Numerical Study of Stochastic Disturbances on the Behavior of Solutions of Lorenz System

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Abstract: Nowadays interest of the deterministic differential system of Lorenz equations is still primarily due to the problem of gas and fluid turbulence. Despite a large number of existing systems for calculating turbulent flows, new modifications of already known models are constantly being investigated. In this paper we consider the effect of stochastic additive perturbations on the Lorenz convective turbulence model. To implement this and subsequent interpretation of the results obtained, a numerical simulation of the Lorenz system perturbed by adding a stochastic differential to its right side is carried out using the programming capabilities of the MATLAB programming environment.

1 INTRODUCTION

Hydrodynamic turbulence (turbulent flow) is the movement of a fluid characterized by chaotic changes in pressure and flow velocity. This is the main difference from laminar flow, which occurs when a fluid flows in parallel layers, with no gap between those layers.

Typically, turbulence is seen in everyday phenomena such as surf, fast-flowing rivers, billowing thunderclouds, and so on. In general terms, in a turbulent flow, unsteady vortices of different sizes arise, which interact with each other.

Turbulence for a long time did not lend itself to detailed physical analysis, since it has a very complex character. At one time, Richard Feynman described turbulence as the most important unsolved problem in classical physics.

This thorny issue attracted new scientists year-by-year and as a result of their studies the so-called Lorenz strange attractor was discovered.

It was the first example of deterministic chaos. The Lorenz model (Lorenz, 1963) was created in 1963 owing to a series of transformations of the Navier–Stokes equation.

Its solutions were interesting because of their quasi-stochastic trajectories and absence of external sources of noise. Such solutions for the first time appeared in a deterministic system.

Overall, the Lorenz model is based on a two-dimensional thermal convection. For the stochastic part of the model, a stochastic differential equation (SDE) will be used. Such differential equations contain a stochastic term, and therefore their solution is also a stochastic process.

This study focuses on modeling and analysis of the stability of the Lorenz system under the influence of stochastic disturbances. In order to realize it and to interpret results, a simulation of the additively disturbed Lorenz system was carried out with MATLAB software package.

2 PROPERTIES OF THE LORENZ SYSTEM

Consider the following classical Lorenz equations:

$$\begin{cases} x_t = \sigma\,(y-x), \\ y_t = x\,(r-z)-y, \\ z_t = xy-bz, \end{cases} \tag{1}$$

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where the variable x represents the rotation rate of the Rayleigh-Benard convection cells, y characterizes the temperature difference ΔT between rising and descending fluid and z shows the deviation of the vertical temperature profile from the linear relationship. The model parameters σ, r and b reflect the values of the Prandtl number, the Rayleigh number, and the coefficient linked to the geometry of the area respectively.

As well known the Lorenz system has the following properties:

1. Homogeneity: the first and most obvious property.
2. Symmetry: in the phase space symmetry is obvious after: $x \to -x$, $y \to -y$.
3. Dissipation: in three-dimensional phase space (x, y, z) we will consider vector of speeds $\vec{L}(x_t, y_t, z_t)$.

Its negative divergence characterizes dissipative system:

$$\nabla \cdot \vec{L} = \frac{\partial}{\partial x}(\sigma y - \sigma x) + \frac{\partial}{\partial y}(x(r-z) - y) + \\ + \frac{\partial}{\partial z}(xy - bz) = -\sigma - 1 - b < 0 \quad (2)$$

Let's look at set of Lorenz systems with different initial conditions. They take volume ΔV while $t = 0$. During the evolution of the system volume declines according to $\Delta V = V_0 \exp(-\sigma - b - 1)$.

At $t \to \infty$ all phase-space trajectories are concentrated inside a compact attractor.

