High-Order Networks and Stock Market Crashes

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Abstract: Network analysis has proven to be a powerful method to characterize complexity in socio-economic systems, and to understand their underlying dynamical features. Here, we propose to characterize the temporal evolution of higher-order dependencies within the framework of high-order networks. We test the possibility of financial crashes identification on the example of the Dow Jones Industrial Average (DJIA) index. Regarding topological measures of complexity, we see drastic changes in the complexity of the system during crisis events. Using high-order network analysis and topology, we show that, unlike traditional tools, the presented method is the most perspective, comparing to traditional methods of financial time series analysis.

1 INTRODUCTION

The growing availability of extensive data, often with time resolution, and coming from very different complex systems, has led to the possibility of a detailed study of their behavior, and in some cases also their internal mechanisms. Complex systems of various nature (biological, technical, financial, economic, etc. (Barabási and Pósfai, 2016; Latora et al., 2017) consist of numerous elementary units that interact heterogeneously with each other and in almost all cases exhibit emergent properties at the macroscopic level. Complex networks have become a powerful basis for studying the structure and dynamics of such systems (Newman, 2010). However, despite notable successes, their tools are limited to describing interactions between two units (or nodes) at the same time, which clearly contradicts the growing empirical data on group interactions in many cases of heterogeneous systems (Battiston and Petri, 2022). It turns out that connections and relationships take place not only between pairs of nodes, but also as collective actions of groups of nodes (Battiston et al., 2021; Sun and Bianconi, 2021), having a significant impact on the dynamics of interacting systems (Battiston et al., 2020; Majhi et al., 2022).

The idea of higher-order interactions is well known in the framework of solid-state physics when the approximation of paired interactions was replaced by multiparticle potentials or quantum mechanical calculations. Or in thermodynamics and statistical physics, Tsallis' efforts have built a theory of nonextensive interactions (Lyra and Tsallis, 1998; Bielinskyi et al., 2022). However, in all these cases, representations of higher-order interactions are simple in the sense that they do not contribute to the emerging complexity of the problem. In complex systems, usually described as networks, the situation is different, and in many cases these interactions need to be taken into account using more complex mathematical structures, such as hypergraphs and simplicial complexes.

To date, various models of higher-order networks have been developed (Bobrowski and Krioukov, 2022), the number of which, including modifications, is growing rapidly, given the relevance and topicality of the study. Let's briefly consider the main

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models that have shown themselves positively (Benson et al., 2018; Bick et al., 2021; Lambiotte et al., 2019).

Multiplex Networks. Multiplex, multilayering, and networks of networks have been proposed as modeling paradigms for systems in which there are different types of interactions (Boccaletti et al., 2014). They are designed to account for links of different types. However, in most cases, interactions are dyadic in nature and therefore can be represented by traditional networks (Skardal et al., 2021). The use of multiplex networks for financial analysis tasks is described in detail in (Bardoscia et al., 2021; del Rio-Chanona et al., 2020; Serguieva, 2016; Brummitt and Kobayashi, 2015; Cao et al., 2021; Xie et al., 2022; Gao, 2022; Aldasoro and Alves, 2018; Squartini et al., 2018; del Rio-Chanona et al., 2020), and higher-order networks in (Stavroglou et al., 2019; Jackson and Pernoud, 2021; Saha et al., 2022; Battiston et al., 2016; Huremovic et al., 2020; Franch et al., 2022; Bartesaghi et al., 2022; Han et al., 2022).

Hypergraphs and Simplicial Complexes. Computational methods from algebraic topology, hypergraphs, and simplicial complexes, which are sets of nodes and hyperlinks, allow encoding any number of units to explicitly consider systems beyond pairwise interactions and extract any "shape" of the data (Battiston et al., 2020; Santoro et al., 2022; Battiston et al., 2021; Berge, 1976).

Higher-order Markov Models for Sequential Data. Markov models defined in networks have become a popular way to describe and model the flows of information, energy, mass, money, etc. between various objects. If evolution is given by a Markov process (of the first order), this process can be considered as a random walk through the graph (Masuda et al., 2017). However, many empirically observed flows in networks have some dependence on the path. Thus, higher-order Markov chain models are required (Lambiotte et al., 2019).

