Storage Fees in a Container Yard with Multiple Customer Types

Yahel Giat[a]
Department of Industrial Engineering, Jerusalem College of Technology, HaVaad HaLeumi 21, Jerusalem, Israel

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Abstract: We consider a small container storage yard that is located in the outskirts of a marine shipping port. Customers are classified according to the number of storage space they need. The yard has a limited number of storage spots, and serves customers only if there is sufficient space to fulfill their needs. We define the yard's state space and derive its steady-state probability matrix for Markovian arrivals and service times. We consider the profit function of two services that the yard offers its customers. In the first, storage fees are independent of the storage duration and in the second they are proportional to it. We show when these pricing schemes are equivalent and demonstrate numerically how profits depend on customer arrivals rates and yard size.

1 INTRODUCTION

In the shipping industry, container yards may have to store containers for some time before they are transported to their next destination. We consider a small container yard located near a marine port that provides two service schemes. In the first, the yard owner manages the entire container-shipping process and therefore the client is unmindful and not responsible for the storage time in the yard, whereas in the second scheme, it is the customer that orders the entry and exit of the container from the yard. Accordingly, in the first service scheme, customer fees are independent of the storage duration ("one-time fee"). In contrast, under the second scheme, storage fees are proportional to the storage duration ("proportional fee").

Customers' arrivals to the yard is assumed to follow a Poisson process. In the simplest setting, each customer requires storage for a single container, if such is available. If there is no vacant spot then the customer is rejected and receives service elsewhere (i.e., no backlogging). The general setting extends to multiple customer types. Here, the arrivals of type-k customers follow a Poisson process with rate λ_k and they require k units of storage space for a time that is exponentially distributed with mean μ_k.

Since the two services are offered to customers, the yard owner is interested to determine whether one service is more profitable than the other. In other words, given a proportional and one-time fees, which service provides a greater profit to the owner? Alternatively, we wish to find whether it is possible to set specific fees for each service such that their pricing schemes are equivalent. Furthermore, when the yard owner has the ability to change the yard's size (for example, through leasing to and from neighboring lots), then the question rises what is the optimal lot size?

To address these questions we describe the state space of the yard and the probability of the yard to be in each state. We then show that if for each customer type the proportional fee is set to be equal to the one-time fee multiplied by the expected service time, then the two pricing schemes are equivalent.

We use a numerical illustration to show properties of the profit function, in particular to demonstrate optimal yard size and the effects of customer arrivals rate on the profit function. While the motivation of this paper are container yard operations, we note that the model can be applied to many other settings. For example, parking lots serve vehicles of varying sizes (e.g., mini cars; passenger cars; vans; minibuses; buses). Another notable example is the telecommunication industry, when customers with larger bandwidth demands require more servers than those with smaller bandwidth needs (Moscholios et al., 2016). Managers of such systems may benefit from applying our model to their particular settings.

[a] https://orcid.org/0000-0001-7296-8852

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2 LITERATURE REVIEW

Queueing systems in which a customer may require more than one server arise naturally in many cases. The queueing system that is most related to our model are the $M/M/m$ queues. The notation $M/M/m$ describes a queuing systems with Markovian arrival times, Markovian service time with m servers. Green (1980) and Green (1981) studies an $M/M/m$ version assuming that the servers assigned to the same customer do not end service simultaneously, that is, these servers become available independently. They derive analytic expressions for the distribution of the waiting time in the queue and the distribution of the number of busy servers. A similar model is developed in Federgruen and Green (1984) assuming that each server has a general service time distribution. They use an approximation for the queue-length distribution. In contrast to these papers, in Fletcher et al. (1986) the servers become available simultaneously. Hassin et al. (2015) consider an M/M/K/K system with a fixed budget for servers. The system owner's problem is choosing the price, and selecting the number and quality of the servers, in order to maximize profits, subject to a budget constraint. In their model, however, there is only one customer type. In contrast, our model is similar to Kaufman (1981) and allows for multiple customer types where each type differs in the number of servers that it requires.