Then we will check the Lorenz system for fixed points:

$$\begin{cases} \sigma(y-x) = 0 \\ x(r-z) - y = 0 \\ xy - bz = 0 \end{cases} \Leftrightarrow \begin{cases} x = y \\ x(r-1-z) = 0 \\ x^2 = bz \end{cases} \Leftrightarrow \\ \Leftrightarrow \begin{cases} x = y \\ \left[\begin{array}{l} x = 0 \\ z = r-1 \end{array} \right. \\ x^2 = bz \end{cases} \quad (3)$$

The Lorenz system always has fixed stationary point $P_0 = (0, 0, 0)$. Also when $r > 1$ two other fixed points appear $P_1 = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $P_2 = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$.

Point $r = 1$ is a bifurcation point. At $r < r_1 \approx 13{,}926$ separatrices S_1 and S_2 attract to the nearest fixed points P_1 and P_2. At $r = r_1$ separatrices transform into a homoclinic loops, i.e. trajectories which complete a full orbit around one of the fixed points and join initial point. They afterwards transform into the saddle orbits, borders of attraction area of P_1 and P_2. Also separatrices S_1 and S_2 approaches to P_1 and P_2 accordingly. The most interesting situation appears at $r = r_2 \simeq 24{,}06$. It corresponds to well-known Lorenz strange attractor, which has property of strong dependence on initial conditions. It means that any small change in the coordinates of the initial point leads to completely different solution.

More detailed information about the structure of the Lorenz system can be found in various books (Sparrow, 1982), (Danilov, 2017), (Leonov and Kuznetsov, 2015).

The effective variation method for obtaining the necessary (and sufficient) stability conditions for the perturbed solutions of Lorenz system was used in (Isaev et al, 2022).

The method uses a variational technique based on the idea of determining the maximum rate of change of the Euclidean metric, assuming that the solution does not leave the ε neighborhood of the equilibrium point. This method is effective for obtaining the necessary stability conditions and makes it possible to continue research (in order to determine sufficient conditions). The method is effective even in cases where the application of the classical Lyapunov method causes difficulties associated with the construction of the Lyapunov function or inaccuracies in Taylor linearization, which is typical for high-dimensional dynamical systems. In a number of cases, this method can be applied to find regions of phase variables in which the necessary stability conditions coincide with the Lyapunov sufficient stability conditions (asymptotic stability). Thus, for the system of Lorenz equations, the efficiency of applying the variational method for obtaining the necessary conditions for stability in the sense of Lyapunov and determining the regions of phase variables in which these conditions become sufficient is shown. This method allows us to conclude that this approach is universal for a wide class of dynamical systems.

3 ITO'S STOCHASTIC CALCULUS

We will describe stochastic differential equations (SDE) with Ito's stochastic calculus. It is based on a stochastic Wiener process. Overall, stochastic process is a set of random variables that has been indexed by some parameter such as time.

Initially we consider division $\{\tau_j^{(N)}\}$ of a $[0,T]$, which corresponds to

$$0=\tau_0^{(N)}<\tau_1^{(N)}<...<\tau_N^{(N)}=T$$

with $\Delta=\max\limits_{0<j<N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0$.

Then we determine sequence of functions in the following way: $\xi^{(N)}(t,\omega)=\xi(\tau_j^{(N)},\omega)$ at $t\in[\tau_j^{(N)},\tau_{j+1}^{(N)})$, $j=0,1,...,N-1$.

Definition: Stochastic Ito's integral for ξ_t is a convergence in quadratic mean of following expression, where f_τ is a Wiener process (Rozanov, 2012):

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}\xi^{(N)}(t,\omega)\left(f(\tau_{j+1}^{(N)},\omega)-f(\tau_j^{(N)},\omega)\right) \underset{def}{=}\int_0^T\xi_\tau\, df_\tau. \tag{4}$$

As a result, we need to determine multiple stochastic integrals for introduction of a numerical scheme. Let's determine them by the following expression:

$$I_{l_1...l_{k_{s,t}}}^{(i_1...i_k)}=\begin{cases}\int_\tau^s(s-\tau_k)^{l_k}...\int_\tau^{\tau_2}(s-\tau_1)^{l_1}\, df_{\tau_1}^{(i_1)}...df_{\tau_k}^{(i_k)}, \\ \text{if } k>0; \\ 1, \text{if } k=0.\end{cases} \tag{5}$$