Higher-order graphical Models and Markov Random Fields. Markov random fields, such as the Ising model and more general graphical models, have also been extended to higher-order models that take into account the interaction between several objects (Shemesh et al., 2013; Komodakis and Paragios, 2009).

Finally, more recently, Santoro et al. (Santoro et al., 2022) proposed a new structure for characterizing instantaneous patterns of signal co-fluctuation of all orders of interaction (pairs, triangles, etc.). To study the global topology of such co-fluctuations, they combined time series analysis, the theory of complex networks, and the analysis of topological data (Wasserman, 2018). They were able to show that, unlike traditional time series analysis tools, higher-order measures are able to reveal the subtleties of different space-time regimes in the case of three different studies: brain activity at rest (measured by fMRI data), stock option prices, and epidemic tasks.

In this paper, we consider the possibility of applying multiplex and higher-order network techniques to modeling crisis states in the stock market. Section 2 presents a graph representation of time series based on the classically paired visibility – visibility graph. Section 3 presents the theory of multiplex networks, which makes it possible to study systems of subgraphs (layers) and their inter- and intra-layer connections. Measures based on them are also provided. Section 4 describes high-order network extensions, various approaches to encoding high-order connections, and measures that will be used for both classical and high-order networks. Section 5 presents empirical results, together with which a comparative analysis of measures based on classical networks, multiplex, and higher-order networks is carried out. Section 6 presents the conclusions of the work done and further prospects.

2 VISIBILITY GRAPH

Visibility graph (VG), which was proposed by Lacasa et al. (Lacasa et al., 2008) is typically constructed from a univariate time series. In a visibility graph, each moment in the time series maps to a node in the network, and an edge exists between the nodes if they satisfy a "mutual visibility" condition.

"Mutual visibility" can be understood by imagining two points x_i at time t_i and x_j at time t_j as two hills of a time series, which can be understood as a landscape, and these two points are "mutually visible" if x_i has no any obstacles in the way on x_j. Formally, two points are mutually visible if, all values of x_k between t_i and t_j satisfy:

$$x_k < x_i + \frac{t_k - t_i}{t_j - t_i}\left[x_j - x_i\right], \quad \forall k : i < k < j \qquad (1)$$

Horizontal visibility graph (HVG) (Luque et al., 2009) is a restriction of usual visibility graph, where two points x_i and x_j are connected if there can be drawn a *horizontal* path that does not intersect an intermediate point x_k, $i < k < j$. Equivalently, node x_i at time t_i and node x_j at time t_j are connected if the horizontal ordering criterion is fulfilled:

$$x_k < \inf(x_i, x_j), \quad \forall k : i < k < j. \qquad (2)$$

Figure 1 is an approximate illustration of the construction of visibility graphs.

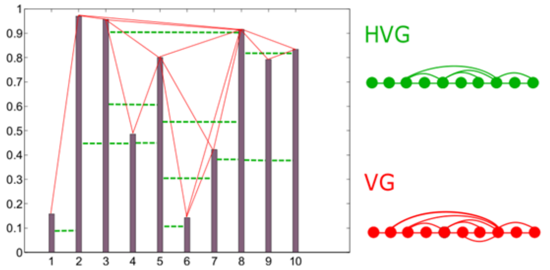

Figure 1: Schematic illustration of the VG (red lines) and the HVG (green lines). Adapted from (Iacovacci and Lacasa, 2016).

3 MULTIPLEX ORDERNESS AND MEASURES OF COMPLEXITY

Multiplex network (Kivelä et al., 2014) is the representation of the system which consists of the variety of different subnetworks with inter-network connections. For working with multiplex financial networks, we set two tasks:

- convert separated time series into network that represent a layer of a multiplex network. The procedure of conversion is presented in section 2;
- create intra-layer connection between each subnetwork.

Figure 2 represents an algorithm for creating a three-layered multiplex visibility graph.

