_8{}^2}} \\ & a_3 = \begin{cases} a_{3m}, \text{when } 1 - \sqrt{(\lambda_3 + \lambda_7)^2 + (\lambda_4 + \lambda_8)^2} < 0 \\ 0, \text{else} \end{cases} \\ & \cos u_3 = \tfrac{\lambda_3 + \lambda_7}{\sqrt{(\lambda_3 + \lambda_7)^2 + (\lambda_4 + \lambda_8)^2}}, \sin u_3 = \tfrac{\lambda_4 + \lambda_8}{\sqrt{(\lambda_3 + \lambda_7)^2 + (\lambda_4 + \lambda_8)^2}} \end{aligned} \tag{9}$$

Suppose that at time t_2 and time t_1, P2 and P1 attain their minimum distance l with E, that is $x_1^2(t_1) + y_1^2(t_1) = l^2$ and $x_2^2(t_2) + y_2^2(t_2) = l^2$. Without loss of generality, we assume that $t_1 \le t_2$.

a) Case $t_1 < t_2$

Since there are no terminal constraints on the state variables, we conclude that $\lambda_i(T) = 0 (i = 1,2,...,8)$. Based on (8), in the time interval $(t_2, T]$, the derivatives of the co-states approach zero as k_1 and k_2 approach $+\infty$. Therefore, the co-states in the time interval $(t_2, T]$ are written as:

$$\lambda_i(\tau) = 0, i = 1,2,...,8 \tag{10}$$

where $\tau \in (t_2, T]$.

At time t_2, it can be seen from (8) that $\dot{\lambda}_5$ and $\dot{\lambda}_6$ are not zero when k_2 approaches $+\infty$. Thereby, in the time interval $(t_1, t_2]$, the co-states are derived as:

$$\begin{aligned}
&\lambda_i(\tau) = 0, i = 1,2,3,4 \\
&\lambda_5(\tau) = -2k_2 x_2(t_2)\mathrm{d}t \\
&\lambda_6(\tau) = -2k_2 y_2(t_2)\mathrm{d}t \\
&\lambda_7(\tau) = -2k_2 x_2(t_2)\mathrm{d}t(t_2 - \tau) \\
&\lambda_8(\tau) = -2k_2 y_2(t_2)\mathrm{d}t(t_2 - \tau)
\end{aligned} \tag{11}$$

where $\tau \in (t_1, t_2]$.

At time t_1, it can be seen from (8) that $\dot{\lambda}_1$ and $\dot{\lambda}_2$ are not zero when k_1 approaches $+\infty$. Thereby, in the time interval $[0\ t_1]$, the co-states are derived as:

$$\begin{aligned}
&\lambda_1(\tau) = -2k_1 x_1(t_1)\mathrm{d}t \\
&\lambda_2(\tau) = -2k_1 y_1(t_1)\mathrm{d}t \\
&\lambda_3(\tau) = -2k_1 x_1(t_1)\mathrm{d}t(t_1 - \tau) \\
&\lambda_4(\tau) = -2k_1 y_1(t_1)\mathrm{d}t(t_1 - \tau) \\
&\lambda_5(\tau) = -2k_2 x_2(t_2)\mathrm{d}t \\
&\lambda_6(\tau) = -2k_2 y_2(t_2)\mathrm{d}t \\
&\lambda_7(\tau) = -2k_2 x_2(t_2)\mathrm{d}t(t_2 - \tau) \\
&\lambda_8(\tau) = -2k_2 y_2(t_2)\mathrm{d}t(t_2 - \tau)
\end{aligned} \tag{12}$$

where $\tau \in [0, t_1]$.

Since P1 and P2 respectively attain minimum distance with E at time t_1 and t_2, the derivatives of $x_1^2(t_1) + y_1^2(t_1)$ and $x_2^2(t_2) + y_2^2(t_2)$ are zero, yielding

$$\begin{aligned}
x_1(t_1)v_{1x}(t_1) + y_1(t_1)v_{1y}(t_1) = 0 \\
x_2(t_2)v_{2x}(t_2) + y_2(t_2)v_{2y}(t_2) = 0
\end{aligned} \tag{13}$$

b) Case $t_1 = t_2$

In this case, the co-states in the time interval $[0, t_1]$ are derived as:

