Computer Mathematics Systems and Tasks with Parameters

Yurii V. Horoshko[1][a], Tetiana V. Pidhorna[2][b], Petro F. Samusenko[3][c] and Hanna Y. Tsybko[1][d]

[1] *T. H. Shevchenko National University "Chernihiv Colehium", 53 Hetman Polubotka Str., Chernihiv, 14013, Ukraine*

[2] *State University of Trade and Economics, 19 Kyoto Str., Kyiv, 02156, Ukraine*

[3] *National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", 37 Peremohy Ave., Kyiv, 03056, Ukraine*

Keywords: Computer Mathematics Systems, Tasks with Parameters, GeoGebra, GRAN.

Abstract: Methodological aspects of using software GeoGebra and GRAN1 for solving tasks with parameters are considered in the paper. Criteria of software selection are developed and a comparative analysis of the specified software for solving tasks with parameters is given.

1 INTRODUCTION

Modeling of various processes and phenomena is one of the main general methods used in scientific research.

Learning to solve tasks with parameters considered in the process of teaching mathematics is one of the preparatory stages for mathematical modeling, where models are studied under different conditions, in particular, under different values of the parameters of mathematical models.

For decades, solving tasks with parameters was usually included in the program of entry exams to higher education institutions of Ukraine, currently this skill is required for a successful completion of an external independent evaluation in mathematics, which has been held in Ukraine for more than 10 years. As evidenced by the practice and results of pedagogical research, solving tasks with parameters causes many difficulties for students (Ilany and Hassidov, 2014), more than 85% entrants at the external independent evaluation in mathematics do not even attempt to solve such tasks (Botuzova, 2019).

A number of publications are devoted to the teaching method of solving tasks with parameters (Amelkin and Rabtsevich, 2004; Gornshteyn et al., 1992; Prus and Shvets, 2016; Gonda, 2018; Zakirova et al., 2019).

With the development of computer technology and corresponding software, the range of such problems, means and methods of learning how to solve them have expanded. Among the most famous free educational software, that provide rational solving of tasks with parameters, GeoGebra, Wolfram|Alpha, SageMath and GRAN can be distinguished (Bhagat and Chang, 2015; Krawczyk-Stańdo et al., 2013; Gunčaga, 2011; Kramarenko et al., 2019; Kashitsyina, 2020; Hrybiuk, 2017; Kramarenko, 2005; Pokryshen, 2007; Ivashchenko, 2015; Zhaldak, 2016).

The topic of research publications on the use of software for the analysis of mentioned tasks covers various aspects of teaching methods for solving tasks with parameters: studying the forms of graphs of functions for different values of the parameter (Ilhan, 2013; Božić et al., 2021); using a computer to illustrate analytical solutions (Pokryshen, 2007; Gunčaga, 2011); the method of organizing the research activity of pupils and students in the process of preliminary graphic analysis of tasks solutions with the further analytical solution (Kramarenko, 2005; Hrybiuk, 2017; Kramarenko et al., 2019; Krawczyk-Stańdo et al., 2013; Gornshteyn et al., 1992); obtaining solutions of the tasks based on detailed graphical analysis (Ivashchenko, 2015; Zhaldak, 2016). As a rule, the cited works consider examples of tasks, where solutions can be obtained analytically, but this is an exceptional case in practice.

Despite the fact that a large number of studies on mathematics teaching methods have been devoted to this topic, in particular with the use of modern

[a] https://orcid.org/0000-0001-9290-7563
[b] https://orcid.org/0000-0002-1414-3489
[c] https://orcid.org/0000-0002-4241-6173
[d] https://orcid.org/0000-0002-1861-3003

Horoshko, Y., Pidhorna, T., Samusenko, P. and Tsybko, H.
Computer Mathematics Systems and Tasks with Parameters.
DOI: 10.5220/0012067700003431
In *Proceedings of the 2nd Myroslav I. Zhaldak Symposium on Advances in Educational Technology (AET 2021)*, pages 768-780
ISBN: 978-989-758-662-0

computer-oriented technologies, its importance is undoubtful, since the number of mentioned tasks and their types is constantly increasing. It is clear that for practical use it is important to know not so much the exact solution value of the task, which describes the mathematical model of a real process or phenomenon, but whether the task is compatible and stable. Then, with the use of modern software, it is possible to find an approximate value of a certain solution of the task with a predetermined accuracy, and that is quite sufficient for practice.

In this work there have been developed criteria for the selection of software, advisable for the use in the process of solving tasks with parameters.

The work contains examples of solving such tasks by both analytical and graphical methods. Considerable attention is paid to the tasks that cannot be solved by analytical methods. At that, the plotting of quite complex graphs of functions for different values of parameters with the use of appropriate software helps to avoid errors in the plotting of such graphs, to focus attention on the analysis of their form and to find the answer to the task question. Examining tasks that can be solved only approximately with the use of the graphic method expands the students' understanding that in the process of describing mathematical models of various objects and phenomena, such solutions are used as well.

2 COMPUTER MATHEMATICS SYSTEMS FOR SOLVING TASKS WITH PARAMETERS

Let's consider the conditions of selection and use of computer mathematics systems (CMS) for solving tasks with parameters.

Today, web-oriented software, including CMS, is being used more and more. As already mentioned, the software tools GeoGebra, Wolfram|Alpha, SageMath, etc are among the most popular freely distributed computer mathematics systems.

Wolfram|Alpha is a knowledge base of various scientific fields, including mathematical ones. It is based on various algorithms and technologies of artificial intelligence. The web-based version can be accessed at https://www.wolframalpha.com/.

SageMath is a free and open source math system licensed under the GPL. The web-based version can be accessed via the link https://www.sagemath.org/.

One of the most widespread educational computer mathematics systems in Ukraine is the software complex GRAN, as evidenced by a significant number of scientific and pedagogical publications devoted to various aspects of the organization and implementation of the educational process in modern learning conditions.

The software complex GRAN was developed at the National Pedagogical Dragomanov University under the leadership of M. I. Zhaldak. This complex consists of three programs: GRAN1, GRAN2D, GRAN3D.

The GRAN1 program is intended for graphic analysis and solving tasks related to the plotting of graphs of functions on the Cartesian plane, defined explicitly and implicitly, parametrically, tabularly, in the polar coordinate system; for processing statistical data, plotting graphs of the probability distribution functions of random variables, calculating definite integrals, the length of curves, the area of curved trapezoids, the area of surfaces and volumes of bodies of rotation, etc.

The first version of the GRAN1 was developed for the Yamaha personal computer back in 1990 by A. V. Penkov (Zhaldak et al., 2016). Later, GRAN1 was improved and adapted for use under the operating system of the Windows family by Y. V. Horoshko. In 2019, this software tool was laid out on a remote desktop, which allows it to be used through a browser remotely (Zhaldak et al., 2021), and not only on local computers. The GRAN-2D program is intended for graphical analysis of systems of geometric objects on a plane, and the GRAN-3D program is intended for graphical analysis of systems of three-dimensional geometric objects. The first versions of the GRAN-2D and GRAN-3D programs were developed in 2002. The complex is freely distributed and can be downloaded from the website https://zhaldak.fi.npu.edu.ua/.