In 1968, the International Organization for Standardization (ISO) introduced the ISO 668 standard, detailing classification, dimensions and ratings for freight containers (ISO, 2020). While standard ISO shipping containers have an assortment of size the vast majority come in one of two lengths twenty feet (6.06m) and forty feet (12.2m). For example, as of 2012, 84 percent of the global feet containers were either twenty or forty feel long (Notteboom et al., 2020). The twenty foot container is typically used as the unit measure, also called Twenty Equivalent Unit Container (TEU). Accordingly, the forty feet container is sometimes called a two TEU container or a Forty Equivalent Unit Container (FEU). The TEU measure is used mainly in the marine shipping industry, with ship sizes measured according to their TEU capacity.

Pricing schemes for container storage service has been explored in various settings. Yu et al. (2011) focuses on the pricing of incoming containers. Our model is more related to Woo et al. (2016) who analyzes pricing storage of outgoing containers. Their pricing structure is nonlinear where there is a limited free storage time that is followed by a per day storage fee. Our study emphasizes a single aspect of the container freight shipping cycle, whereas other studies focus on other steps in the process. For example, Zhang et al. (2014) optimize the repositioning of empty containers, Chan et al. (2019) attempt to forecast ports' container throughput, and Dong and Song (2012) consider the optimal leasing of the containers themselves.

Container storage is similar to autonomous vehicle storage and retrieval systems. These systems share two critical features with the container storage problem. First, they park vehicles with varying sizes and therefore may demand a different number of storage units. Second, vehicles can be easily moved around so that vacant storage units can be located adjacently to accommodate large vehicles that require multiple units. See Marchet et al. (2012) for an analysis of such a system. Similarly, our model can be applied to the management of recharging docks in electric vehicle charging stations (Dreyfuss and Giat, 2017). In this setting recharging docks are the system's servers and vehicles with larger batteries may require more recharging docks than smaller batteries.

3 A CONTAINER YARD

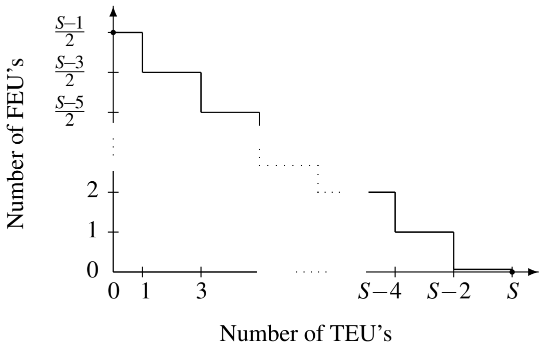
We begin with description of a simple yard model in which there are only two types of customers. In Section 3.2 we show that these results can be extended to any number of customer types.

3.1 Two Customer Types

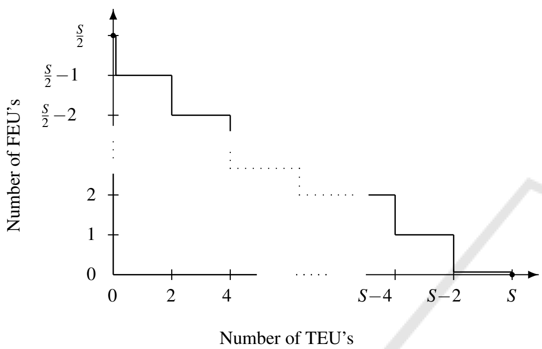
A yard owner provides short-term container storage. The yard has S storage spots and customers arrive with a single container that is either a TEU or FEU. TEU's require a single storage spot and FEU's require two spots. It is relatively easy to move containers around ("remarshalling"), and therefore any two available spots can be made to store an FEU. We assume no backlogging and therefore if there is no room to store a container, then the customer stores their container elsewhere.

Customers' arrivals and storage times are independent of each other. We assume that the arrivals of the FEU's and TEU's follow a Poisson process with rates λ_F and λ_T, respectively. Storage times of containers in the yard are exponential with means μ_F and μ_T for the FEU's and the TEU's, respectively.

The yard's current state is the number of FEU's and TEU's that are currently stored in it. Let (i,j) denote the state in which there are i FEU's and j TEU's stored in the yard and let $\mathcal{K}$ denote the state space of



(a) S is an odd number.



(b) S is an even number.

