Analytical Solution of Homogeneous Groundwater Flow Equation using Method of Separation of Variables

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Abstract: This paper presents an analytical solution for predicting the one-dimensional (1D) time-dependent groundwater flow profile in an unconfined system. This hydraulic charge prediction problem is modeled as a boundary value problem governed by the heat diffusion equations. The solution technic employs the separation of variables method, the results are compared to the numerical solution, and the solution displays a reasonable flow head during different periods.

1 INTRODUCTION

The groundwater equation governed through Darcy law and the continuity equation was the subject of a set of research. Among the first researchers concerned with this equation (Bansal and Das, 2011; Bear, 2013; Chapman, 1980; Childs, 1971; Glover, 1960; Hantush, 1967; McDowell - Boyer et al., 1986; Simmons et al., 2001; Verhoest and Troch, 2000; Wooding and Chapman, 1966). Initially, these researches focused on trying to understand the behavior of groundwater and its flow mechanisms in porous media. These researches have focused to find solutions to the groundwater equation, the researchers analyzed the mechanisms of evolution and regular groundwater flow regeneration in the aquifers. as a result, it has been proposed and developed a set of analytical solutions (Manglik et al., 1997; Pauwels et al., 2002; Rai and Manglik, 1999; Verhoest and Troch, 2000).

In the same context, some recherche has focused on finding in obtaining analytical solutions to the linear Boussinesq equation by adopting the uniform recharge of the rainfall rate, therefore these solutions are exploited to estimate the groundwater levels change and drainage flow (Pauwels et al., 2002; Serrano, 1995; Verhoest and Troch, 2000).

Other research focused on researching analytical solutions for the same equation, taking into account the hypothesis of temporal variation of the level of rainfall (Dralle et al., 2014; Park and Parker, 2008; Pauwels et al., 2002; Ram and Chauhan, 1987; Su, 1994; Thomas, 2013).

Some other research has worked on the Laplace Transform method to develop an analytical solution to express the distribution of groundwater levels (Bansal and Das, 2011; Kim and Ann, 2001; Kumar et al., 2016; Pauwels et al., 2002; Sun et al., 2011).

On the other hand, special studies have worked on the use of some numerical solutions as a mechanism for studying and developing models of groundwater flow equation (Draoui et al.; El Mansouri and El Mezouary, 2015; El Mezouary, 2016; El Mezouary, El Mansouri, and El Bouhaddioui, 2020; El Mezouary et al., 2015; El Mezouary, El Mansouri, Moumen, et al., 2020; EL MEZOUARY et al., 2016; Sadiki et al., 2019) as a model of forecasting and simulating the dynamic behavior based on boundary conditions.

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The present study focused on the method of Separation of Variables for solving the groundwater equation using Darcy's law as a theoretical basis and applied the principle of mass conservation (the continuity equation) to govern the groundwater flow.

2 GROUNDWATER FLOW EQUATION

The general groundwater flow equation is deducted from Darcy's law and the continuity equation. The net rate of penetration of a fluid in a control volume is exactly equal to the net rate of change of storage of the mass of fluid in the same control volume.

We begin by examining the last groundwater flow phenomena (Diffusion), which are treated similarly with a linear diffusion partial differential equation.

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial}{\partial x} h(x,y,z,t)\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial}{\partial y} h(x,y,z,t)\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial}{\partial z} h(x,y,z,t)\right) = S_s \frac{\partial}{\partial t} h(x,y,z,t) - q(x,y,z,t) \quad (1)$$

Where K_x, K_y, K_z are the hydraulic conductivity, S_s is specific storage, h is the piezometric head.

We consider the equations in one dimension that the medium is uniform. which means that the coefficient of permeability K is spatially invariant, we can write to them as simple constants. Then, equation 1 is simplified by:

$$k\left(\frac{\partial^2}{\partial x^2} h(x,t)\right) = \frac{\partial h(x,t)}{\partial t} \quad (2)$$

Where k is the groundwater flow diffusivity or hydraulic diffusivity of the medium, are L^2/T:

$$k = \frac{K}{S_s} \quad (3)$$

3 ANALYTICAL SOLUTION

In this level, we will proceed to solve the groundwater flow equation 2 by the variable separation method:

We note that x represents the position in the one-dimensional medium (an aquifer) that we can identify with the interval $[0, L]$. The hydraulic height in this aquifer at time t and location x is $h(x,t)$.