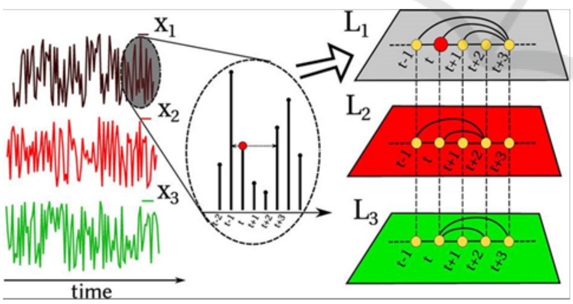

Figure 2: Illustration of the multiplex VG formation on the example of three layers. Adapted from (Lacasa et al., 2015).

Multiplex network is the representation of a pair $M = (G,C)$, where $\{G_\alpha \,|\, \alpha \in 1,\ldots,M\}$ is a set of graphs $G_\alpha = (X_\alpha, E_\alpha)$ that called layers and

$$C = \left\{E_{\alpha\beta} \subseteq X_\alpha \times X_\beta \,|\, \alpha,\beta \in 1,\ldots,M, \alpha \neq \beta\right\} \quad (3)$$

is a set of intra-links in layers G_α and G_β ($\alpha \neq \beta$). E_α is intra-layer edge in M, and each $E_{\alpha\beta}$ is denoted as inter-layer edge.

A set of nodes in a layer G_α is denoted as $X_\alpha = \{x_1^\alpha,\ldots,x_{N_\alpha}^\alpha\}$, and an intra-layer adjacency matrix as $A^{[\alpha]} = (a_{ij}^\alpha) \in \mathrm{Re}^{N_\alpha \times N_\alpha}$, where

$$\alpha_{ij}^\alpha = \begin{cases} 1, & (x_i^\alpha, x_j^\alpha) \in E_\alpha, \\ 0. & \end{cases} \quad (4)$$

for $1 \leq i \leq N_\alpha$, $1 \leq j \leq N_\beta$ and $1 \leq \alpha \leq M$. For an inter-layer adjacency matrix, we have $A^{[\alpha,\beta]}(a_{ij}^{\alpha\beta}) \in \mathrm{Re}^{N_\alpha \times N_\beta}$, where

$$\alpha_{ij}^{\alpha\beta} = \begin{cases} 1, & (x_i^\alpha, x_j^\beta) \in E_{\alpha\beta}, \\ 0. & \end{cases} \quad (5)$$

A multiplex network is a partial case of inter-layer networks, and it contains a fixed number of nodes connected by different types of links. Multiplex networks are characterized by correlations of different nature, which enable the introduction of additional multiplexes.

For a multiplex network, the node degree k is already a vector

$$k_i = (k_i^{[1]},\ldots,k_i^{[M]}), \quad (6)$$

with the degree $k_i^{[\alpha]}$ of the node i in the layer α, namely

$$k_i^{[\alpha]} = \sum\nolimits_j a_{ij}^{[\alpha]}, \quad (7)$$

while $a_{ij}^{[\alpha]}$ is the element of the adjacency matrix of the layer α. Specificity of the node degree in vector form allows describing additional quantities. One of them is the *overlapping degree* of node i:

$$o_i = \sum_{\alpha=1}^{M} k_i^{[\alpha]}. \quad (8)$$

The next measure quantitatively describes the inter-layer information flow. For a given pair (α,β) within M layers and the degree distributions $P(k^{[\alpha]}), P(k^{[\beta]})$ of these layers, we can defined the so-called *interlayer mutual information*:

$$I_{\alpha,\beta} = \sum\sum P(k^{[\alpha]},k^{[\beta]}) \log \frac{P(k^{[\alpha]},k^{[\beta]})}{P(k^{[\alpha]}),P(k^{[\beta]})}, \quad (9)$$

where $P(k^{[\alpha]},k^{[\beta]})$ is the joint probability of finding a node degree $k^{[\alpha]}$ in a layer α and a degree $k^{[\beta]}$ in a layer β. The higher the value of $I_{\alpha,\beta}$, the more correlated (or anti-correlated) is the degree distribution of the two layers and, consequently, the structure of a time series associated with them. We also find the mean value of $I_{\alpha,\beta}$ for all possible pairs of layers – the scalar $\langle I_{\alpha,\beta}\rangle$ that quantifies the information flow in the system.