Figure 1: State space of the yard.

the yard. Since there are S storage spots,

$$\mathcal{K} = \{(i,j)|0 \leq i \leq \lfloor \frac{S}{2} \rfloor, 0 \leq j \leq S-2i\},$$

where $\lfloor x \rfloor$ represents the floor function.

In Figure 1b we depict the state space when S is odd and in Figure 1a when S is even. When S is even the number of states in the yard is $(\frac{S+2}{2})^2$, whereas if S is odd then the number states is $\frac{S+1}{2} \cdot \frac{S+3}{2}$.

The transition rates between states depend on the number of FEU's and TEU's in the system. Consider the state (i,j). If $i < \lfloor S/2 \rfloor$ then the transition rate to state $(i+1,j)$ is the arrivals rate of FEU's, λ_F and if $j < S$ the transition rate to state $(i, j+1)$ is the arrivals rate of TEU's λ_T. If $i > 0$ then the transition rate to state $(i-1,j)$ is $i\mu_F$ and if $j > 0$ then the transition rate to state $(i, j-1)$ is $j\mu_T$. The system's transition rates diagram is depicted in Figure 2.

Let

$$I_{(i,j)} = \begin{cases} 1 & \text{if} \quad \{(i,j) \in \mathcal{K}\} \\ 0 & \text{otherwise} \end{cases} .$$

For each state $(i,j) \in \mathcal{K}$, using the transition rates described above, the state's balance equation is given by

$$\begin{aligned} \pi_{i,j} \Big(\lambda_F I_{\{(i+1,j)\in\mathcal{K}\}} + \lambda_T I_{\{(i,j+1)\in\mathcal{K}\}} \\ + i\mu_F I_{\{(i-1,j)\in\mathcal{K}\}} + j\mu_T I_{\{(i,j-1)\in\mathcal{K}\}} \Big) \\ = \quad \pi_{i+1,j}(i+1)\mu_F I_{\{(i+1,j)\in\mathcal{K}\}} \\ + \pi_{i,j+1}(j+1)\mu_T I_{\{(i,j+1)\in\mathcal{K}\}} \\ + \pi_{i-1,j}\lambda_F I_{\{(i-1,j)\in\mathcal{K}\}} \\ + \pi_{i,j-1}\lambda_T I_{\{(i,j-1)\in\mathcal{K}\}}, \end{aligned} \quad (1)$$

and the normalizing condition is

$$\sum_{(i,j)\in\mathcal{K}} \pi_{i,j} = 1. \quad (2)$$

To compute statistical properties of the system (e.g., probability for rejection, expected number of FEU's and TEU's) we must first compute the system's steady state probabilities. These are given in the following proposition.

Proposition 1. *Let $a_F := \frac{\lambda_F}{\mu_F}$ and let $a_T := \frac{\lambda_T}{\mu_T}$. Let $\vec{\pi}$ denote the system's steady state probability matrix such that $\pi_{i,j}$ is the probability for the system to be in state $(i,j) \in \mathcal{K}$ in steady state. Then*

$$\begin{aligned} \pi_{0,0} &= \left(\sum_{i=0}^{\frac{S}{2}} \sum_{j=0}^{S-2i} \frac{{a_F}^i}{i!} \frac{{a_T}^j}{j!} \right)^{-1} \\ \pi_{i,j} &= \frac{{a_F}^i}{i!} \frac{{a_T}^j}{j!} \pi_{0,0} \end{aligned} \quad (3)$$

Proof: We need to show that $\vec{\pi}$ is the unique solution to the system (1 - 2). Consider the following system of equations:

$$\begin{aligned} i\mu_F \pi_{i,j} &= \lambda_F \pi_{i-1,j} & (i,j),(i-1,j) \in \mathcal{K} \\ j\mu_T \pi_{i,j} &= \lambda_T \pi_{i,j-1} & (i,j),(i,j-1) \in \mathcal{K} \\ (i+1)\mu_F \pi_{i+1,j} &= \lambda_F \pi_{i,j} & (i+1,j),(i,j) \in \mathcal{K} \\ (j+1)\mu_T \pi_{i,j+1} &= \lambda_T \pi_{i,j} & (i,j+1),(i,j) \in \mathcal{K} \\ \sum_{(i,j)\in\mathcal{K}} \pi_{i,j} &= 1. & (4) \end{aligned}$$