$$\begin{aligned}
&\lambda_1(\tau) = -2k_1 x_1(t_1)\mathrm{d}t \\
&\lambda_2(\tau) = -2k_1 y_1(t_1)\mathrm{d}t \\
&\lambda_3(\tau) = -2k_1 x_1(t_1)\mathrm{d}t(t_1 - \tau) \\
&\lambda_4(\tau) = -2k_1 y_1(t_1)\mathrm{d}t(t_1 - \tau) \\
&\lambda_5(\tau) = -2k_2 x_2(t_1)\mathrm{d}t \\
&\lambda_6(\tau) = -2k_2 y_2(t_1)\mathrm{d}t \\
&\lambda_7(\tau) = -2k_2 x_2(t_1)\mathrm{d}t(t_1 - \tau) \\
&\lambda_8(\tau) = -2k_2 y_2(t_1)\mathrm{d}t(t_1 - \tau)
\end{aligned} \tag{14}$$

(11), (12) and (14) provide the expressions of the co-states. Based on the control equation in (9), the equilibrium strategies are obtained as below.

Based on (9), (11), (12) and (14), we infer that the pursuers adopt constant control. The magnitude and heading of the pursuer's acceleration are invariant. Specifically, the best strategies of the pursuers are written as

$$\begin{aligned}
a_1 &= a_{1m} \\
u_1 &= \vartheta_1 \\
a_2 &= a_{2m} \\
u_2 &= \vartheta_2
\end{aligned} \tag{15}$$

where $\vartheta_1 = \arctan 2[-y_1(t_1), -x_1(t_1)], \vartheta_2 = \arctan 2[-y_2(t_2), -x_2(t_2)]$.

Based on (9), the control of the evader a_3 and u_3 depend on the values of $\lambda_3 + \lambda_7$ and $\lambda_4 + \lambda_8$. The strategy of the evader is given in two cases as below.

a) Case $t_1 < t_2$

In the time interval $(t_1, t_2]$, based on (11) and Eq. (9), the control a_3 and u_3 are written as:

$$\begin{aligned}
&a_3 = \begin{cases} a_{3\mathrm{m}}, \text{when } 1 - |2k_2\mathrm{d}t(t_2 - \tau)l| < 0 \\ 0, \text{else} \end{cases} \\
&u_3 = \vartheta_2
\end{aligned} \tag{16}$$

where $\tau \in (t_1, t_2]$.

In the time interval $[0, t_1]$, based on (12) and (9), the control a_3 and u_3 are derived as:

$$\begin{aligned}
&a_3 = \begin{cases} a_{3m}, \text{when } 1 - 2l\mathrm{d}t \sqrt{\begin{matrix} k_1^2(t_1-\tau)^2 \\ +k_2^2(t_2-\tau)^2 \\ +2k_1k_2(t_1-\tau) \\ (t_2-\tau)\cos(\vartheta_1-\vartheta_2) \end{matrix}} < 0 \\ 0, \text{else} \end{cases} \\
&u_3 = \arctan 2[n, m] \\
&\text{where :} \\
&m = 2k_1 l\mathrm{d}t(t_1 - \tau)\cos\vartheta_1 + 2k_2 l\mathrm{d}t(t_2 - \tau)\cos\vartheta_2 \\
&n = 2k_1 l\mathrm{d}t(t_1 - \tau)\sin\vartheta_1 + 2k_2 l\mathrm{d}t(t_2 - \tau)\sin\vartheta_2
\end{aligned} \tag{17}$$

where $\tau \in [0, t_1]$.

b) Case $t_1 = t_2$

In the time interval $[0, t_1]$, based on (14) and (9), the controls are derived as:

$$\begin{aligned}
&a_3 = \begin{cases} a_{3m}, \text{when } 1 - 2l\mathrm{d}t \sqrt{(t_1-\tau)^2 \begin{bmatrix} k_1^2 + k_2^2 \\ +2k_1k_2 \\ \cos(\vartheta_1-\vartheta_2) \end{bmatrix}} < 0 \\ 0, \text{else} \end{cases} \\
&u_3 = \arctan 2[n, m] \\
&\text{where :} \\
&m = (2k_1 l\mathrm{d}t\cos\vartheta_1 + 2k_2 l\mathrm{d}t\cos\vartheta_2)(t_1 - \tau) \\
&n = (2k_1 l\mathrm{d}t\sin\vartheta_1 + 2k_2 l\mathrm{d}t\sin\vartheta_2)(t_1 - \tau)
\end{aligned} \tag{18}$$

where $\tau \in [0, t_1]$. It can be seen that u_3 is constant in this case.

In a quick summary, (15), (16), (17) and (18) present the equilibrium strategy structure of the differential game. To solve the game when given an initial state, a two point boundary problem formulated by the state equation (3), the co-state equation (8) and the control equation (9) needs to be calculated.