The *multiplex degree entropy* is another multiplex measure which quantitatively describes the distribution of a node degree i between different layers. It can be defined as

$$S_i = -\sum_{\alpha=1}^{M} \frac{k_i^{[\alpha]}}{o_i} \ln \frac{k_i^{[\alpha]}}{o_i}. \quad (10)$$

Entropy is close to zero if ith node degree is within one special layer of a multiplex network, and it has the maximum value when ith node degree is uniformly distributed between different layers.

4 HIGH-ORDER EXTENSION OF TEMPORAL NETWORKS

4.1 Time-Respecting Paths

Financial networks are strongly influenced by the ordering and timing of links. In their context of their temporality, we must consider *time-respecting paths*, an extension of the concept of paths in static network topologies which additionally respects the timing and ordering of time-stamped links (Holme and Saramäki, 2012; Kempe et al., 2002; Pan and Saramäki, 2011). For a source node v and a target node w, a time-respecting path can be presented by any sequence of time-stamped links

$$(v_0, v_1; t_1), (v_1, v_2; t_2), \ldots, (v_{l-1}, v_l; t_l), \quad (11)$$

where $v_0 = v, v_l = w$ and $t_1 < t_2 < \ldots < t_l$. Time ordering of temporal financial networks is important since it implies causality, i.e. a node i is able to influence node j relying on two time-stamped links (i,k) and (k,j) only if edge (i,k) has occurred before edge (k,j).

Apart the restriction on networks to have the correct ordering, it is common to impose a maximum time difference between consecutive edges (Scholtes et al., 2016), i.e. there is a maximum time difference δ and, example, two time-stamped edges $(i,k;t)$ and $(k,j;t')$ that contribute to a time-respecting path if $0 \leq t' - t \leq \delta$. If $\delta = 1$, we are usually interested in paths with short time scales. For $\delta = \infty$, we impose no restrictions on time-range and consider a path definition where links can be weeks or years apart.

4.2 High-Order Networks

The key idea behind this abstraction is that the commonly used time-aggregated network is the simplest possible time-aggregated representation, whose weighted links capture the frequencies of time-stamped links. Considering that each time-stamped link is a time-respecting path of length one, it is easy to generalize this abstraction to higher-order time-aggregate networks in which weighted links capture the frequencies of longer time-respecting paths.

There are several variants for encoding high-order interactions (Majhi et al., 2022). The first concept of high-order links represent *hyperlink*, which can contain any number of nodes. *Hypergraph* is the generalized notion of network which is composed of nodeset V and hyper-edges E that specify which nodes from V participate in which way.

Simplex is another mathematical abstraction to accomplish high-order interaction. Formally, a *k-simplex* σ is a set of $k+1$ fully interacting nodes $\sigma = [v_0, v_1, \ldots, v_k]$. Essentially, a node is 0-simplex, a link is 1-simplex, a triangle is 2-simplex, a tetrahedron is 3-simplex, etc. Since a standard graph is a collection of edges, *simplicial complexes* are collections of simplices $K = \{\sigma_0, \sigma_1, \ldots, \sigma_n\}$.

Figure 3 demonstrates examples of simplices and hyperlinks of orders 1, 2, and 3.