A typical problem is to consider that the distribution of the hydraulic height over the entire length of the aquifer is known at time $t = 0$ (initial condition) and that the flow of groundwater through the ends $x = 0$ and $x = L$ are given values (boundary conditions). Therefore we can imagine that the hydraulic height is determined for $x \in (0, L)$ and $t > 0$. The conditions imposed on the ends are often of the form:

- $h(0,t) = 0$ or, $\frac{\partial h(0,t)}{x} = 0$ or $\frac{\partial h(0,t)}{x} = 0 = ah(0,t)$
- $h(L,t) = 0$ or, $\frac{\partial h(L,t)}{x} = 0$ or $\frac{\partial h(L,t)}{x} = 0 = -ah(L,t)$

Where $a > 0$ is also a physical constant. The solution of problem (2) given by the method of separation of variable it is as follows:

$$h(x,t) = \sum_{n=0}^{\infty} sin\left(\left[\left(\frac{|\lambda_n|}{k}\right)^{1/2} x\right]\right) Q_n e^{\lambda_n t} \quad (4)$$

Since $\lambda_n = -k(n + \frac{1}{2})^2 \left(\frac{\pi}{L}\right)^2$ the equation can be written in this formula:

$$h(x,t) = \sum_{n=0}^{\infty} sin\,(\alpha\pi x) Q_n e^{-k[\alpha\pi]^2 t} \quad (5)$$

Where $\alpha = \frac{(2n+1)}{2L}$, and the $Q_n$is:

$$Q_n = \frac{2}{L}\int_0^L sin\left(\left[(\beta)^{1/2} x\right]\right)\varphi(x)dx, \forall n \in \mathbb{N} \quad (6)$$

Where $\beta = \frac{|\lambda_n|}{k}$, $Q_n$also can be written :

$$Q_n = \frac{2}{L}\int_0^L sin\,(\alpha\pi x)\varphi(x)dx \quad (7)$$

4 SIMULATION OF SOLUTION

Consider the case of a $1D$ flow problem on an unconfined aquifer that a river and a lake run parallel to each other (figure 1) with $L = 500m$ apart. They fully penetrate aquifer with a hydraulic conductivity $K = 400\ m/day$, and specific yield $S_s = 22\%$. To demonstrates the feasibility of the analytical solution given by method of separation of variable, we are compared it by a two numerical profile simulated using the CrankNicholson implicit method (8)

(Thomas, 2013) and the Forward Time Centered Space method (FTCS) (9) (Anderson et al., 1997).

The CrankNicholson implicit method consists of replacing the second derivative $\frac{\partial^2 h}{\partial x^2}$ in the equation (2) by the average of its discrete representations at times n and $n+1$.

$$\left[\frac{\partial^2 h}{\partial x^2}\right]_i^{n+1} = \frac{1}{h^2}\left[\frac{1}{2}(h_{i+1}^{n+1} - 2h_i^{n+1} + h_{i-1}^{n+1}) + \frac{1}{2}(h_{i+1}^{n} - 2h_i^{n} + h_{i-1}^{n})\right] \quad (8)$$

While for the Forward Time Centered Space (FTCS) or forward/backward space method is an implicit single-stage finite difference method that can use for numerically solving the heat equation and similar parabolic partial differential equations.

This scheme is unconditionally stable. then the equation (2) can be represented by the flowing scheme:

$$h_i^{n+1} = h_i^n + \alpha(h_{i+1}^n - 2h_i^n + h_{i-1}^n) \quad (9)$$

With $\alpha = \frac{k\Delta t}{\Delta x^2}$

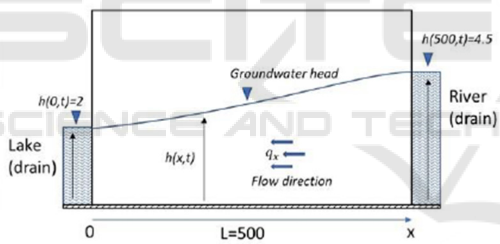

Figure 1: Groundwater flow conceptual model on unconfined, horizontal aquifer with dirichlet boundary condition.