One of the most common educational computer mathematics systems is the GeoGebra system. The first version of GeoGebra was developed in 2001-2002 by M. Hohenwarter (Hohenwarter and Fuchs, 2004). In December 2021, GeoGebra was acquired by the conglomerate Byju's (Singh, 2021). This software tool can be used both remotely and on a local computer by downloading the appropriate program modules. A component of the GeoGebra system are programs for graphic analysis and solving tasks related to the plotting graphs of functions on the Cartesian plane, defined in explicit or implicit form, in the polar coordinate system, tasks in the theory of probability, planimetry and stereometry. Using CMS GeoGebra, one can also create didactic materials for different users, provide access to them for others; create an educational classroom environment for students to use.

The table 1 analyzes the presence of program functions, which, in our opinion, are needed for graphical analysis of solving tasks with parameters.

Thus, it is advisable to use, first of all, GRAN1 and GeoGebra from the listed software tools for solving tasks with parameters. We will give appropriate examples.

3 SOME EXAMPLES

1. Solve the equation

$$1-\frac{3}{x+a-1}=\frac{5a}{(x+a-1)(x+1)}. \qquad (1)$$

To analyze the task, we will use CMS GRAN1. Let's plot graphs of functions

$$f(x)=1-\frac{3}{x+a-1}$$

and

$$g(x)=\frac{5a}{(x+a-1)(x+1)}$$

for specific values of the parameter a. To do this, we specify the parameter a by $p1$ (the default parameter name). Changing the values of the $p1$ parameter, for example, from -5 to 5 with a step 0.1 (these values are set by default) leads to a corresponding shape change of the graphs of the specified functions. The equation (1) solutions are the abscissas of the points of intersection of the graphs of the functions $f(x)$ and $g(x)$. Since for values of the parameter $p1$ from -4.9 to -3.1, from -2.9 to -0.1 and from 0.1 to 5 the graphs of the functions $f(x)$ and $g(x)$ intersect at two points, the equation (1) for such $p1$ has two solutions (figure 1).

If $p1=-5$, then visually the graphs of the functions intersect at one point, that is an exceptional case for the equation (1), since it, generally speaking, reduces to a quadratic equation, and therefore, under certain conditions, has two solutions. Therefore, it is advisable to consider the $p1$ parameter, for example, from -6 to 5.

Considering the shape of the graphs of the functions $f(x)$ and $g(x)$ for $p1=-5.2$, $p1=-5.1$, $p1=-4.9$ and $p1=-4.8$ and taking into account, that even increasing the scale for $p1=-5$ it is not possible to get a clear answer about the number of roots of the equation (1) (figure 2), we come to the conclusion about the need of analytical equation study (1) when the value of the parameter a is changing in a neighborhood of the point -5.

In the case when $p1$ equals -3 or 0 the graphs of the functions $f(x)$ and $g(x)$ intersect at one point, that is, the equation (1) has a unique solution (figure 3, figure 4).

Taking into account the continuity of the functions $f(x)$ and $g(x)$ on the corresponding intervals, it is possible to hypothesize about the existence of two solutions of the equation (1) for $a\in(-\infty;-5)\cup(-5;-3)\cup(-3;0)\cup(0;\infty)$ and about the existence of one solution of the equation (1) for $a=-5$, $a=-3$ and $a=0$. To confirm or refute it, as a rule, it is necessary to make an analytical study of the problem.

We present the analytical solution of the given equation. It is clear that $x\neq-1$, $x\neq1-a$. Then

$$(x+a-4)(x+1)-5a=0$$

or $x^2+x(a-3)-4a-4=0$.

According to Viet's theorem

$$x_1=4,\ x_2=-a-1.$$

From the restrictions imposed on the variable x, it follows that

$$x_1\neq-1,\ x_1\neq1-a,\ x_2\neq-1,\ x_2\neq1-a.$$

The first and fourth conditions are obvious. Therefore, let's consider the rest of the conditions in more detail. From the inequality $x_1\neq1-a$ we get that $a\neq-3$. That is for $a=-3$ the value $x_1=4$ is not a root of the given equation (in this case, the root will be $x_2=2$) (figure 2). The third inequality $x_2\neq-1$ is equivalent to $a\neq0$. Therefore, for $a=0$ the root of the equation will be $x_1=4$ (figure 3).

Thus, if $a=-3$, then $x=2$ (figure 2), if $a=0$, then $x=4$ (figure 3), and if $a\neq-3$ and $a\neq0$, then $x=4$ or $x=-a-1$ (figure 1).

Thus, the previously proposed hypothesis about the number of roots of the equation (1) was partially confirmed.

In figures 1-3 the parameter value respectively is 0.4, -5 and -3. Note that the parameter value can also be selected using a slider. At that, the same drawing can contain images of graphs of functions that correspond to different fixed values of the parameter.

As it is known, there exists a slightly different approach to constructing a geometric interpretation of the task (1). Namely, it is necessary to find the intersection points of the graph of the function

$$h(x)=1-\frac{3}{x+a-1}-\frac{5a}{(x+a-1)(x+1)}$$

for different values of the parameter a with the abscissa axis. It is clear that solving such a task using appropriate software is similar in complexity to the presented above. In figure 5 the graph of the function $h(x)$ is plotted under the condition that $a=0.4$. In this case the graph of the function $h(x)$ intersect the abscissa axis at two points, that is, the equation (1) has two solutions.

Table 1: Program comparison.

Program functions	GRAN1	GeoGebra	Wolfram|Alpha	SageMath
Plotting graph of a function given in explicit form	+	+	+	+
Plotting graph of a function given in implicit form	+	+	+	+
Using a parameter in the function definition and its change, automatic change of a graph depending on the parameter value, the possibility to change the parameter changing step	+	+	–	–
Plotting of a tangent to a curve at a point	+	+	+	+
Ability to change the scale	+	+	-	–
Determination of the coordinates of the intersection of graphs of functions	+	+	+	–

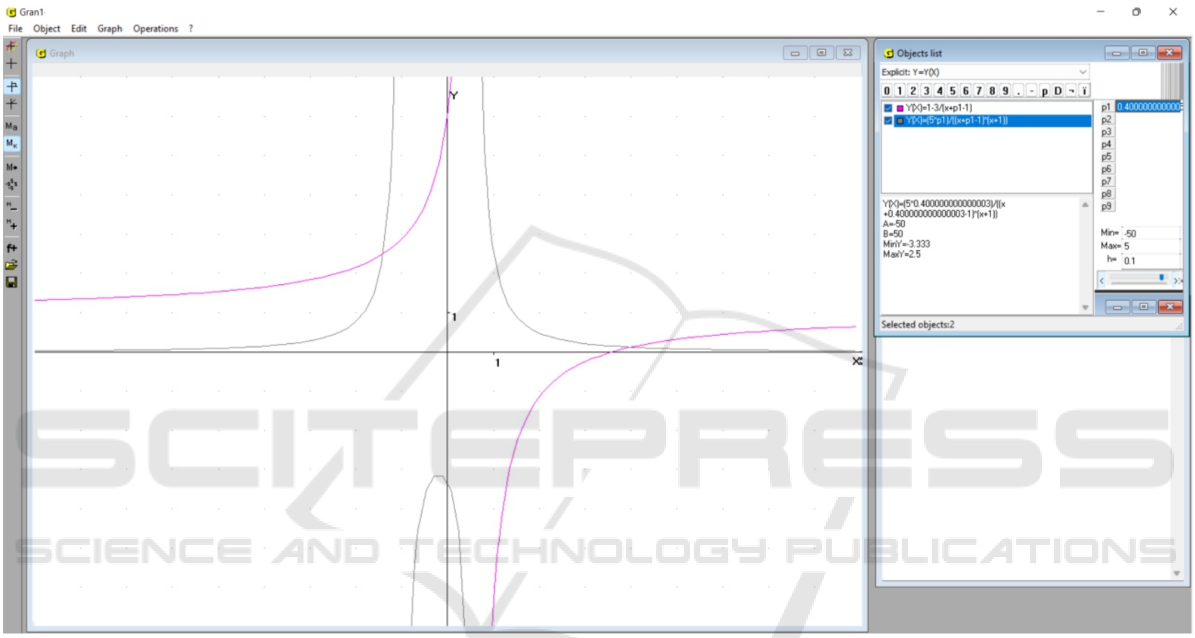

Figure 1.