It can be easily verified that $\vec{\pi}$ is a solution to the system (4). Summing the first four equations in (4) gives (1) and therefore $\vec{\pi}$ satisfies (1 - 2). Uniqueness follows from the fact that (1 - 2) is an ergodic Markov chain and therefore has a unique steady state probability. $\square$

Many mathematical softwares compute the Poisson distribution ($Poisson(\lambda, k)$)very efficiently and therefore for computational purposes, it may be convenient to use the following relationship: $\frac{(\lambda)^k}{k!} = Poisson(\lambda, k)e^{\lambda}$ in (12). When computational considerations are critical and only lowerbound probabilities are needed then the following lemma may be useful.

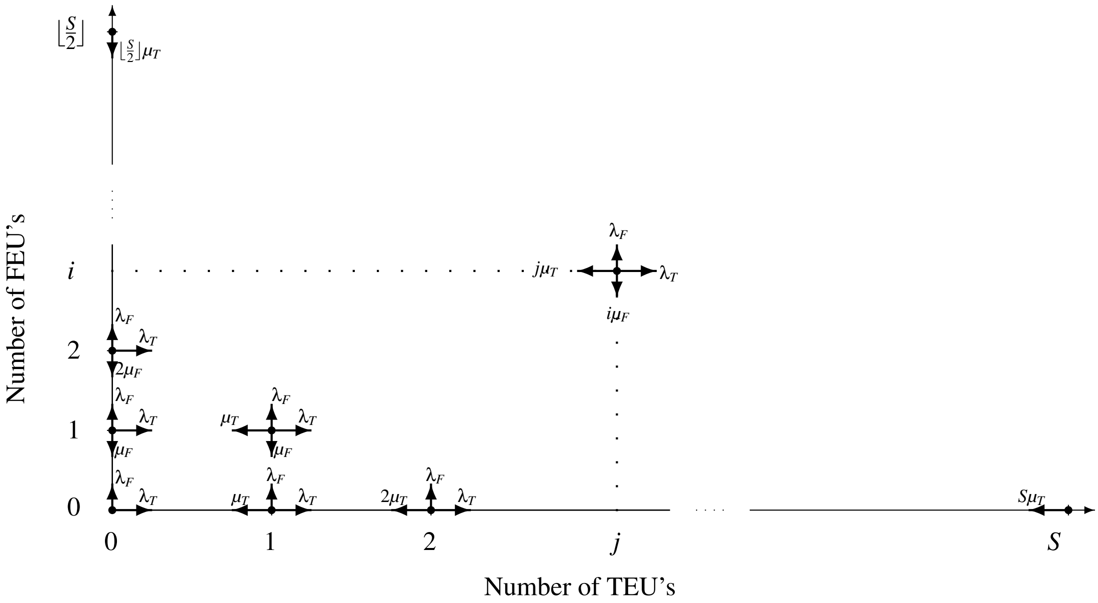


Figure 2: The yard's states' transition rates.

Lemma 1. *$e^{-(a_T+a_F)}$ is a lower bound of $\pi_{0,0}$.*

Proof: Since $\sum_{i=0}^{N} \frac{\lambda^k}{k!} < e^{\lambda}$ for any finite N, we have that

$$(\pi_{0,0})^{-1} = \sum_{i=0}^{\frac{S}{2}} \sum_{j=0}^{S-2i} \frac{a_F{}^i}{i!} \frac{a_T{}^j}{j!} < \sum_{i=0}^{\frac{S}{2}} \frac{a_F{}^i}{i!} e^{a_T} < e^{a_F} e^{a_T}$$

. Thus, $\pi_{0,0} > e^{-(a_T+a_F)}$. $\square$

The accuracy of the lower bound in Lemma 1 increases with S whereas the computational time to compute $\pi_{(0,0)}$ is $O(S^2)$. Thus, when S is very large, it is convenient to use the lower bound in lieu of the direct formula without any appreciable loss of accuracy.

