A Unity-norm Preserving Quaternion Extended Kalman Filter

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Keywords: Extended Kalman Filter, Quaternion, Robots.

Abstract: Preserving the quaternion unity norm has significant importance when combined with the extended Kalman filter (EKF) for state estimation of nonlinear systems perturbed by noise using noisy measurements. Although this unity challenge is solved numerically on the estimated quaternion, it is normalized out of the EKF algorithm. Projecting this constraint on the derivation of the EKF algorithm to preserve the quaternion unity norm is the purpose of this paper. The quaternion unity norm constraint is derived in two forms and projected on the EKF gain derivation. Then this gain is used to update the quaternion vector keeping its unity norm. The results show that the unity norm is preserved using the proposed constrained quaternion EKF (CQEKF) even though sudden changes occur.

1 INTRODUCTION

The state of art of estimation based on the extended Kalman filter EKF integrates the Bayesian approach with the observer theory. The system nonlinear model along with stochastic models form the EKF structure (Simon, 2006); (Grewal, 2001); (Maybeck, 1982). The time behavior of the system is described by the former while the later describe the noise properties. This filter is used extensively in many applications (Ulrich, 2011); (Jassemi-Zargani, 2002); (Hedberg, 2017); (Bussi, 2017); (Xu, 2018). The EKF works in two steps, the first one predicts the states of the nonlinear system based on the previous known states and input. The next step updates the predicted states based on the measurement residual.

In robotic applications, the quaternion representation combines the advantages of both Euler based orientation and direct cosine matrix representations (Phuong, 2009). Therefore it is used for orientation representation. However, it is used with the constraint of unity norm (Murray, 1994).

Quaternion based Kalman filters are reported in the literature (Choukroun, 2006); (Kim, 2004); (Suh, 2010). The quaternion EKF is implemented for rigid body orientation estimation based on the measurements of the angular velocity (Hashlamon, 2010). The quaternion vector forms the EKF states. However, the quaternion is constrained to unity norm which is not preserved by the EKF (Leffert, 1982). To overcome this problem, numerical techniques are applied on the post estimated quaternion to maintain its unity norm after the estimation is finished . However, this is an approximation and the constraint is not included into the filter derivation. Kalman filtering with state constraints is reported in (Simon, 2009).

This paper proposes a systematic method of including the unity norm constraint into the filter derivation to form the constrained quaternion extended Kalman filter (CQEKF). The constraint is derived in two forms. Then the EKF gain is derived based on minimizing the state covariance subjected to the given constraint.

The rest of the paper is organized as follows: The conventional EKF is introduced in section II. The unit quaternion is in section III. The model derivation is in section IV. The problem is defined in section V. The CQEKF derivation is in section VI The results are presented in section VII and the paper is concluded in section VIII.

2 CONVENTIONAL EKF AND DESIGN ASSUMPTIONS

Consider the discrete-time nonlinear state space model

$$x_k = f\left(x_{k-1}, u_{k-1}\right) + \Phi_{k-1} \upsilon_{k-1}, \tag{1}$$

Hashlamon, I.
A Unity-norm Preserving Quaternion Extended Kalman Filter.
DOI: 10.5220/0010966900003260
In *Proceedings of the 4th International Conference on Applied Science and Technology on Engineering Science (iCAST-ES 2021)*, pages 1435-1439
ISBN: 978-989-758-615-6; ISSN: 2975-8246

$$y_k = Hx_k + v_k , \quad (2)$$

where $x \in \mathbb{R}^n$ is the vector of states, u is the input, $y \in \mathbb{R}^d$ represents the measurement vector, and k is the time index.$\upsilon \in \mathbb{R}^m$ and $v \in \mathbb{R}^d$ are the Gaussian process and measurement noises with covariances Q and R respectively. It is assumed that they are independent and uncorrelated. $\Phi \in \mathbb{R}^{n\times m}$ maps the noise to the states space. H is the output matrix and given as $H \equiv \begin{bmatrix} I_4 & \mathbf{0}_{4\times3} \end{bmatrix}$. I_4 is identity matrix of size four and $\mathbf{0}_{4\times3}$ is 4×3 zero matrix.

Then for the given system in (1), the conventional EKF algorithm is composed of two steps:

The prediction step

$$\hat{x}_k^- = f\left(\hat{x}_{k-1}, u_{k-1}\right), \quad (3)$$

$$P_k^- = A_{k-1}P_{k-1}A_{k-1}^T + \Phi_{k-1}Q_{k-1}\Phi_{k-1}^T , \quad (4)$$

where

$$A_{k-1} = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k-1}, u_{k-1}} , \quad (5)$$

$\hat{x}$ and P are the posterior estimated state and the estimation error covariance matrix respectively. $\hat{x}^-$ and P^- are the prior estimates.