Note that the use of this approach to solving tasks with parameters can lead to a more adequate geometric interpretation of the task. So, for $a = -5$ the graph of the function $h(x)$ at the point with abscissa $x = 4$ touches the abscissa axis (figure 6).

This means that

$$h(4) = 0,\ h'(4) = 0,$$

i.e. the equation

$$h(x) = 0,$$

and therefore the equation (1) has two identical roots $x = 4$. Indeed, by construction

$$h(x) = (x-4)^2 \widetilde{h}(x),\ \widetilde{h}(4) \neq 0.$$

2. Find the number of roots of the equation

$$a^x = \log_a x. \tag{2}$$

In this case, we use GeoGebra software for graphic illustrations. Let's plot the graphs of the functions $f(x) = a^x$ and $g(x) = \log_a x$ for specific values of the a parameter. Increasing the value of the parameter a from 0 with a step of 0.1, we come to conclusion – if the parameter a takes values from 0.1 to 0.9, then the equation (2) has one root, from 1.1 to 1.4 – the equation (2) has two roots, and finally, if the parameter value is greater than 1.4, then the equation (2) has no roots.

It is clear that the use of only a graphical way of solving the equation (2) does not only not allow to make the correct conclusion about the number of roots of the equation for $a \in (0;1) \cup (1;\infty)$, but also does not allow us to express an adequate hypothesis about the length of the intervals of change of the parameter a, where the equation (2) has the same number of roots. And the situation is not improved by a significant decrease in the step of changing the parame-

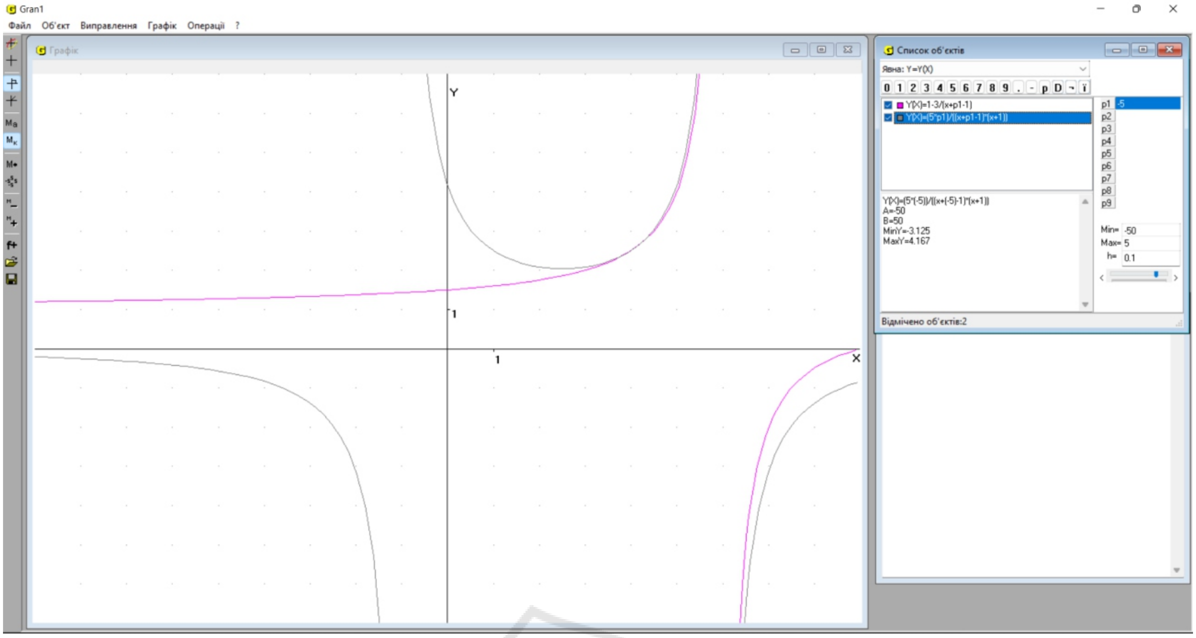

Figure 2.

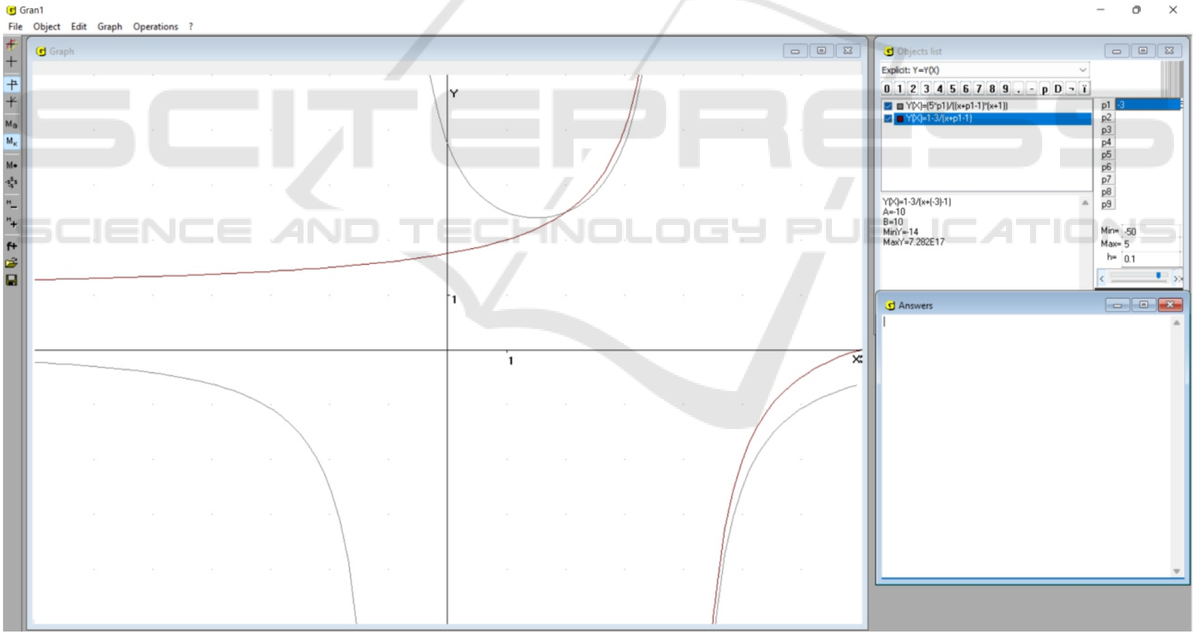

Figure 3.

ter a, since the values of the parameter, when passing through which the number of roots of the equation (2) changes, are transcendental numbers.