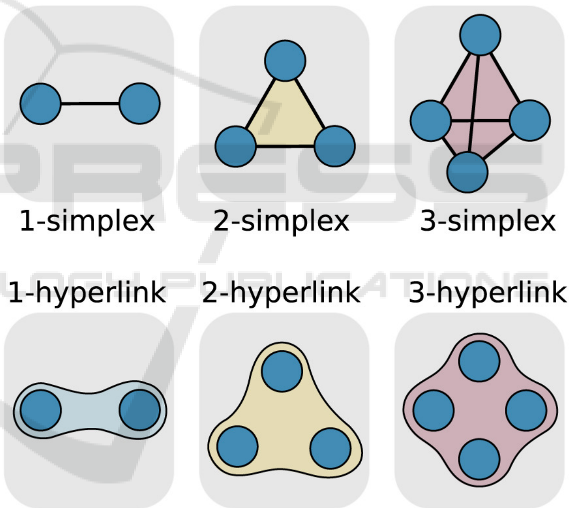

Figure 3: High-order connections in terms of simplices and hyperlinks. Adapted from (Battiston et al., 2020).

For a temporal network $G^T = (V^T, E^T)$ we thus formally define a kth order time-aggregated (or simply aggregate) network as a tuple $G^{(k)} = (V^{(k)}, E^{(k)})$ where $V^{(k)} \subseteq V^k$ is a set of node k-tuples and $E^{(k)} \subseteq V^{(k)} \times V^{(k)}$ is a set of links. For simplicity, we call each of the k-tuples $v = v_1 - v_2 - \ldots - v_k$ $(v \in V^{(k)}, v_i \in V)$ a kth order node, while each link $e \in E^{(k)}$ is called a kth order link. Between two kth order nodes v and w exists kth order edge (v,w) if they overlap in exactly $k-1$ elements. Resembling so-called De Bruijn graphs (De Bruijn, 1946), the basic idea behind this construction is that each kth order link represents a possible time-respecting path of length k in the underlying temporal network, which connects node v_1

to node w_k via k time-stamped links

$$(v_1, v_2 = w_1; t_1), ..., (v_k = w_{k-1}, w_k; t_k). \quad (12)$$

Importantly, and different from a first-order representation, kth order aggregate networks allow to capture *non-Markovian* characteristics of temporal networks. In particular, they allow to represent temporal networks in which the kth time-stamped link $(v_k = w_{k-1}, w_k)$ on a time-respecting path depends on the $k-1$ previous time-stamped links on this path. With this, we obtain a simple static network topology that contains information both on the presence of time-stamped links in the underlying temporal network, as well as on the ordering in which sequences of k of these time-stamped links occur.

4.3 Degree Centrality

Network centralities are node-related measures that quantify how "central" a node is in a network. There are many ways in which a node can be considered so: for example, it can be central if it is connected to many other nodes (degree centrality), or relatively to its connectivity to the rest of the network (path based centralities, eigenvector centrality). One of the simplest centrality measure is the *degree of a node*, which counts the number of edges incident to an ith node.

For any adjacency matrix the degree of a node i can be defined as

$$D_i = \sum_j A_{ij}. \quad (13)$$

High-order degree centrality counts the number of kth-order edges incident to the kth-order node i. To get a scalar value which will serve as an indicator of high-order dynamics, we obtain mean degree D_{mean}:

$$D_{mean} = \frac{1}{N} \sum_{i=1}^{N} D_i. \quad (14)$$

Except this measure, we can calculate nth moment of the degree distribution, which can be defined as

$$\langle k^n \rangle = \sum_{k_{\min}}^{\infty} k^n p_k \approx \int_{k_{\min}}^{\infty} k^n p_k dk. \quad (15)$$

In this study we will present the dynamics of the first moment, which is the mean weighted degree of a network, and its high-order behavior.

4.4 Assortativity Coefficient

Assortativity is a property of network nodes that characterizes the degree of connectivity between them. Many networks demonstrate "assortative mixing" on their nodes, when high-degree nodes tend to be connected to other high-degree nodes. Other networks demonstrate disassortative mixing when their high-degree nodes tend to be connected to low-degree nodes. Assortativity of a network can be defined via the Pearson correlation coefficient of the degrees at either ends of an edge. For an observed network, we can write it as

$$r = \left(M^{-1} \sum_i j_i k_i - \left[M^{-1} \sum_i \frac{1}{2}(j_i + k_i) \right]^2 \right) \Bigg/ \left(M^{-1} \sum_i \frac{1}{2}(j_i^2 + k_i^2) - \left[M^{-1} \sum_i \frac{1}{2}(j_i + k_i) \right]^2 \right), \quad (16)$$

where $-1 \leq r \leq 1$; j_i, k_i are the degrees of the nodes at the ends of the ith edge, with $i = 1, ..., M$, where M is the number of edges of a network.

