Mathematical Modelling of One-dimensional Fluid Flows Bounded by a Free Surface and an Impenetrable Bottom

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Keywords: One-dimensional fluid flows, free surface, impenetrable bottom, shoreline, breakup of discontinuity, shallow water equation, non-stationary self-similar variable, analytical solution, convergent series.

Abstract: The paper investigates a one-dimensional model of a wave coming ashore with a subsequent collapse. For modelling, a system of shallow water equations is taken, which considers the effect of gravity. A non-stationary self-similar variable is introduced in the system of shallow water equations. For a system of equations written in new variables, a boundary condition on the sound characteristic is formulated. The power series is used to construct the solution. Algebraic and ordinary differential equations are solved to find the coefficients of the series. The convergence of this series is proved. The locally analytical solution of the problem of wave overturning in the space of physical variables is constructed. The obtained analytical solutions can be useful for setting boundary and initial conditions in numerical simulation of a tsunami wave over a long period of time.

1 INTRODUCTION

Approximate shallow water equations are often used in numerical modelling of tsunami waves coming ashore. In such models, problems with a movable boundary are solved, in which the shoreline (the water-land boundary) moves to the shore. Since the water depth becomes zero at the shoreline, a feature appears in the system of equations (Vol'cinger, Klevannyj, Pelinovskij, 1989). To correctly account for this feature in calculations, it is necessary to construct an analytical solution in the vicinity of the shoreline (Hibberd, Peregrine, 1979). Earlier in (Carrier, Greenspan, 1958), analytical solutions of a system of one-dimensional shallow water equations were obtained to describe the output to a flat slope of non-collapsing standing waves. In (Carrier, Wu, Yen, 2003) and (Kanoglu, 2004), the dependence of the trajectory of the point of shoreline on the initial waveform was considered. The formula for calculating the maximum value of the wave height on a flat angular slope was obtained in (Sanolakis, 1987). The models obtained in (Carrier, Greenspan, 1958; Carrier, Wu, Yen, 2003; Kanoglu, 2004; Sanolakis, 1987) are approximate, since the coastal slope in them is a flat slope, and not a curved surface as it is observed in nature. The movement of the wave on such a surface has a more complex form. The main difficulty here is to model the motion along the curved surface of the water-land boundary for crashing waves. This work is devoted to solving this problem.

Note that the first approximation of the system of shallow water equations exactly coincides with the equations of motion of a polytropic gas with the polytropic exponent $\gamma = 2$. In this case, the shoreline for shallow water equations in the system of gas dynamics equations is the gas-vacuum boundary. In (Bautin, Deryabin, 2005), solutions of one-dimensional and multidimensional problems of modelling gas motion in vacuum are given. In (Bautin, Deryabin, 2005) the problem of the breakup of a special discontinuity is solved. Here is the formulation of this problem.

It is assumed that the surface Γ separates the gas from the vacuum. If the density of the gas on one side of the impenetrable surface of the gas is strictly greater than zero, and on the other is equal to zero, then they say that this is the problem of the breakup of a special discontinuity. In the problem, it is

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Deryabin, S. and Mezentsev, A.
Mathematical Modelling of One-dimensional Fluid Flows Bounded by a Free Surface and an Impenetrable Bottom.
DOI: 10.5220/0011582800003527
In *Proceedings of the 1st International Scientific and Practical Conference on Transport: Logistics, Construction, Maintenance, Management (TLC2M 2022)*, pages 266-271
ISBN: 978-989-758-606-4

required to describe the movement of gas after the instant destruction of the wall Γ. The self-similar solution of such a problem in the one-dimensional case was first found by B. Riemann for plane-symmetric flows. In the future, the solution of this problem was constructed in special functional spaces after replacing the dependent and independent variables. Such a solution method became possible in (Bautin et al., 2011) to describe the overturning of the wave. The constructed solution has the form of a power series converging in the vicinity of the boundary. Using spatial variables, the law of motion of the water-land boundary is obtained and the values of the velocity of the liquid on it are found. In this paper, locally converging series are also constructed – the solution of the wave overturning problem, but unlike (Bautin et al., 2011), the solution is constructed using non-stationary self-similar variables in physical space.