The simulated stochastic Lorenz system is demonstrated below:

$$d\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\sigma(y-x)\\x(r-z)-y\\xy-bz\end{pmatrix}dt+c\cdot\begin{pmatrix}0\\0\\dW_3^{(t)}\end{pmatrix} \tag{6}$$

In this paper we used the version of unified Taylor-Ito expansion gained by Kulchitsky (Kulchitski and Kuznetsov, 1998). The main problem is that this expansion contains multiple stochastic integrals, which are not easily approximated. We will use the fundamental results of Kuznetsov (Kuznetsov, 2010) to approximate these integrals properly. He discovered expansions of our multiple stochastic integrals using independent random variables ξ_j.

We will use several of them (more details see (Kulchitski and Kuznetsov, 1998)):

$$I_{0_{T,t}}^{(i_1)}=\sqrt{T-t}\,\xi_0, \tag{7}$$

$$I_{1_{T,t}}^{(i_1)}=-\frac{(T-t)^{3/2}}{2}\left(\xi_0+\frac{1}{\sqrt{3}}\xi_1\right), \tag{8}$$

$$I_{2_{T,t}}^{(i_1)}=\frac{(T-t)^{5/2}}{2}\left(\xi_0+\frac{\sqrt{3}}{2}\xi_1+\frac{1}{2\sqrt{5}}\xi_2\right). \tag{9}$$

Using them in the Taylor-Ito expansion in the Kloeden-Platen form (Kloeden and Platen, 1995), we get the explicit numerical scheme directly from this expansion. For the sake of brevity, we only present here the final result. Initially let us denote step of division $\{\tau_j\}_{j=0}^N$ as h, $j=\overline{1,N}$.

The explicit numerical scheme, which we have implemented, is as follows:

$$x_{j+1}=x_j+he+\frac{h^2}{2}(-he+\sigma g)+ +\frac{h^3}{6}e_1-h^{5/2}\sigma x_j c v_1, \tag{10}$$

$$y_{j+1}=y_j+hg+\frac{h^2}{2}\left((r-z_j)e-g-x_jf\right)- -\frac{h^3}{6}g_1-h^{3/2}cx_jv_2+ +h^{5/2}\left(-e+(1+b)x_j\right)cv_1+h^{5/2}ecv_3, \tag{11}$$

$$z_{j+1}=z_j+hf+\frac{h^2}{2}ey_j+gx_j-bf+ +\frac{h^3}{6}f_1+h^{1/2}c\xi_1-h^{3/2}bcv_2+ +h^{5/2}c(b^2-x_j-2)v_1. \tag{12}$$

In the scheme (10)-(12) we made a number of some designations to simplify the recording of the scheme that was written above:

$e=-\sigma x_j+\sigma y_j$, $g=rx_j-y_j-x_jz_j$,

$f=-bz_j+x_jy_j$,

$$g_1=e\left((-\sigma-1)(r-z_j)\right)+bz_j-2x_jy_j+ +g\left(\sigma(r-z_j)+1-x_j^2\right)+f\left(-e+(b+1)x_j\right),$$

$$f_1=e\left(2x_j(r-z_j)-(b+1+\sigma)y_j\right)+ +g\left(-(b+1+\sigma)x_j+2\sigma y_j\right)+f\left(b^2-x_j^2\right),$$

$v_1=\frac{\xi_1}{6}-\frac{\xi_2}{4\sqrt{3}}+\frac{\xi_3}{6\sqrt{20}}$, $v_2=-\frac{\xi_1}{2}-\frac{\xi_2}{2\sqrt{3}}$,

$v_3=-\frac{\xi_1}{6}-\frac{\xi_3}{3\sqrt{20}}$.