4 SIMULATION CASE

In this section, we present a simulation case. The end time $T = 1(s)$, the collision distance $l = 1(m)$, the bounds of the accelerations $a_{1m} = 15(m/s^2), a_{2m} = 15(m/s^2), a_{3m} = 10(m/s^2)$. The initial position of the evader is (0.4 m, 36.5 m), the initial velocity of the evader is (0 m/s, -100 m/s). The initial positions of the pursuers are (-2 m, 0 m) and (2 m, 0 m), the initial velocities of the pursuers are (0 m/s, 0 m/s). By solving the two point boundary problem, the parameters are calculated:

$$\begin{aligned} \vartheta_1 &= -0.086 \\ \vartheta_2 &= -3.006 \\ k_1 \mathrm{d}t &= 0.1472 \\ k_2 \mathrm{d}t &= 2.3073 \\ t_1 &= 0.365 \\ t_2 &= 0.366 \end{aligned} \tag{19}$$

Based on the control equation of the evader, it is derived that $a_3 = 0$ after time 0.135 s. The trajectory of the evader is shown in Fig. 1. The evader turns at first and then goes straight. The trajectory almost passes by the point (0, 0), which locates at the middle of the initial positions of the two pursuers.

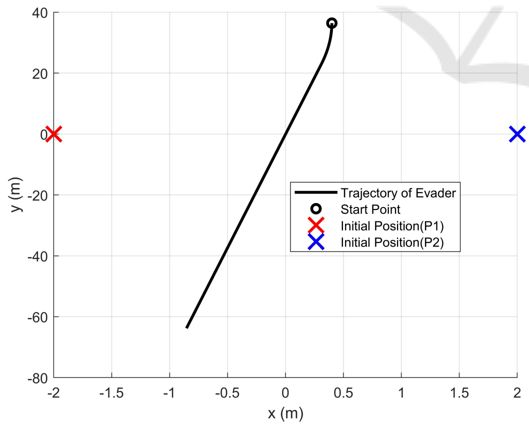

Figure 1: The trajectory of the evader.

The magnitude of the acceleration of the evader a_3 with respect to time is shown in Fig. 2. The evader adopts $a_3 = 10$ at first. After time 0.135 s, the evader adopts $a_3 = 0$, indicating that the evader travels on a straight line. Thus, the control effort (the integral of a_3 in (4)) equals to 1.35.

The variation of the acceleration vector of the evader – $(a_3 \cos u_3, a_3 \sin u_3)$ – with respect to time is shown in Fig. 3. It is almost a constant vector before time 0.135 s.

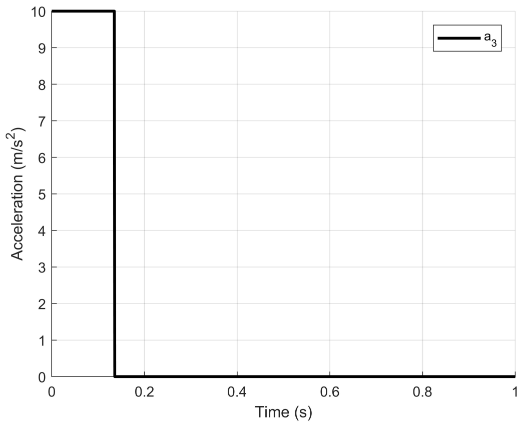

Figure 2: The magnitude of the acceleration of the evader.

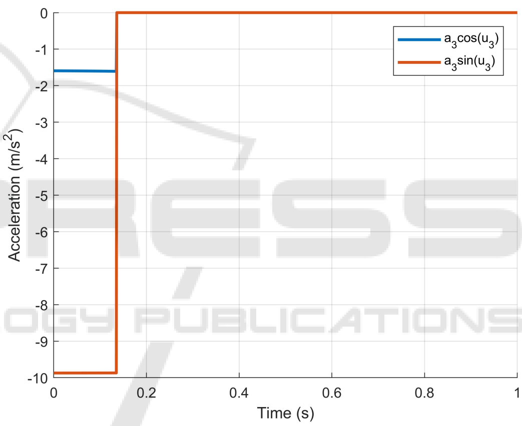

Figure 3: The variation of the acceleration vector of the evader with respect to time.

The distance of the evader and the two pursuers are shown in Fig. 4. The two distances attain their minimum at the time 0.365 s and 0.366 s, which is consistent with the values of t_1 and t_2 in (19). The minimum of the distance is almost 1 m.