This correlation function is zero for no assortative mixing. If $r = 1$, then we have perfect assortative mixing pattern. For $r = -1$, we can observe perfect disassortativity.

Studying financial networks, with time-respecting paths, we can consider four type of assortativity: $r(in,in), r(in,out), r(out,in), r(out,out)$, which will correspond to tendencies to have similar in and out degrees. We can denote one of the studied in/out pairs as (α, β). Suppose, for a given ith edge, we have got the source (i.e. tail) node of the edge and target (i.e. head) node of the edge. We can denote them as α-degree of the source (j_i^{α}) and β-degree of the target (k_i^{β}). Assortativity coefficient for degrees of a specific type can be defined as

$$r(\alpha, \beta) = \frac{\sum_i \left(j_i^{\alpha} - \overline{j^{\alpha}} \right) \left(k_i^{\beta} - \overline{k^{\beta}} \right)}{\sqrt{\sum_i \left(j_i^{\alpha} - \overline{j^{\alpha}} \right)^2} \sqrt{\sum_i \left(k_i^{\beta} - \overline{k^{\beta}} \right)^2}}, \quad (17)$$

where $\overline{j^{\alpha}}$ and $\overline{k^{\beta}}$ are the average α-degree of sources and β-degree of targets.

5 EMPIRICAL RESULTS

To build indicators (indicators-precursors) based on multiplex and high-order networks, the following is done:

- databases of 6 most influential stock market indices for the period from 02.01.2004 to 18.10.2022 were selected for multiplex analysis (see figure 4). The data were extracted using Yahoo! Finance API based on Python programming language (Aroussi, 2022);

- the indicators described in the previous sections were calculated using the sliding window procedure (Bielinskyi et al., 2022; Soloviev et al., 2020; Bielinskyi et al., 2021c,b; Kiv et al., 2021; Bielinskyi and Soloviev, 2018; Bielinskyi et al., 2020). The essence of this procedure is that: (1) a fragment (window) of a series of a certain length w was selected; (2) a network measure was calculated for it; (3) the measure values were stored in a pre-declared array; (4) the window was shifted by a predefined time step h, and the procedure was repeated until the series was completely exhausted; (5) further, the calculated values of the network measure were compared with the dynamics of the stock index. Subsequently, conclusions were drawn regarding the further dynamics of the market. In our case, window length $w = 500$ days and time step $h = 10$ day. The choice of step was limited by the counting time for high-order networks;
- multiplex and high-order indicators are compared with the Dow Jones Industrial Average (DJIA) index.

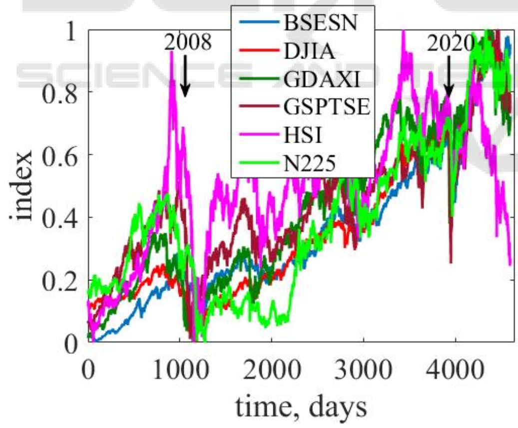

Figure 4: The dynamics of stock market indices for studying multiplex characteristics.

In figure 5 presented the dynamics of inter-layer mutual information (I) and multiplex degree entropy (S) along with the DJIA index.