4 RESULTS OF NUMERICAL MODELING

It was decided to start with intermediate values to understand how the system as a whole would behave. First the parameter $r = 20$ was fixed and two situations were modelled: at $c = 0$ and at $c = 2$. Parameter c shows the intensity of stochastic influence. The state at $c = 0$ is given for comparison (Figure 1).

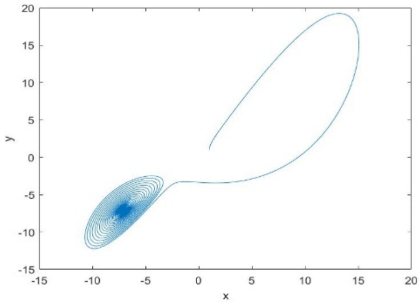

Figure 1: $r = 20, c = 0$.

At $c = 2$ the trajectory loses its regularity, which is reasonably predictable (Figure 2).

Further, let us increase c to 3 (Figure 3).

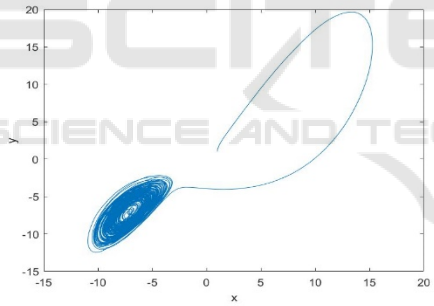

Figure 2: $r = 20, c = 2$.

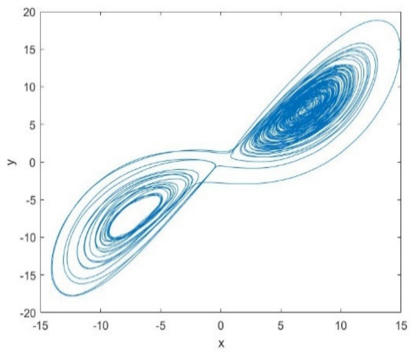

Figure 3: $r = 20, c = 3$.

Our numerical simulation using the special techniques described above, shows that the trajectory of the stochastically perturbed system seems like the Lorenz attractor while parameter r is sufficiently far from classical value 24,06.

Next, let us increase the parameter c to 4, to test this assumption, and get a picture that is even more similar to Lorenz attractor (Figure 4).

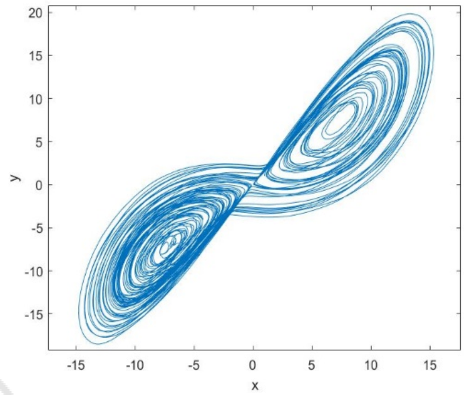

Figure 4: $r = 20, c = 4$.

Then consider a different state of the system at $r = 13$ and look at the effect of noise, but in three-dimensional space.

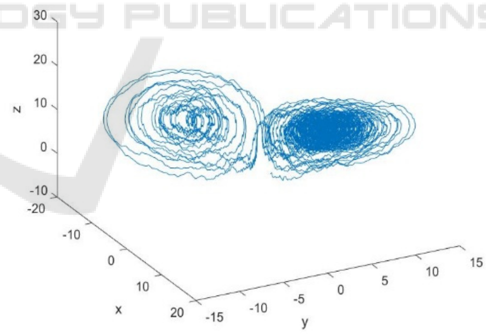

Figure 5: $r = 13, c = 4$.

As be seen from the graph, with less r perturbed systems also demonstrate similar behavior. Under these conditions, the change of attractor occurs much earlier than in a classic system. As stochastic intensity increases, the stochastic analogue of the Lorenz attractor with substantially smaller r can be observed. Overall, there is a negative relationship between the stochastic factor c and the bifurcation values of r. It is interesting to see how the system works with large values of r. We start with $r = 200$

and build a determine system (blue color with $c = 0$) and interfered system (red color with $c = 5$).