The measurement update step updates the estimates using the measurement residual e and Kalman gain K as

$$\hat{x}_k = \hat{x}_k^- + K_k e_k , \quad (6)$$

where K

$$K_k = P_k^- H^T S_k^{-1} , \quad (7)$$

and

$$e_k = Y_k - H\hat{x}_k^- , \quad (8)$$

Y is the measurement vector with the same dimension as y and S is

$$S_k = HP_k^- H^T + R_k , \quad (9)$$

The time update of P is given by

$$P_k = \left(I - K_k H\right)P_k^- , \quad (10)$$

where I is the identity matrix, and The Kalman gain in (7) can be expressed as

$$K_k = P_k H^T R_k^{-1} \quad (11)$$

3 UNIT QUATERNION

The quaternion is a four elements vector, one element is scalar represented by $q_0 \in R$ and the remaining three elements form a vector represented by $\vec{n} \in R^3$. The quaternion is written mathematically as

$$\mathbf{q} = q_0 + \vec{n} \quad (12)$$

or can be extended using the three imaginary axes $\hat{i}, \hat{j}$, and $\hat{k}$ as

$$\begin{aligned} \mathbf{q} &= q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k} \\ &\equiv \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T \end{aligned} \quad (13)$$

The quaternion norm is

$$\|q\|_2^2 = \sum_{i=1}^{3} q_i^2 \quad (14)$$

For rigid bodies, representing the rotations and orientation is of significant importance. The quaternion $\mathbf{q}$ subjected to the constraint of unit ℓ_2^2 norm ($\|\mathbf{q}\|_2^2 = 1$) is employed to represent the orientation of a rigid body with respect to a reference frame (Murray, 1994). This necessitates deriving the quaternion model.

4 MODEL DERIVATION

There is a direct relation between the measured angular velocity $\boldsymbol{\Omega} \in \mathbb{R}^3$ and the quaternion time derivative $\dot{\mathbf{q}}$. This relation utilizes the quaternion multiplication $\otimes$ as

$$\dot{\mathbf{q}} = \mathbf{q} \otimes \boldsymbol{\Omega} . \quad (15)$$

However, the measured angular velocity has bias $\boldsymbol{b} \in \mathbb{R}^3$and contaminated white zero mean noise $\boldsymbol{v} \in \mathbb{R}^3$ (Roetenberg, 2006)

$$\boldsymbol{\Omega} = \boldsymbol{\omega} + \mathbf{b} + \mathbf{v} , \quad (16)$$

where $\boldsymbol{\omega}$ is the true angular velocity and $\mathbf{b}$ is given by

$$\mathbf{b}_k = \mathbf{b}_{k-1} + \mathbf{v}_{b,k-1} . \quad (17)$$

where $\boldsymbol{v}_b \in \mathbb{R}^3$ is white zero mean noise.

Equation (15) is nonlinear and perturbed by noise. Therefore the EKF is employed to estimate the quaternion and bias.

Define the state vector as $x_k \equiv \begin{bmatrix} \mathbf{q}_k^T & \mathbf{b}_k^T \end{bmatrix}^T$, then after mathematical manipulation and discretization, the model (1) is obtained with

$$f(x_{k-1}, u_{k-1}) \equiv \begin{bmatrix} \left(I_4 + \frac{1}{2} T\mathbf{F}(\mathbf{\Omega}_{k-1} - \mathbf{b}_{k-1}) \right) \mathbf{q}_{k-1} \\ \mathbf{b}_{k-1} \end{bmatrix}, \tag{18}$$

$$\upsilon_{k-1} \equiv \begin{bmatrix} \mathbf{v}_{k-1} \\ \mathbf{v}_{b,k-1} \end{bmatrix}, \tag{19}$$

where

$$\mathbf{F}(\lambda) = \begin{bmatrix} 0 & -\lambda_x & -\lambda_y & -\lambda_z \\ \lambda_x & 0 & \lambda_z & -\lambda_y \\ \lambda_y & -\lambda_z & 0 & \lambda_x \\ \lambda_z & \lambda_y & -\lambda_x & 0 \end{bmatrix}, \tag{20}$$

and

$$\Phi_{k-1} = \begin{bmatrix} -\frac{1}{2} T \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}_{k-1} & \mathbf{0}_{4\times 3} \\ \mathbf{0}_{3\times 3} & I_3 \end{bmatrix}. \tag{21}$$

This model is used in the EKF algorithm and the quaternion $\mathbf{q}$ is considered to be the output.