2 MATERIALS AND METHODS

The characteristic Cauchy problem is taken as the object of research. In (Bautin, 2009), one can find formulations and proofs of theorems about the existence and uniqueness of solutions to such problems. The method of constructing solutions is as follows. A nonlinear system of differential equations describing the physical conservation laws is chosen. Boundary and initial conditions are set for it using analytical functions. A local theorem on the existence and uniqueness of the solution of the initial boundary value problem is proved. The analytical solution is constructed in the form of a power series, and its convergence is proved.

2.1 Statement of the Problem

The flow of an incompressible inviscid fluid without vortices under the action of gravity is considered. Let the layer of such a liquid be bounded by a free surface and an impermeable bottom. The Cartesian coordinate system is introduced so that the line $z = 0$ corresponds to the level of the stationary liquid. The bottom is given by the function $z = -h(x)$.

At time $t = 0$, the liquid wave is separated from the land by the point Γ (the shoreline). Moreover, there is a dry shore to the left of Γ, and the sea to the right (fig. 1).

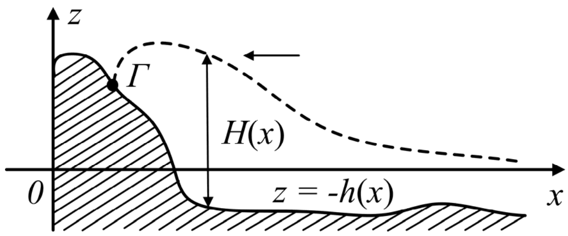

Figure 1: Wave, shore and shoreline (point) Γ.

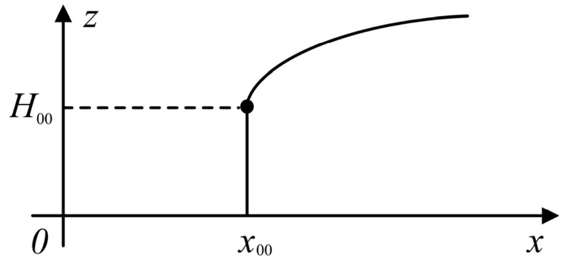

Figure 2: Waveform at time $t = 0$.

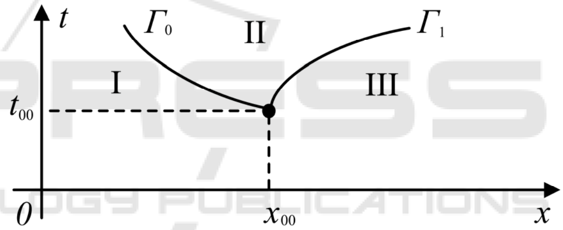

Figure 3: I — dry shore, II — disturbed wave, III — undisturbed wave.

Consider the system of shallow water equations in the first approximation (Ovsyannikov, 2003; Khakimzyanov et al., 2001):

$$\begin{aligned} &H_t + uH_x + Hu_x = 0, \\ &u_t + uu_x + gH_x = gh_x, \end{aligned} \tag{1.1}$$

where g is the acceleration of gravity, and the unknown functions: H is the height of the liquid measured from the bottom to the upper level of the liquid, u is the velocity of the liquid. It is also assumed that at time $t = 0$ the waveform has the form of a step with a straight vertical part (fig. 2). The vertical equation has the form $x = x_{00}$, and the height of the vertical part is equal to $H_{00} = H_0(x_{00})$. At the initial moment of time, the analytical functions are known:

$$u = u_0(x),\ \ H = H_0(x),\ \ x > x_{00}.$$

Moreover, it is assumed that $u(x_{00}) = u_{00} < 0$ and for all $x \geq x_{00}$, the values of the function $H_0(x) > 0$ are strictly positive. The fluid flow defined by such functions is called the background flow (undisturbed wave). After overturning the step at time $t = 0$, a disturbed wave is formed, which, at $t > 0$, is separated from the undisturbed wave by the line Γ_1 (the line of weak discontinuity), from the land by the boundary Γ_0 — the shoreline (the water-land boundary) (fig. 3):