5 CONCLUSIONS

This paper investigated how to avoid collision with two inertial obstructive objects by minimum energy cost. Thanks to the payoff augmentation method proposed by previous literature, the optimal control structure is obtained, as well as a two point boundary value problem established. A simulation was presented, and the result agree with the theoretical analysis.

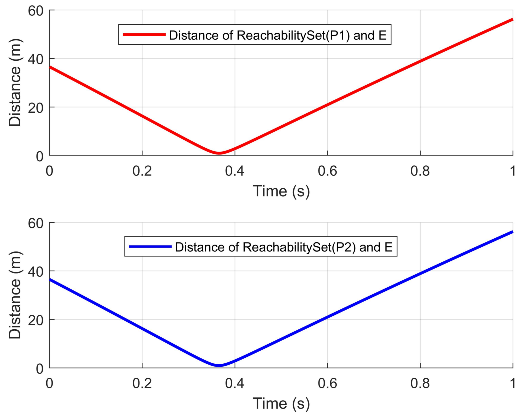

Figure 4: The distance of the evader and the reachability sets of the two pursuers.

In the future, we may focus on studying the collision avoidance issue with more sophisticated dynamic models.

ACKNOWLEDGMENTS

This work is supported, in part, by the NSFC 62088101 Autonomous Intelligent Unmanned Systems, and by the Zhejiang Provincial Natural Science Foundation of China (Grant No. LR20F030003).

REFERENCES

Başar, T. and Zaccour, G. (2018). *Handbook of dynamic game theory*. Springer.

Exarchos, I., Tsiotras, P., and Pachter, M. (2015). On the suicidal pedestrian differential game. *Dynamic Games and Applications*, 5(3):297–317.

Exarchos, I., Tsiotras, P., and Pachter, M. (2016). Uav collision avoidance based on the solution of the suicidal pedestrian differential game. In *AIAA Guidance, Navigation, and Control Conference*, page 2100.

Friedman, A. (2013). *Differential games*. Courier Corporation.

Ganebny, S. A., Kumkov, S. S., Ménec, S. L., and Patsko, V. S. (2012). Model problem in a line with two pursuers and one evader. *Dynamic Games and Applications*.

Garcia, E., Fuchs, Z. E., Milutinovic, D., Casbeer, D. W., and Pachter, M. (2017). A geometric approach for the cooperative two-pursuer one-evader differential game. *Ifac Papersonline*, 50(1):15209–15214.

Hagedorn, P. and Breakwell, J. V. (1976). A differential game with two pursuers and one evader. *Journal of Optimization Theory & Applications*, 18(1):15–29.

Ho, Y., Bryson, A., and Baron, S. (1965). Differential games and optimal pursuit-evasion strategies. *IEEE Transactions on Automatic Control*, 10(4):385–389.

Isaacs and Philip, R. (1966). Differential games : a mathematical theory with applications to warfare and pursuit, control and optimization. *Wiley*.

Kumkov, S. S., Ménec, S. L., and Patsko, V. S. (2014). Solvability sets in pursuit problem with two pursuers and one evader. *Ifac Proceedings Volumes*, 47(3):1543–1549.

Le Ménec, S. (2011). Linear differential game with two pursuers and one evader. In *Advances in dynamic games*, pages 209–226. Springer.

Makkapati, V. R., Sun, W., and Tsiotras, P. (2018). Pursuit-evasion problems involving two pursuers and one evader. In *2018 AIAA Guidance, Navigation, and Control Conference*, page 2107.

Pachter, M., Moll, A. V., Garcia, E., Casbeer, D., and Milutinovi, D. (2020). Cooperative pursuit by multiple pursuers of a single evader. *Journal of Aerospace Information Systems*, (2):1–19.

Pachter, M., Von Moll, A., Garcia, E., Casbeer, D. W., and Milutinović, D. (2019). Two-on-one pursuit. *Journal of Guidance Control & Dynamics*, pages 1–7.

Sun, W., Tsiotras, P., Lolla, T., Subramani, D. N., and Lermusiaux, P. F. (2017). Multiple-pursuer/one-evader pursuit–evasion game in dynamic flowfields. *Journal of guidance, control, and dynamics*, 40(7):1627–1637.

Zhang, Y., Zhang, P., Wang, X., Song, F., and Li, C. (2022). A payoff augmentation approach to two pursuers and one evader inertial model differential game. *IEEE Transactions on Aerospace and Electronic Systems*.