5 PROBLEM DEFINITION

Maintaining the quaternion unity norm during the estimation is a challenge. The conventional EKF does not preserve this constraint. In the previous studies, this challenge is solved numerically by normalizing the posterior estimates $\hat{\mathbf{q}}$ out of the EKF algorithm as in

$$\hat{\mathbf{q}}_N = \frac{\hat{\mathbf{q}}}{\|\hat{\mathbf{q}}\|_2^2}, \tag{22}$$

or as reported in (Hashlamon, 2010)

$$\hat{\mathbf{q}}_N = \hat{\mathbf{q}} + \left(1 - \|\hat{\mathbf{q}}\|_2^2\right)\hat{\mathbf{q}}, \tag{23}$$

where $\hat{\mathbf{q}}_N$ is the normalized estimated quaternion vector. It is a low cost method of unity preserving done out of the EKF algorithm. Here the unity norm constraint is projected on the Kalman gain derivation and included into the EKF algorithm.

6 CQEKF DERIVATION

The unity norm of the estimated quaternion $\hat{\mathbf{q}}$ is given as

$$f(\hat{\mathbf{q}}) = \|\hat{\mathbf{q}}\|_2^2 = \sum_{i=0}^{3} \hat{q}_i^2 = 1, \tag{24}$$

however, $\hat{\mathbf{q}}$ is unknown before the updating step in the EKF. The best known quaternion vector is the predicted quaternion vector $\hat{\mathbf{q}}_k^-$ obtained from (3). Then (24) can be written in terms of $\hat{\mathbf{q}}_k^-$ as $f(\hat{\mathbf{q}}^-) = \|\hat{\mathbf{q}}^-\|_2^2 = 1$ and linearized about $\hat{\mathbf{q}}_k^-$ using Taylor series in two forms.

The first form is

$$f(\hat{\mathbf{q}}_k^-) = f(\hat{\mathbf{q}}_k^-) + \dot{f}(\hat{\mathbf{q}}_k^-)(\hat{\mathbf{q}}_k^- - \hat{\mathbf{q}}_k^-) = 1. \tag{25}$$

then (25) can be simplified as

$$\dot{f}(\hat{\mathbf{q}}_k^-)\hat{\mathbf{q}}_k^- = 1 - f(\hat{\mathbf{q}}_k^-) + \dot{f}(\hat{\mathbf{q}}_k^-)\hat{\mathbf{q}}_k^-, \tag{26}$$

which has only $\hat{\mathbf{q}}^-$ as unknown, (26) is written as

$$G\hat{\mathbf{q}}_k^- = d, \tag{27}$$

where

$$G = \dot{f}(\hat{\mathbf{q}}_k^-) = \frac{\partial f(\hat{\mathbf{q}}_k^-)}{\partial(\hat{\mathbf{q}}_k^-)} = 2\begin{bmatrix} \hat{q}_0^- & \hat{q}_{1,}^- & \hat{q}_2^- & \hat{q}_3^- \end{bmatrix}_k, \tag{28}$$

and

$$d = 1 + \sum_{i=0}^{3} (\hat{q}_i^-)_k^2. \tag{29}$$

The second form used the previous known value of the estimated quaternion as

$$f(\hat{\mathbf{q}}_k^-) = f(\hat{\mathbf{q}}_k^-) + \dot{f}(\hat{\mathbf{q}}_k^-)(\hat{\mathbf{q}}_k^- - \hat{\mathbf{q}}_{k-1}) = 1. \tag{30}$$

then (30) can be simplified as

$$\dot{f}(\hat{\mathbf{q}}_k^-)\hat{\mathbf{q}}_k^- = 1 - f(\hat{\mathbf{q}}_k^-) + \dot{f}(\hat{\mathbf{q}}_k^-)\hat{\mathbf{q}}_{k-1}, \tag{31}$$

For this case, G and d in (27) are as expressed as in (28) and

$$d = 1 - \sum_{i=0}^{3} \left(\hat{q}_i^-\right)_k^2 + G\hat{\mathbf{q}}_{k-1} . \qquad (32)$$

respectively.