$$H(t,x)|_{\Gamma_0} = 0.$$

In the problem, it is required to construct analytical functions describing the motion of a fluid in the region of disturbed and undisturbed waves and the motion of the boundary Γ_1. Note that the system (1.1) and the initial data satisfy the conditions of Kovalevskaya's theorem. According to this theorem, its only solution has the form (Bautin, 2009):

$$u = u^0(t,x),\ \ H = H^0(t,x).$$

In system (1.1), we introduce a new unknown function:

$$C(t,x) = H^{1/2}(t,x),\ \ (H = C^2).$$

After the transformations, we get:

$$\begin{aligned} &C_t + uC_x + \frac{1}{2}Cu_x = 0, \\ &u_t + uu_x + 2gCC_x = gh_x. \end{aligned} \tag{1.2}$$

In these new designations, the background flow will have the form:

$$\begin{aligned} &u = u^0(t,x), \\ &C = C^0(t,x) = \sqrt{H^0(t,x)},\ C_{00} = \sqrt{H_{00}}. \end{aligned}$$

Let's write down the differential equation and the initial condition for the motion of Γ_1: $x = x_1(t)$ (Ovsyannikov, 2003):

$$x_{1t} = u^0(t,x_1) + \sqrt{g}C^0(t,x_1),\ \ x_1(0) = x_{00}. \tag{1.3}$$

Problem (1.3) satisfies the conditions of Kovalevskaya's theorem. According to this theorem, its solution can be represented as:

$$x_1(t) = \sum_{k=0}^{\infty} x_{1k}\frac{t^k}{k!}. \tag{1.4}$$

Let's find the coefficients of the series (1.4). The zero and first coefficients of the series are from (1.3):

$$x_{10} = x_{00},\ \ x_{11} = u_{00} + \sqrt{g}C_{00}.$$

The following coefficients of the series are found by successive differentiation of equation (1.3):

$$x_{1tt} = u_t^0 + \sqrt{g}\,C_t^0.$$

Then we get

$$x_{12} = u_t^0(0,x_{00}) + \sqrt{g}\,C_t^0(0,x_{00}).$$

According to the obtained formulas, as well as in (Bautin, 2009), the law of motion $x_1(t)$ is written using the analytical function $x_2(t)$ Γ_1:

$$x = x_1(t) = x_{00} + t x_2(t).$$

The boundary conditions on Γ_1 are given by the equations:

$$\begin{aligned} u(t,x)|_{x=x_1(t)} &= u^0(t,x_1(t)), \\ C(t,x)|_{x=x_1(t)} &= C^0(t,x_1(t)). \end{aligned} \tag{1.5}$$

To construct a disturbed wave in the system (1.2), we introduce non-stationary self-similar variables according to the following formulas:

$$t' = t,\ \ y = \frac{x - x_{00}}{t}.$$

The stroke sign is not used in the future.

After the transformations, we get the system:

$$\begin{aligned} tC_t + (u-y)C_y + \frac{1}{2}Cu_y &= 0, \\ tu_t + (u-y)u_y + 2gCC_y &= tgh_x(x_{00} + ty), \end{aligned} \tag{1.6}$$

with conditions on the characteristic Γ_1:

$$\begin{aligned} u(t,y)|_{y=x_2(t)} &= u^0(t,x_1(t)), \\ C(t,y)|_{y=x_2(t)} &= C^0(t,x_1(t)). \end{aligned} \tag{1.7}$$

2.2 Construction of a Solution in Physical Space

To construct a solution of the problem (1.6), (1.7), we write the power series (Bautin, Deryabin, 2005):

$$\boldsymbol{f}(t,y)=\sum_{k=0}^{\infty}\boldsymbol{f}_k(y)\frac{t^k}{k!},\quad \boldsymbol{f}=\{u,C\}. \qquad (2.1)$$