From figure 5 we can see that multiplex mutual information increases before the crisis of 2008. Also, it noticeably becomes higher before COVID-19 crash. For the last months, it demonstrates decreasing pattern, which indicates that the economies of different countries may be experiencing different evolutions now. Nevertheless, it can be seen that, as a rule, this

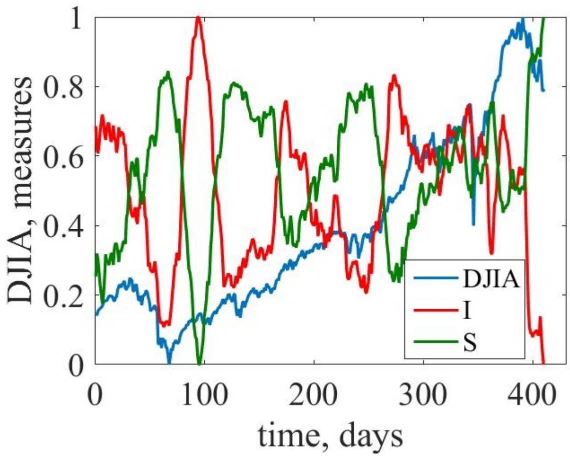

Figure 5: The dynamics of inter-layer mutual information (I) and multiplex degree entropy (S) along with the DJIA index.

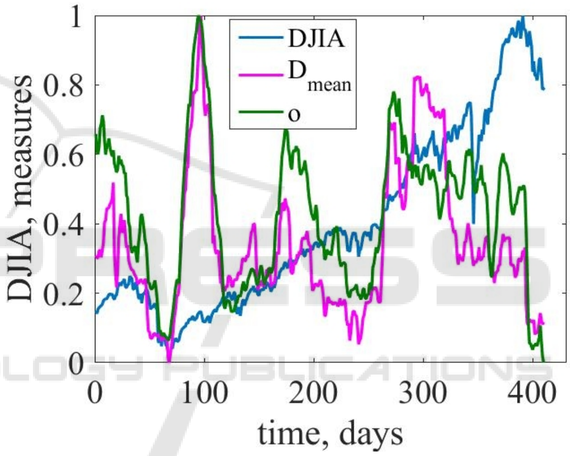

Figure 6: The dynamics of the mean degree (D_{mean}) and overlapping degree (o) along with the DJIA index.

indicator is characterized by growth, indicating an increase in the interconnection of the economies of different countries. In a crisis, this indicator usually declines, demonstrating different resistance to the collapse events of the stock markets of countries and the difference in the actions that they take. Entropy indicator shows asymmetric behavior

Next, we compare one of the multiplex measure, overlapping degree (o), with the mean degree of a network (D_{mean}). Figure 6 represents this result.

In figure 6 we can see that both D_{mean} and o are characterized by similar dynamics. These indicators increase near the crash, which indicates an increase in the concentration of connections for some network nodes, and further, based on the indicators during the crisis, there is a decline in concentration both in the dynamics of the DJIA and the inter-layer connectedness of stock indices. We may see that the multiplex approach does not significantly change the dynamics of the concentration degree indicator in comparison with the indicator based on the classical univariate graph.

Figure 7 demonstrates the dynamics of mean weighted degree (equation (15)) for order 1 and 2 along with the DJIA index.

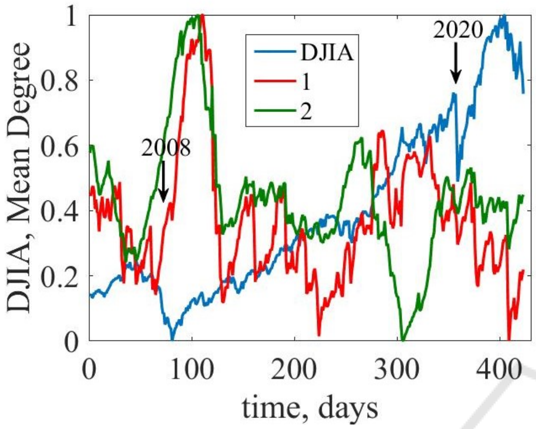

Figure 7: The dynamics of first- and second-order mean (weighted) degree along with the DJIA index.