The gain in (7) is the solution of (33)

$$K_k = \arg\min_{K_k} Tr\left(\begin{array}{r} \left(I - K_k H_k\right) P_k^- \left(I - K_k H_k\right)^T \\ + K_k R_k K_k^T \end{array} \right), \qquad (33)$$

In the same way, minimizing (33) subjected to the constraint $G\hat{\mathbf{q}}_k^- = d$ yields to the constraint system Kalman gain $\widehat{K}_k$

$$\widehat{K}_k = K_k - \begin{bmatrix} \delta_k \\ \mathbf{0}_{3\times4} \end{bmatrix}, \qquad (34)$$

where K_k is given in (7) and δ is

$$\delta_k = G^T \left(GG^T\right)^{-1} \left(G\hat{\mathbf{q}}_k^- - d\right)\left(e_k S_k^{-1} e_k^T\right)^{-1} e_k^T S_k^{-1} . \qquad (35)$$

where d is as in (29) or (32).

As a final estimation, the CQEKF has the equations(3)-(9) and

$$K_k = P_k^- H_k^T S_k^{-1} , \qquad (36)$$

$$\widehat{K}_k = K_k - \begin{bmatrix} \delta_k \\ \mathbf{0}_{3\times4} \end{bmatrix}, \qquad (37)$$

$$e_k = \begin{bmatrix} 1 \\ \mathbf{0}_{3\times1} \end{bmatrix} - \begin{bmatrix} q_0 \hat{q}_0^- + n^T \hat{n}^- \\ q_0 \hat{n}^- - \hat{q}_0^- n - n \times \hat{n}^- \end{bmatrix}, \qquad (38)$$

$$\hat{x}_k = \hat{x}_k^- + \widehat{K}_k e_k , \qquad (39)$$

$$P_k = \left(I - \widehat{K}_k H_k\right) P_k^- , \qquad (40)$$

where $\mathbf{q} \equiv \left[q_0 \quad \vec{n}\right]^T$ is considered as the measured quaternion. The zero error in quaternion means that both of the quaternions, the measured and the estimated, are coinciding on each other and in the same direction i.e. $q_0 \hat{q}_0 + n^T \hat{n} = 1$ and $q_0 \hat{n} - \hat{q}_0 n - n \times \hat{n} = \mathbf{0}_{3\times1}$. Accordingly, e_k in quaternion representation is as in (38).

The derived CQEKF can be projected directly on the adaptive EKF proposed in (Hashlamon,2020).

7 RESULT

In this part, the CQEKF is tested. MATLAB Simulink is used as an experimental platform. The noise with the specified covariance matrices are generated using the Simulink Gaussian noise generator. The input u in (1) is the measured angular velocity Ω . To form Ω , the bias and process noise with covariance Q are added to ω . The measured vector Y in (8) has contaminated noise with covariance ∴. The model equations (18) -(21) are used to form the CQEKF algorithm. The true bias values are generated based on time t as in (41) .

$$\mathbf{b} = \begin{cases} \left[0.5 \quad 0.1 \quad 0\right]^T & t \le 20 \\ \left[1 \quad 2 \quad 1\right]^T & 20 < t \le 50 \\ \left[0 \quad 0 \quad 2\right]^T & else \end{cases} . \qquad (41)$$

The initialization, simulation parameters, and the corresponding constants values are listed in Table 1. The experiment duration is 500 sec. The performance of the CQEKF displays its ability to preserve the quaternion constraint as depicted in Fig. 1. This also shows that though sudden changes in the bias take place, the norm still unity. Both forms in (28), (29) or (32) show the same steady state behavior.

Table 1: Initialization and simulation parameters.

Parameter	Value
T	0.01 sec
R	$10^{-6} I_4$
Q	$10^{-2} I_6$
P_0	$10 I_7$
$\mathbf{q}_0$	$\left[0.5 \quad 0.5 \quad 0.5 \quad 0.5\right]^T$
$\mathbf{b}_0$	$\left[0 \quad 0 \quad 0\right]^T$

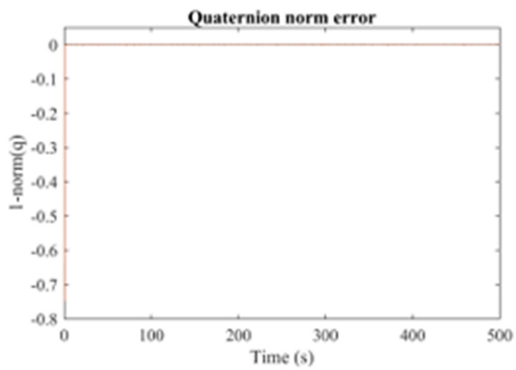

Figure 1: The estimated quaternion norm error.

8 CONCLUSIONS

A constrained quaternion extended Kalman filter CQEKF is proposed. The norm constraint is projected on the derivation of the EKF gain. Two forms of the constraint are obtained, both have the same effect. The obtained CQEKF preserves the unity norm constraint for the quaternion during the running of the algorithm. The results show that unity norm is preserved even sudden changes to the states may occur.

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