Therefore, an analytical study of the task is necessary.

On the condition of the task $x \in (0;\infty)$ and $a \in (0;1) \cup (1;\infty)$. Let it first $a \in (1;\infty)$. We find a tangent point M of graphs of the functions $y = a^x$ and $y = \log_a x$. It is clear that the abscissa of the tangent point will be the solution of the equation (2). Since these functions are mutually turned, their graphs are symmetrical relative to the line $y = x$. Taking into account the strict monotony of functions $y = a^x$ and $y = \log_a x$ and the immutability of the type of convexity of their graphs, we come to the conclusion that if the tangent point of the graphs of these functions

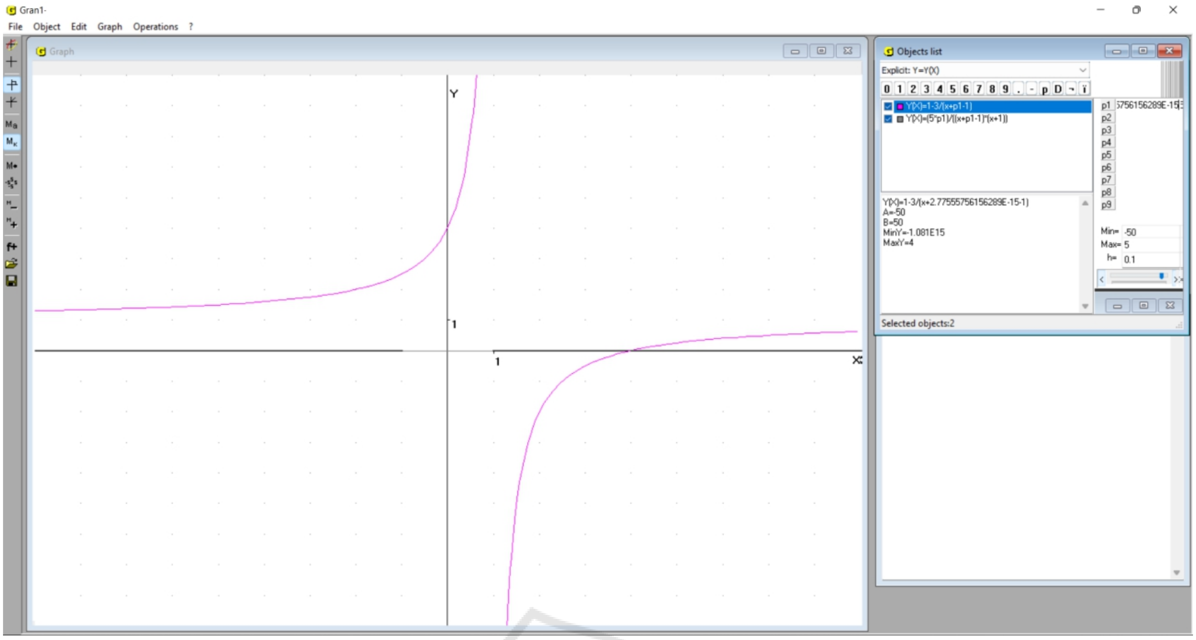

Figure 4.

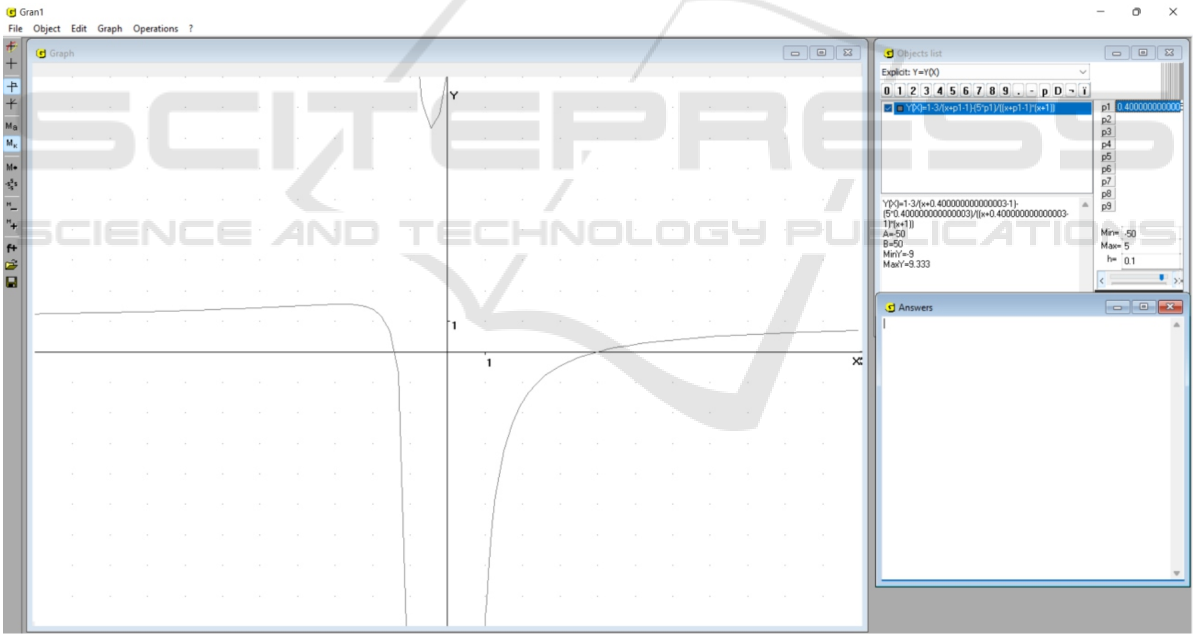

Figure 5.

exists, then it is unique and lies on the line $y = x$ (figure 6).

It is known that the coordinates of the tangent point of graphs of the functions $y = \varphi(x)$ and $y = \psi(x)$ meet the system of equations

$$\begin{cases} \varphi(x) = \psi(x), \\ \varphi'(x) = \psi'(x), \end{cases}$$

that in our case will look like

$$\begin{cases} a^x = x, \\ a^x \ln a = 1. \end{cases}$$

From here we get $a = \sqrt[e]{e}$, $M(e, e)$.

Suppose, that $a > \sqrt[e]{e}$. Then graphs of the functions $y = a^x$ and $y = x$ do not intersect, that is, the equation (2) has no solution (figure 8).