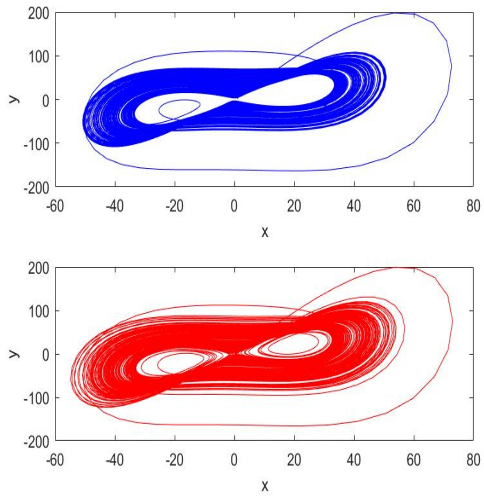

Figure 6: $r = 200, c = 0, c = 5$.

The graphs are quite similar, and here we clearly see auto-oscillating mode. By increasing r to 300 (Figure 7), and then up to 500 (Figure 8), we can obtain a predictable result, based on fact that r is an analogue of the Rayleigh number.

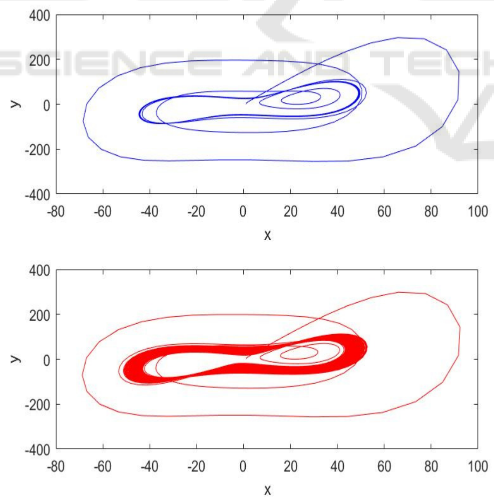

Figure 7: $r = 300, c = 0, c = 5$.

As parameter r increases, the role of noise will gradually decrease. The system will be a stochastic analogue of the auto-oscillating movement, which will differ from the unperturbed system only by a slight irregularity of the trajectory.

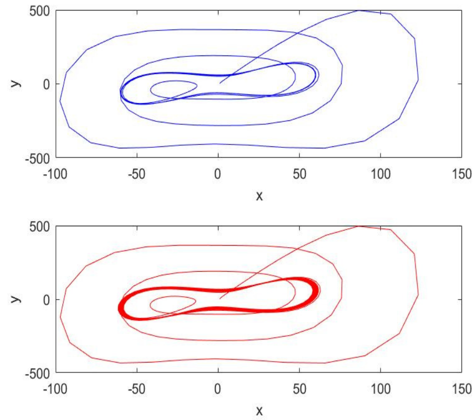

Figure 8: $r = 500, c = 0, c = 5$.

5 CONCLUSIONS

In conclusion we would like to make the following observations and draw a parallel with the real physical system. All in all, it seems quite logical that stochastic interferences strengthen quasi-stochastic oscillations around equilibrium positions. As a result, a trajectory similar enough to the Lorenz strange attractor appears at smaller r . The same changes can be observed, for example, in real physical systems, where turbulence occurs earlier in the presence of some noise source than without it. Then, gradually, the noise reduces effect on the system, because the Rayleigh number is already high enough. The behavior of the system after the noise appearance demonstrates quite clearly that stochastic interference plays a significant role in describing turbulence. Lorenz wanted to use his model for long-term weather forecasting (Lorenz, 1963). Moreover, he wanted to prove the theoretical existence of such a method. By and large, due to the significant impact of additive interference, it is unlikely that such a method will ever be developed.

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