This solution example corresponds to the following mathematical problem with nonhomogeneous Dirichlet boundary conditions (Figure 2).

$$\begin{cases} k\left(\frac{\partial^2}{\partial x^2}h(x,t)\right) = \frac{\partial h(x,t)}{\partial t}, \\ \quad 0 \le x \le L, and \ \ 0 \le t \le T; \\ h(0,t) = 2, h(L,t) = 4.5, \quad 0 \le t \le T; \\ h(x,0) = (2 + \frac{x^3}{5 \times 10^7}), \quad 0 \le x \le L \end{cases} \quad (10)$$

Following the solution represented by equation 4, the simulation of the solution illustrate on problem (2) is shown in Figure 2, the figure show comparison of the evaluated exact solution with implicit numerical methods of CrankNicholson and FTCS, while Figure 3 shows the evaluated exact solution at various times $t = 10, t = 20, t = 30, t = 40$.

It should be remembered that this simulation took into account the imposed boundary conditions, based on the conceptual model shown in Figure 1, it is obvious that the figure's results provide a good match solution at different times.

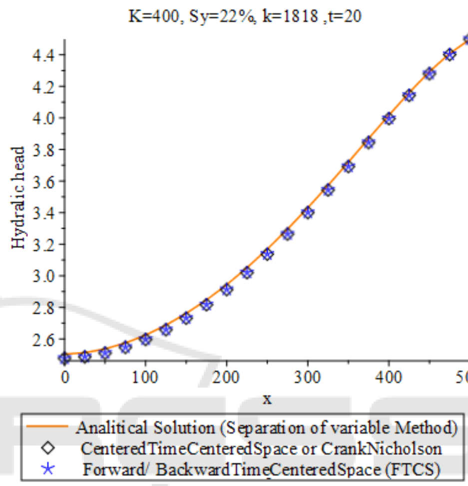

Figure 2: Comparison of the evaluated exact solution with implicit numerical methods of CrankNicholson and FTCS.

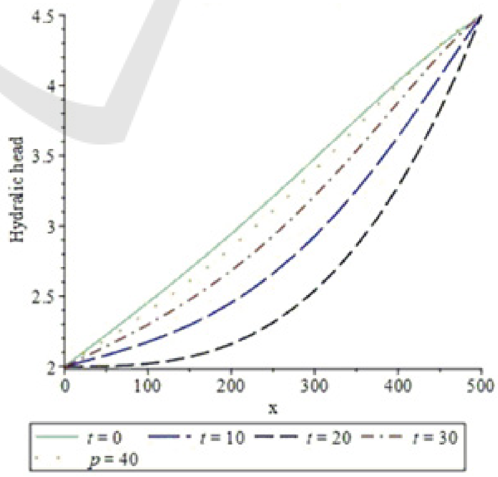

Figure 3: evaluated exact solution at various times.

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5 DISCUSSION

Analytical solution for the prediction of the one-dimensional ($1D$) time-dependent groundwater flow profile in an unconfined system evaluated for a setting corresponding case to $L = 500\,m$, $\mathrm{K} = 400\,\mathrm{m}$, $\mathrm{k} = 1818\,m^2/\mathrm{d}$, $S_s = 22\%$. The solution uses a uniform domain in different time step lengths. The simulation results obtained according to equation 4, the results showing note that the solution produced by the method of separation of the variable is acceptable, as well as that in all the cases in which the solution was applied, it was found that there is a match between the solution produced by the separation of variable method and with the other two numerical methods of CrankNicholson and FTCS method.

The module can simulate the same solutions that were given by the CrankNicholson and FTCS methods. It can then be concluded that the solution is given by the separation of the variable method when applied in a homogeneous medium, taking into account the normal boundary conditions, the solution presented can reproduce the behavior of the groundwater in a very acceptable way.

6 CONCLUSIONS

The paper introduces an analytical solution of a one-dimensional groundwater equation for a homogenous porous media. Using the method of separation of variables, this solution precisely reproduces the similar solution given from CrankNicholson and FTCS finite-difference methods. An example is used to verify the proposed solution, considering constant head in boundary conditions (Dirichlet conditions). The analytical solution has been compared with the CrankNicholson and FTCS numerical solutions, for a context with sand-gravel medium characteristics. The correlation is good within the example case. In consequence, the proposed method is valid for the homogenous horizontal unconfined aquifer, also for another similar physical or environmental problem.

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