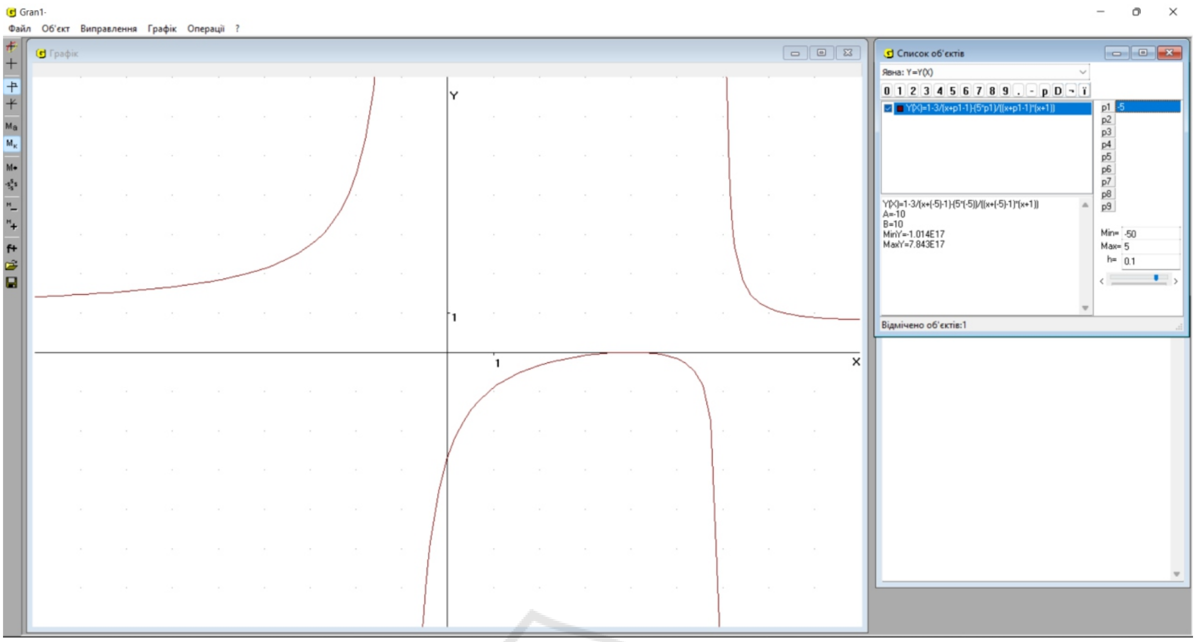

Figure 6.

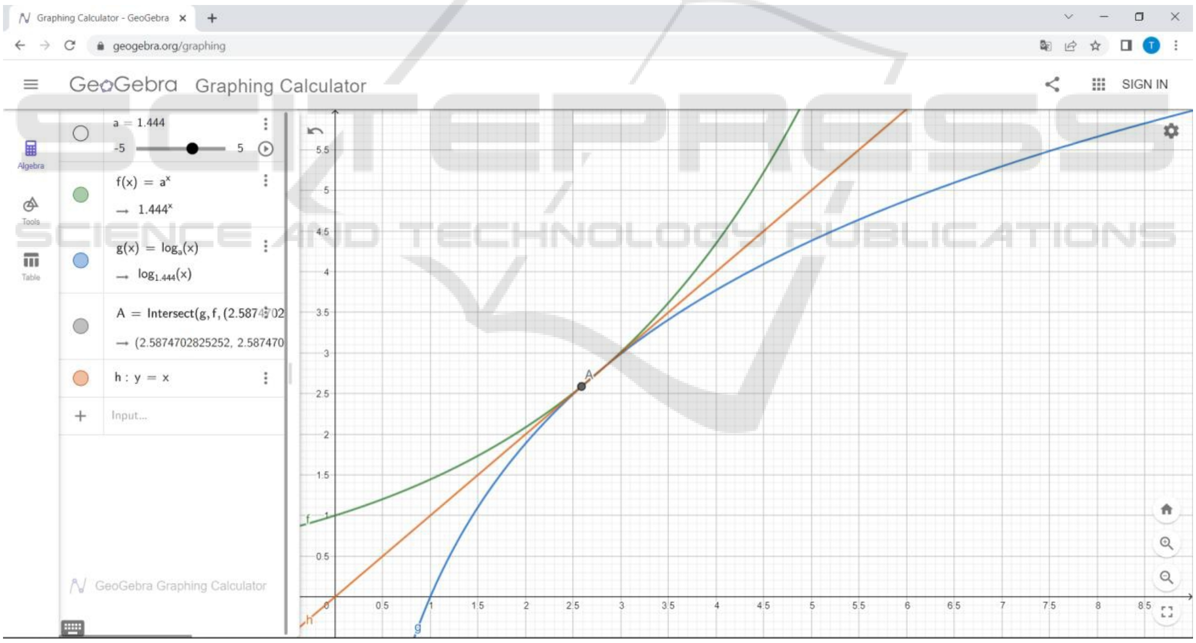

Figure 7.

Indeed, we denote as $h(x) = a^x - x$, $x \geq 0$. Then $h'(x) = a^x \ln a - 1$ and $x = -\frac{\ln \ln a}{\ln a}$ is a stationary point of function $h(x)$. Since $h''(x) = a^x \ln^2 a > 0$, $x \geq 0$, then $x = -\frac{\ln \ln a}{\ln a}$ is a minimum point of the function $h(x)$.

Taking into account that

$$h(0) = 1 > 0$$

and

$$h\left(-\frac{\ln \ln a}{\ln a}\right) = \frac{1 + \ln \ln a}{\ln a} > 0,\ a > \sqrt[e]{e},$$

then $h(x) > 0$, $x \geq 0$, i.e. $a^x > x$, $x \geq 0$.

Let it now be $1 < a < \sqrt[e]{e}$. Then the equation (2) has two solutions.

Indeed, determining the function $h(x)$, like before,

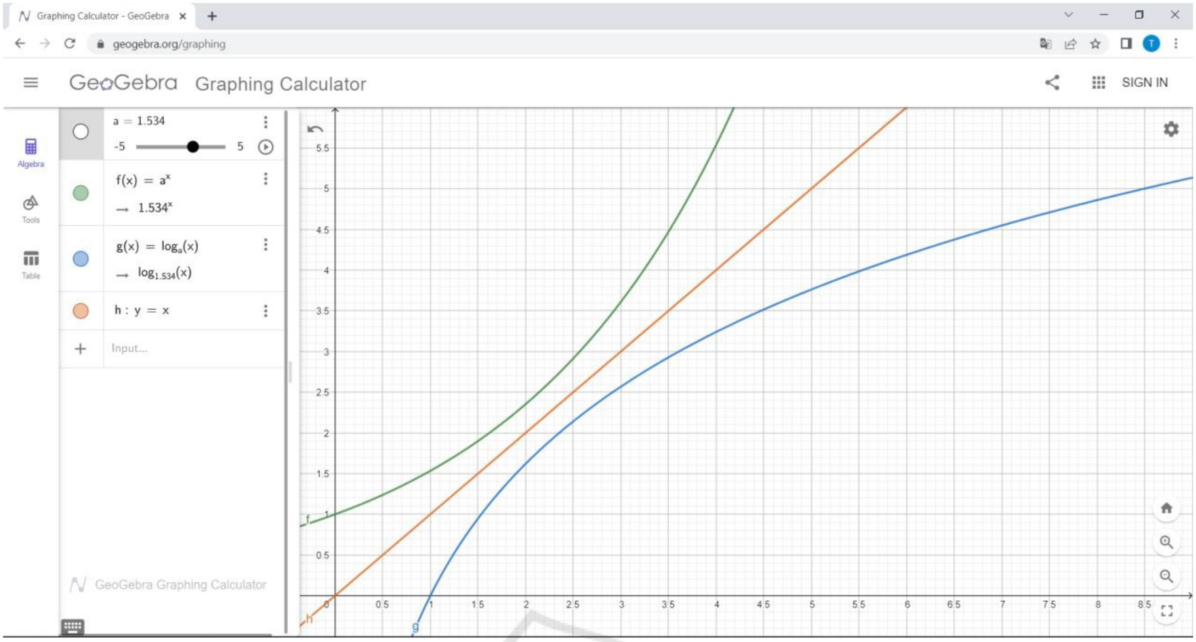

Figure 8.

we get

$$h(0) = 1 > 0,\ h\left(-\frac{\ln\ln a}{\ln a}\right) < 0,$$

$$\lim_{x\to+\infty} h(x) = +\infty.$$

Since graphs of the functions $y = a^x$ and $y = \log_a x$ have no points of inflection, then according to the Bolzano-Koshi theorem, they intersect at two points that are besides contained on the line $y = x$ (figure 9).