In figure 7 we can see that the second-order D_{mean} is slightly different from the first-order one. The second-order D_{mean} starts to increase a slightly earlier before the crisis of 2008. We can see that before crisis of 2020 second-order D_{mean} declines more noticeably comparing to the first-order one. However, this difference between the first and second order is still insignificant, what can we say about the fact that the classical visibility graph can reflect all the information that the series under study can represent.

Next, let us present high-order dynamics of the assortativity coefficient for the DJIA index (see figure 8).

Figure 8 presents the assortativity coefficient for first, second, and third orders. Assortativity declines before crashes and increases during them. We see that high-orderness does not change radically change the dynamics of this indicator. Third-order assortativity responds better for the crash of 2008, but worse for the COVID-19 crisis, comparing to first- and second-order assortativity.

6 CONCLUSIONS

In this article, we have presented methods to measure and model systems with casual, multiplex, and high-order interactions. From our analysis, we have found that typically non-Markovian, non-stationary,

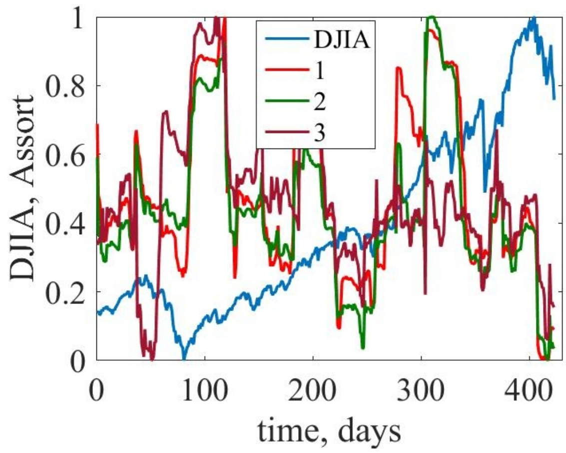

Figure 8: The dynamics of first-, second-, and third-order assortativity along with the DJIA index.

non-linear systems are characterized by long-range spatio-temporal correlations which are better described by the high-order paradigm. Typically, high-order connectivity is described in terms of hypergraphs (Schölkopf et al., 2007; de Arruda et al., 2020; Carletti et al., 2020) or simplicial complexes (Schaub et al., 2020; Torres and Bianconi, 2020; Skardal and Arenas, 2020). Such richer types of links bring new possibilities to go beyond typical nodes and encode into one node edges, triangles, tetrahedra, etc. to investigate higher-order clusters and temporal dependencies.

We have presented indicators (indicators-precursors) based on classic visibility graphs, multiplex networks, and high-order networks. In this study we have used such network measures as the mean degree of a node D_{mean}, first-moment degree (mean weighted degree) of a network, assortativity coefficient, inter-layer mutual information I, multiplex degree entropy S, and mean overlapping degree of a network o. We have constructed the visibility graph relying on the time series of the Dow Jones Industrial Average (DJIA) index. We have studied multiplex network dynamics using a database that consists of 6 of the most developed and capitalized stock indices of different countries and which include companies from different sectors. We have chosen the period from 02.01.2004 to 18.10.2022. Each indicator was calculated using the sliding window algorithm. We have shown that multiplex and high-order networks do not substantially differ dynamically from the traditional pairwise visibility model. This may indicate that the classical visibility graph reflects all possible short-term and long-term dependencies in the values of the DJIA index. All the presented measures work similarly, like indicators (indicators-precursors) of critical financial events, increasing or decreasing before and during them. Although multiplex and high-order network indicators give promising results, it still needs further development and improvements for studying complex financial time series. The solution may lie in the framework that combines Markov chains of multiple, higher orders into a multi-layer graphical model that captures temporal correlations in pathways at multiple length scales simultaneously (Scholtes, 2017). Another perspective lies in the use of neuro-fuzzy forecasting and clustering methods of complex financial systems (Bielinskyi et al., 2021a; Bondarenko, 2021; Kmytiuk and Majore, 2021; Kobets and Novak, 2021; Kucherova et al., 2021; Lukianenko and Strelchenko, 2021; Miroshnychenko et al., 2021).

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