We will find the zero coefficients of the series from the system (1.6) at the value $t = 0$:

$$\begin{aligned}&(u_0-y)C_{0y}+\frac{1}{2}C_0u_{0y}=0,\\&(u_0-y)u_{0y}+2gC_0C_{0y}=0.\end{aligned} \qquad (2.2)$$

For the existence of a non-zero solution of the resulting system (2.2), it is necessary that its determinant is equal to zero, i.e.

$$(u_0-y)^2=gC_0{}^2,\ u_0-y=\pm\sqrt{g}C_0.$$

Since at $t = 0$ on characteristic Γ_1 we have (Ovsyannikov, 2003):

$$y=u_{00}+\sqrt{g}C_{00}.$$

Hence, we have:

$$u_0-y=-\sqrt{g}C_0. \qquad (2.3)$$

Substituting $u_0 - y$ into the second equation of the system (2.2.), we get:

$$u_{0y}=2\sqrt{g}C_{0y},\ u_0=2\sqrt{g}C_0+D,$$

where D is determined from the conditions (1.7):

$$u_0=2\sqrt{g}C_0+u_{00}-2\sqrt{g}C_{00}.$$

Substituting u_0 into (2.3.), we get:

$$\begin{aligned}&C_0=\frac{1}{3\sqrt{g}}(y-u_{00}+2\sqrt{g}C_{00}),\\&u_0=\frac{2}{3}y-\frac{2}{3}\sqrt{g}C_{00}+\frac{1}{3}u_{00}.\end{aligned} \qquad (2.4)$$

The following relations are also valid

$$C_{0y}=\frac{1}{3\sqrt{g}},\ u_{0y}=\frac{2}{3}y. \qquad (2.5)$$

After differentiating (2.2) at $t = 0$, considering (2.4), (2.5), we obtain:

$$\begin{aligned}&C_0u_{1y}-2\sqrt{g}C_0C_{1y}+\frac{2}{3\sqrt{g}}u_1+\frac{8}{3}C_1=0,\\&-C_0u_{1y}+2\sqrt{g}C_0C_{1y}+\frac{5}{3\sqrt{g}}u_1+\frac{2}{3}C_1=D_{10},\end{aligned} \qquad (2.6)$$

where

$$D_{10}=\sqrt{g}\,h_x(x_{00}).$$

When adding the equations of the system (2.6), we obtain:

$$\frac{7}{3\sqrt{g}}u_1+\frac{10}{3}C_1=\sqrt{g}h_x(x_{00})$$

or

$$u_1=-\frac{10}{7}\sqrt{g}C_1+\frac{3}{7}gh_x(x_{00}),$$

$$u_{1y}=-\frac{10}{7}\sqrt{g}C_{1y}.$$

Substituting u_1 and u_{1y} into the second equation (2.6), after the transformations we have

$$\sqrt{g}C_0C_{1y}-\frac{1}{2}C_1=\frac{1}{12}\sqrt{g}h_x(x_{00}).$$

Substituting C_0 in this equation, after the transformations we get:

$$(y-u_{00}+2\sqrt{g}C_{00})C_{1y}-\frac{3}{2}C_1=\frac{1}{4}\sqrt{g}h_x(x_{00}).$$

Integrating the equation, we have:

$$\begin{aligned}&C_1=C_{10}\left(y-u_{00}+2\sqrt{g}C_{00}\right)^{\frac{3}{2}}-\frac{D_{10}}{6},\\&u_1=-\frac{10}{7}\sqrt{g}C_{10}\left(y-u_{00}+2\sqrt{g}C_{00}\right)^{\frac{3}{2}}-\frac{D_{10}}{6}.\end{aligned} \qquad (2.7)$$