Let $0 < a < 1$. We denote as $l(x) = \log_a x - a^x$, $a \in [a_0; 1)$, where a_0 will be determined below. Let's investigate the function $l(x)$ on monotony:

$$l'(x) = \frac{1 - xa^x \ln^2 a}{x \ln a}.$$

The value a_0 we select so that $l'(x) \le 0$, $x \in (0;\infty)$. Since $x\ln a < 0$, $x \in (0;\infty)$, $a \in (0;1)$, then $l'(x) \le 0$, $x \in (0;\infty)$, than and only when

$$1 - xa^x \ln^2 a \ge 0,\ x \in (0;\infty).$$

Select a_0 so that

$$1 - xa^x \ln^2 a \ge 0,\ x \in (0;\infty),\ a \in [a_0;1).$$

We denote as $m(x) = xa^x \ln^2 a$, $x \in (0;\infty)$ and investigate $m(x)$ per extremum. Calculating

$$m'(x) = a^x \ln^2 a(1 + x\ln a),$$

$$m''(x) = a^x \ln^3 a(2 + x\ln a),$$

we are convinced, that the stationary point $x = -\frac{1}{\ln a}$ of the function $m(x)$ is a maximum point.

Since

$$m(0) = 0$$

and

$$m\left(-\frac{1}{\ln a}\right) = -\frac{1}{e}\ln a \le -\frac{1}{e}\ln\frac{1}{e^e} = 1,\ a \in \left[\frac{1}{e^e};1\right),$$

then $a_0 = \frac{1}{e^e}$.

Thus, $l'(x) \le 0$, $x \in (0;\infty)$, and points where $l'(x) = 0$, do not form a segment. Therefore, the function $l(x), x \in (0;\infty), a \in \left[\frac{1}{e^e};1\right)$ is descending. Therefore, the equation (2) has one solution (figure 10).

Let it finally $a \in \left(0;\frac{1}{e^e}\right)$. In this case, the equation (2) has three solutions (figure 11).

Indeed, graphs of the functions $y = a^x$ and $y = \log_a x$ intersect at some point of the line $y = x$. Let us denote the abscissa of this point as x_0. It is clear that $x = x_0$ is a solution of the equation (2). Consider the interval $(0;x_0)$. Note that $\lim\limits_{x\to 0+} l(x) = +\infty$. Determine a sign of the number $l(x_0 - \varepsilon)$, where ε – is a quite small positive constant.

$$l(x_0 - \varepsilon) = \log_a(x_0 - \varepsilon) - a^{x_0 - \varepsilon} = \log_a\left(1 - \frac{\varepsilon}{x_0}\right) +$$

$$+\log_a x_0 - a^{x_0}a^{-\varepsilon} = x_0 + \frac{1}{\ln a}\ln\left(1 - \frac{\varepsilon}{x_0}\right) - x_0 a^{-\varepsilon} =$$

$$= \frac{1}{\ln a}\ln\left(1 - \frac{\varepsilon}{x_0}\right) - x_0\left(a^{-\varepsilon} - 1\right) =$$

$$= \frac{1}{\ln a}\left(-\frac{\varepsilon}{x_0} - \frac{1}{2}\left(\frac{\varepsilon}{x_0}\right)^2 - \frac{1}{3}\left(\frac{\varepsilon}{x_0}\right)^3 - \ldots\right) -$$

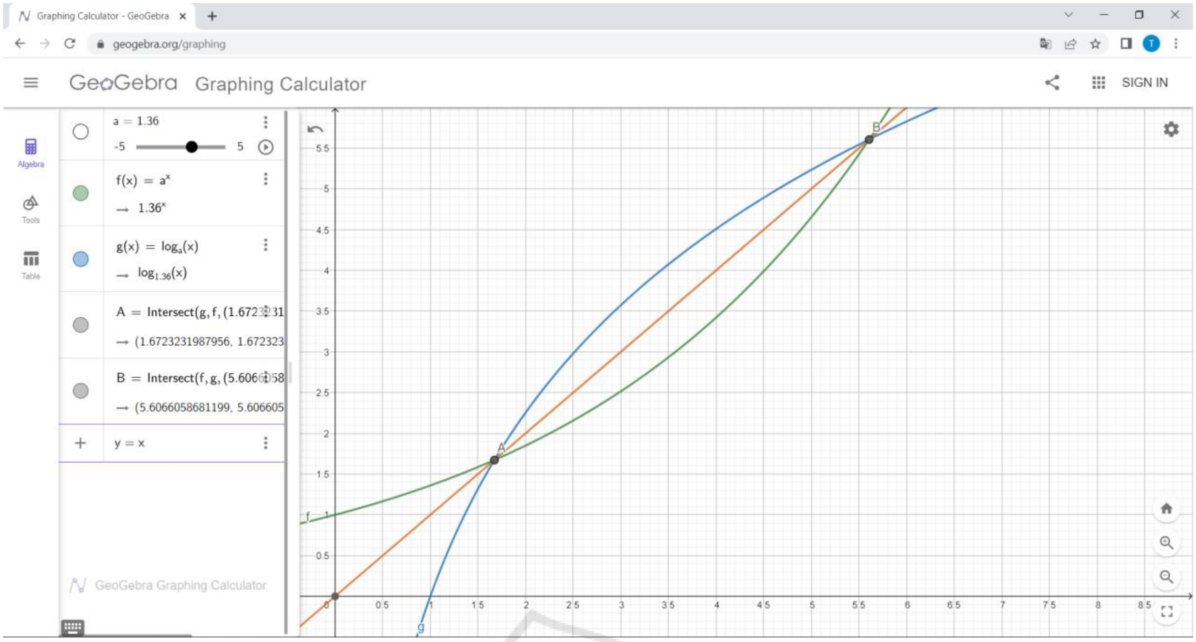

Figure 9.

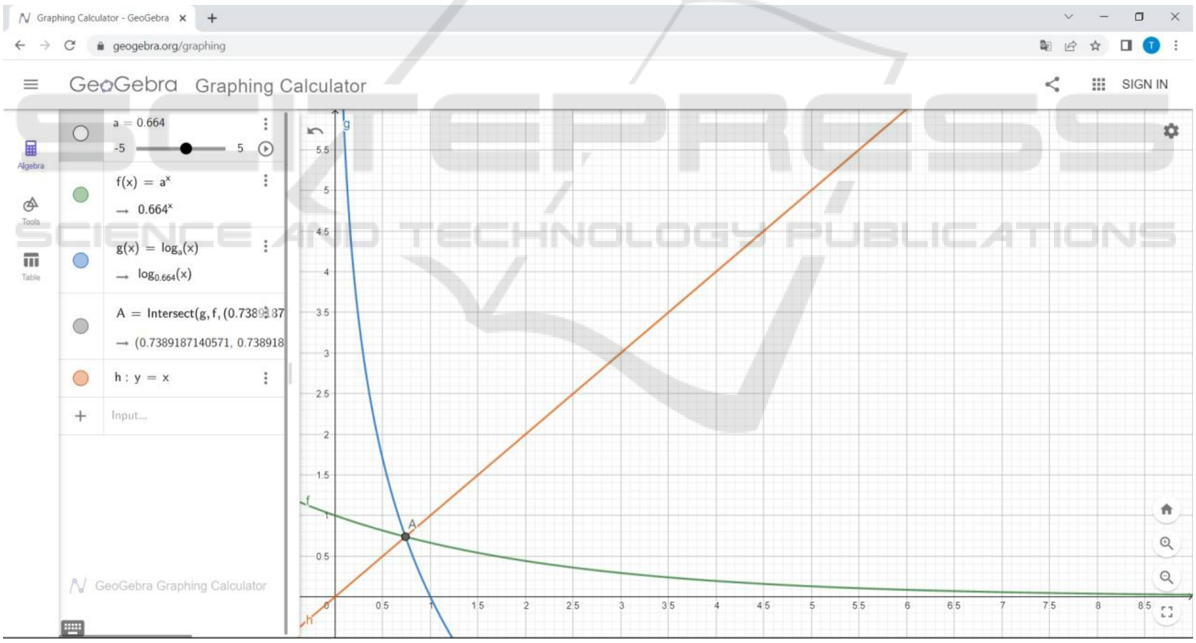

Figure 10.

$$-x_0(-\varepsilon\ln a+O(\varepsilon^2))\leq$$
$$\leq-\frac{1}{\ln a}\left(\frac{\varepsilon}{x_0}+\left(\frac{\varepsilon}{x_0}\right)^2+\left(\frac{\varepsilon}{x_0}\right)^3+\ldots\right)-$$
$$-x_0(-\varepsilon\ln a+O(\varepsilon^2))=$$
$$=\varepsilon\left(-\frac{1}{(x_0-\varepsilon)\ln a}+x_0\ln a\right)+O(\varepsilon^2)<0,$$

since inequality

$$-\frac{1}{x_0\ln a}+x_0\ln a<0 \tag{3}$$

is correct for all $a\in\left(0;\frac{1}{e^e}\right)$. Indeed, inequality (3) is equivalent to

$$x_0^2\ln^2 a>1$$

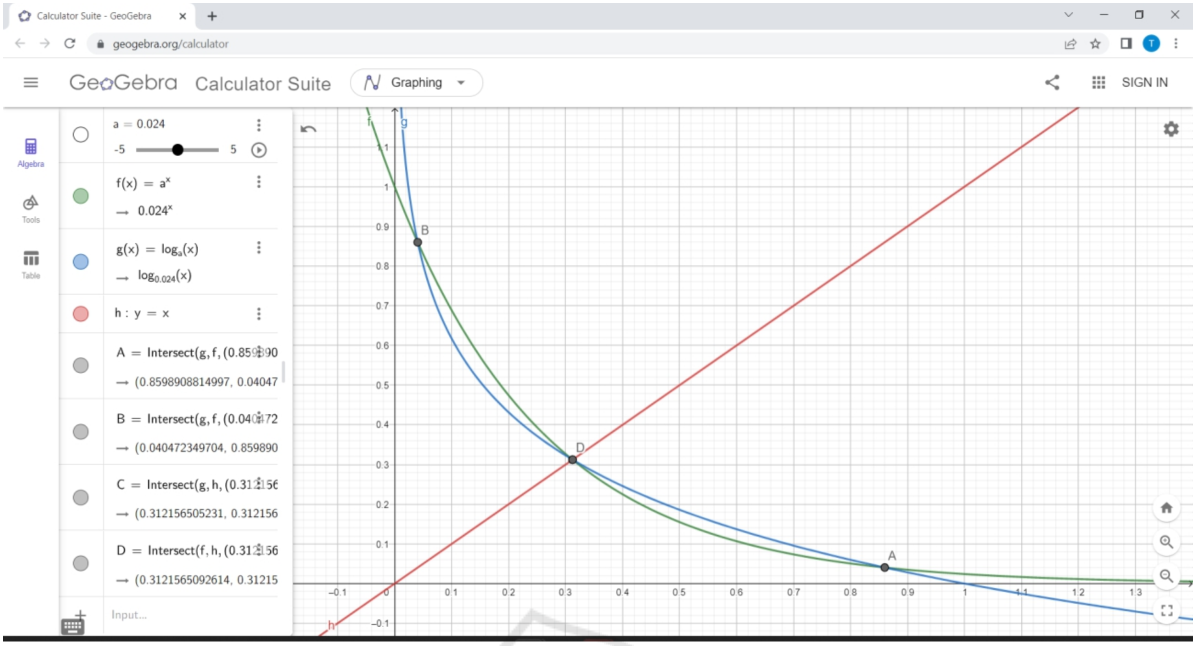

Figure 11.

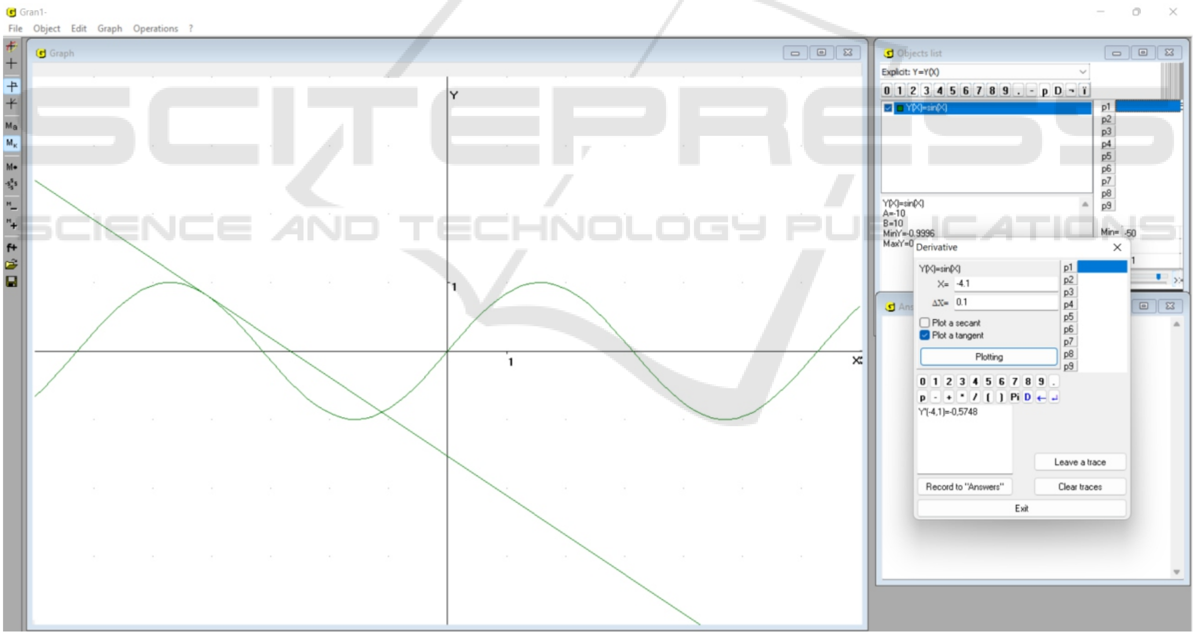

Figure 12.

or to

$$\ln^2 x_0 > 1,$$

i.e. $x_0 \in \left(0; \frac{1}{e}\right) \cup (e; \infty)$.