The integration constant C_{10} is determined from the conditions (1.7). The following coefficients of the series (2.1) are found from (1.6) by differentiating k times. After that, we assume $t = 0$ (Bautin, Deryabin, 2005). So, given (2.3), (2.4), we get:

$$\begin{aligned}&C_0u_{ky}-2g^{\frac{1}{2}}C_0C_{ky}+\frac{2u_k}{3\sqrt{g}}+\frac{6k+2}{3}C_k=F_{1k},\\&-C_0u_{ky}+2g^{\frac{1}{2}}C_0C_{ky}+\frac{3k+2}{3\sqrt{g}}u_k+\frac{2}{3}C_k=F_{2k},\end{aligned} \qquad (2.8)$$

where the functions $F_{1k} = F_{1k}(y)$, $F_{2k} = F_{2k}(y)$ are determined recursively based on the previously found coefficients of the series. Adding the first and second equations of the system (2.8), we get:

$$\frac{1}{\sqrt{g}}\left(k+\frac{4}{3}\right)u_k+\left(2k+\frac{4}{3}\right)C_k=F_k^+(y),$$

or

$$u_k = -\frac{6k+4}{3k+4}\sqrt{g}C_k + \frac{3}{3k+4}\sqrt{g}F_k^+(y),$$

$$u_{ky} = -\frac{6k+4}{3k+4}\sqrt{g}C_{ky} + \frac{3}{3k+4}\sqrt{g}F_{ky}^+(y),$$

where

$$F_{1k}(y) + F_{2k}(y) = F_k^+(y).$$

Substituting u_k and u_{ky} into the second equation (2.8), after the transformations we have:

$$\sqrt{g}\,\frac{12k+12}{3k+4}C_0 C_{ky} + \frac{2}{3}\left(1 - \frac{(3k+2)^2}{3k+4}\right)C_k = F_k.$$

Here the function $F_k = F_k(y)$ has the form:

$$F_k = F_{2k} + \frac{3\sqrt{g}}{3k+4}C_0 F_{ky}^+ - \frac{3k+2}{3k+4}F_k^+.$$

Substituting C_0 into the equation, after the transformations we get:

$$\left(y - u_{00} + 2\sqrt{g}C_{00}\right)C_{k1y} - \frac{3}{2}kC_k = \frac{3k+4}{4k+4}F_k(y).$$

Integrating this equation, we have:

$$C_k = G_{1k}\left(C_{k0} + G_{2k}\right),$$
$$u_k = -\frac{6k+4}{3k+4}\sqrt{g}G_{1k}\left(C_{k0} + G_{2k}\right) + G_{3k}, \quad (2.9)$$

where

$$G_{2k} = \frac{3k+4}{4k+4}\int F_k(y)\left(y - u_{00} + 2\sqrt{g}C_{00}\right)^{-\frac{3}{2}k-1}dy,$$

$$G_{1k} = \left(y - u_{00} + 2\sqrt{g}C_{00}\right)^{\frac{3}{2}k}, G_{3k} = \frac{3\sqrt{g}F_k^+(y)}{3k+4}.$$

The integration constants C_{k0} are found from (1.7). Substitute $C = C^0(t, x_1(t))$, $y = x_2(t)$ in series (2.1). As a result, we have

$$C(t, x_2(t)) = C^0(t, x_1(t)).$$

Differentiating this relation and substituting $t = 0$, we get the equations for finding the coefficients:

$$C_{k0}: \left(3\sqrt{g}C_{00}\right)^{\frac{3}{2}k}C_{k0} = Q_k.$$

Here the function Q_k is a known constant. Since $C_{00} \neq 0$, then C_{k0} are uniquely determined. Thus, the uniqueness of the formal solution of the problem (1.6), (1.7) constructed in the form of a series (2.1) is proved.

Theorem 1. Problem (1.6), (1.7) has a unique analytical solution, which is the convergent series (2.1).