Since $x = \frac{1}{e}$ is a solution of the equation

$$\log_{\frac{1}{e^e}} x = x,$$

then the solution of the equation

$$\log_a x = x$$

is less than $\frac{1}{e}$, if $0 < a < \frac{1}{e^e}$. And therefore $x_0 \in \left(0; \frac{1}{e}\right)$, that is inequality (3) is correct.

Thus, according to the Bolzano-Cosh Theorem the equation (2) in the interval $(0; x_0)$ has a solution. For reasons of symmetry, taking into account the immutability of the type of convexity of graphs of the functions $y = a^x$ and $y = \log_a x$, it follows that for $a \in \left(0; \frac{1}{e^e}\right)$ equation (2) has three solutions.

Summarizing the analysis we come to the conclusion. Equation (2) has three solutions if $a \in \left(0; \frac{1}{e^e}\right)$, two solutions if $a \in (1; \sqrt[e]{e})$, one solution if $a \in \left[\frac{1}{e^e}; 1\right) \cup \{\sqrt[e]{e}\}$ and has no solution if $a \in \left(\frac{1}{e^e}; \infty\right)$.

3. Find the values of the parameters a and b, for which the equation

$$\sin x = ax + b \tag{4}$$

has two solutions.

The equation (4) will have two solutions if graphs of the functions $y = \sin x$ and $y = ax + b$ will intersect at two points. It is clear that one of these points is the tangent point of the graphs of these functions (figure 12).

In the GRAN1 program, it is possible to plot a tangent to a curve at a given point. In this case, the abscissa of the point of tangency can be considered a parameter. Further, by gradually changing the value of the parameter, one can observe the change in the position of the tangent and visually determine the number of roots of the considered equation. So, for which values of the parameters a and b does the equation (4) have 2 solutions?

Let the point with the abscissa x_1 is a tangent point of graphs of the functions $y = \sin x$ and $y = ax + b$. Also suppose that the solutions of the equation (4) belong to the segment $\left[-\frac{3\pi}{2}; 0\right]$.

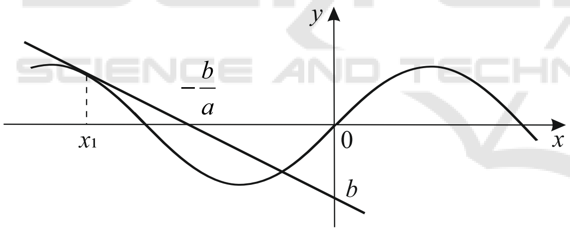

Figure 13.

Then the parameters a and b meet the inequalities

$$\begin{cases} -1 < a < 0, \\ -\pi < b < 0. \end{cases}$$

Since

$$\begin{cases} \sin x_1 = ax_1 + b, \\ \cos x_1 = a, \end{cases}$$

then, using the basic trigonometric identity, we get

$$\cos\left(\sqrt{\frac{1}{a^2} - 1} + \frac{b}{a}\right) = a. \tag{5}$$

Note that the inverse statement is correct as well. Namely, from the condition (5) it follows that at the point with the abscissa

$$x_1 = -\sqrt{\frac{1}{a^2} - 1} - \frac{b}{a}$$

graphs of the functions $y = \sin x$ and $y = ax + b$ touch.

The line $y = ax + b$ intersects the abscissa axis at the point $\left(-\frac{b}{a}; 0\right)$. Then when meeting the condition

$$-\pi < -\frac{b}{a} < 0,$$

the line $y = ax + b$ and the curve $y = \sin x$ will only have two common points.

Thus, if

$$\begin{cases} \cos\left(\sqrt{\frac{1}{a^2} - 1} + \frac{b}{a}\right) = a, \\ a\pi < b < 0, \end{cases}$$

then the equation (4) has two solutions.

Thinking similarly, one can find the conditions, under which there exist two solutions of the equation (4) at an arbitrary interval from the range $(-\infty; +\infty)$ so that at the whole range $(-\infty; +\infty)$ the equation (4) would have two solutions.

Note that the equation (4) can be solved approximately with the use of expansion of the function $y = \sin x$ in a Maclaurin series, namely,

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = ax + b. \tag{6}$$

Then, taking into account the convergence of the series in the left side of equality (5) in the range $(-\infty; +\infty)$, instead of of the equation (5) such equation can be considered

$$\sum_{k=1}^{p} \frac{x^{2k-1}}{(2k-1)!} = ax + b, \tag{7}$$

where the expression in the left part is a polynomial. The value of p is determined by the specified accuracy.

The equation (7) contains a polynomial of an odd degree with real coefficients. Therefore, it has an odd number of real roots. The latter fact in no way contradicts the proven statement of the even number of real roots of equation (4), since the equation (4) unlikely (7) is transcendental and the properties of the function $y = \sin x$ (limited, monotony, bulge) are significantly different from the corresponding properties of the polynomial $\sum_{k=1}^{p} \frac{x^{2k-1}}{(2k-1)!}$.

Using the developed approach to solving the equation (4), it is possible to consider various problems related to finding a given number of their solutions.

4 CONCLUSIONS AND FURTHER RESEARCH PROSPECTS

The graphical method of finding solutions of equations, inequalities and their systems is based on the procedure of plotting graphs of the corresponding functions. In the case of implicit or parametric definition of functions, the process of plotting their graphs is quite complicated. The task becomes even more cumbersome if it contains a parameter. That is why they often try to solve these problems using various software tools.

This paper substantiates the expediency of using GRAN1 and GeoGebra computer mathematics systems for solving tasks with parameter. In the GRAN1 program, it is possible to write the formula of the function with the designation of parameters through the variables $p1$, $p2$, ..., $p10$. At that, the parameter can be changed both by assigning it a certain value and by using a slider. In the process of changing the parameter values, the graph of the function is automatically constructed, taking into account the updates of the parameter values. Similar functions are inherent in the GeoGebra program. It is only needed to define the variables through which the parameters are denoted.

Using the automatic change of the shape of the graph for different values of the parameter, the change of the step of changing the value of the parameter, the change of scale for viewing the graph of the function or its fragment, the automatic determination of the coordinates of the point of intersection of the graphs of the functions, in the work there are established the conditions of compatibility of the geometric character of three illustrative tasks with the parameter and their analytical solutions are given. At that for geometrical support of the process of solving the specified tasks, GRAN1 and GeoGebra computer mathematics systems were used, as already mentioned above. The mentioned software tools are equally convenient for use in the process of solving tasks with parameters.

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