The proof of the theorem is carried out by the majorant method, the application of which to the characteristic Cauchy problem is described in detail in (Bautin, 2009) and is not given in this paper. Using the simplest transformations, the solution (2.1) is written in the physical space of variables t, x:

$$H = C^2\left(t, \frac{x - x_{00}}{t}\right), u = u\left(t, \frac{x - x_{00}}{t}\right).$$

3 RESULTS AND DISCUSSION

Previously, the problem of the breakup of a special discontinuity were solved (Bautin, Deryabin, 2005; Bautin et al., 2011) in the space of specially introduced new independent variables. At the same time, in the space of the initial physical variables, the laws of motion of the surfaces Γ_0, Γ_1 were determined explicitly. But in order to determine the values of gas-dynamic parameters in the space of physical variables at some point in time $t = t_0$, it was necessary to reverse the implicitly specified functions. This procedure is rather cumbersome and difficult for setting the initial data at time $t = t_0 > 0$ between the surfaces Γ_0 and Γ_1 for the subsequent construction of the gas flow by numerical methods. To overcome the difficulty of inverting implicitly given functions this article solves the problem of wave overturning by introducing non-stationary self-similar variables. In this case, the gas parameters at time $t = t_0 > 0$ are determined explicitly in the space of the initial physical independent variables using the initial segments of the converging series (2.1).

Note that the constructed solution (2.1) allows us to obtain an approximation of the initial conditions at time $t = t_0$ in the form of the initial segments of the series. Also, the formulas (1.4), (1.5) give an approximation of the boundary conditions on the line Γ_1.

4 CONCLUSIONS

In this paper, the locally analytical solution of the problem of wave overturning in the space of physical variables is constructed. In the form of a convergent series, the initial conditions at time $t = t_0$ are obtained. In the form of a converging series, the boundary conditions are obtained on the boundary of an undisturbed and disturbed wave.

Thus, the analytical study was carried out for numerical simulation of the flow that arose after the collapse of the wave for a long period of time (Khakimzyanov et al., 2001; Bautin et al., 2011).

REFERENCES

Vol'cinger, N. E., Klevannyj, K. A., Pelinovskij, E. N., 1989. *Long-wave dynamics of the coastal zone*, Gidrometeoizdat. Leningrad, p. 272.

Hibberd, S., Peregrine, D., H., 1979. Surf and run-up on a beach: a uniform bore. *Journal of Fluid Mechanics.* 95 (2), pp. 323–345.

Carrier, G. F., Greenspan, H. P., 1958. Water waves of finite amplitude on a sloping beach. *Journal of Fluid Mechanics.* 4 (1). pp. 97–109.

Carrier, G. F., Wu, T. T., Yen, H., 2003. Tsunami run-up and draw-down on a plane beach. *Journal of Fluid Mechanics.* 475. pp. 79–99.

Kanoglu, U., 2004. Nonlinear evolution and runup–rundown of long waves over a sloping beach. *Journal of Fluid Mechanics.* 513. pp. 363–372.

Sanolakis, C. E., 1987. The runup of solitary waves. *Journal of Fluid Mechanics.* 185. pp. 523–545.

Bautin, S. P., Deryabin, S. L., 2005. *Mathematical modeling of ideal gas outflow into vacuum*, Science. Novosibirsk, p. 390.

Bautin, S. P., Deryabin, S. L., Sommer, A. F., Khakimzyanov, G. S., Shokina, N. Yu., 2011. Use of analytic solutions in the statement of difference boundary conditions on movable shoreline. *Russian Journal of Numerical Analysis and Mathematical Modeling.* 26 (4). pp. 353-377.

Bautin, S. P., 2009. *The characteristic Cauchy problem and its applications in gas dynamics*, Science. Novosibirsk, p. 368.

Ovsyannikov, L. V., 2003. *Lectures on the fundamentals of gas dynamics*, Institute of Computer Research. Moscow, Izhevsk. p. 336.

Khakimzyanov, G. S., Shokin, Yu. I., Barahnin, V. B., Shokina, N. Yu., 2001. *Numerical simulation of fluid flows with surface waves*, SB RAS. Novosibirsk. p. 394.