Let L_F and L_T denote the average number of FEU's and TEU's in the yard, respectively.

$$\begin{aligned} L_F &= \pi_{0,0} \Big(\sum_{i=0}^{\frac{S}{2}} i \frac{a_F{}^i}{i!} \sum_{j=0}^{S-2i} \frac{a_T{}^j}{j!} \Big), \\ L_T &= \pi_{0,0} \Big(\sum_{i=0}^{\frac{S}{2}} \frac{a_F{}^i}{i!} \sum_{j=0}^{S-2i} j \frac{a_T{}^j}{j!} \Big). \end{aligned} \tag{5}$$

Let r_F and r_T denote the probability of an FEU and a TEU to be rejected, respectively. For the TEU's this happens when there is not a single empty slot and is the sum of the probabilities of the states $(0,S),(1,S-2),\ldots,(S/2-1,2),(S/2,0)$. Thus, $r_T = \sum_{i=0}^{S/2} \pi_{i,S-2i}$. For the FEU's, r_F equals r_T plus the probabilities of the states $(0,S-1),(1,S-3),\ldots,(S/2-2,3),(S/2-1,1)$. Thus, $r_F = r_T + \sum_{i=0}^{S/2-1} \pi_{i,S-2i-1}$. Therefore, the probabilities for the FEU's and TEU's to be accepted are:

$$\begin{aligned} 1-r_F &= \sum_{i=0}^{\frac{S}{2}} \sum_{j=0}^{S-2i} \pi_{i,j} - \sum_{i=0}^{\frac{S}{2}} \pi_{i,S-2i} - \sum_{i=0}^{\frac{S}{2}-1} \pi_{i,S-2i-1}, \\ 1-r_T &= \sum_{i=0}^{\frac{S}{2}} \sum_{j=0}^{S-2i} \pi_{i,j} - \sum_{i=0}^{\frac{S}{2}} \pi_{i,S-2i}. \end{aligned} \tag{6}$$

The following relationship between occupancy and rejection holds for both container types:

$$\begin{aligned} L_T &= a_T(1-r_T) \\ L_F &= a_F(1-r_F). \end{aligned} \tag{7}$$

3.2 The General Model

In many (if not most) situations customers arrive with more than just one container. Therefore, we extend the model to a yard that accepts multiple customer types. We let K denote the number of customer types where customer-type $k, k = 1,\ldots,K$ requires k storage units. The customer's use of the k slots is simultaneous, i.e., starts and ends at the same time. Let λ_k denote the arrival rate of customer-type k and let μ_k denote the expected storage time required by customer-type k. Similarly to the two customer types model, we assume that arrivals follow a Poisson process, and that storage times are exponentially distributed and that these times are independent.

A yard's state is denoted by the vector $(i_1,\ldots,i_k,\ldots,i_K)$ where i_k denotes the number of customers with demand for k storage units that are currently in the system. Here, too, we use $\mathcal{K}$ to denote the state space of the yard. A state $(i_1,\ldots,i_K) \in \mathcal{K}$ if and only if $i_k \geq 0$ for all k and $0 \leq \sum_{k=1}^{K} k \cdot i_k \leq S$. Therefore, it is convenient to define the state space, $\mathcal{K}$, can be defined iteratively as follows: $(i_1,\ldots,i_k,\ldots,i_K) \in \mathcal{K}$ if and only if

$$
\begin{aligned}
0 \leq \quad & i_K & \leq N_k(S) \equiv \frac{S}{K} \\
0 \leq \quad & i_{K-1} & \leq N_{K-1}(S) \equiv \frac{S-Ki_K}{K-1} \\
& \vdots & \\
\leq \quad & i_k & \leq N_k(S) \equiv \frac{S-\sum\limits_{j=k+1}^{K} j\cdot i_j}{k} \\
& \vdots & \\
0 \leq \quad & i_1 & \leq N_1(S) \equiv S-Ki_K-(K-1)i_{K-1}-\cdots-2i_2
\end{aligned}
\tag{8}
$$

In the above, $N_k(S)$ is the maximal number of customers of type k that the yard can accommodate given that the yard is serving $i_1,\ldots i_{k-1}$ customers of types $1,\ldots,k-1$, respectively.

Let $\pi_{i_1,\ldots,i_K}$ denote the steady state probability of state $(i_1,\ldots,i_K) \in \mathcal{K}$. For each state $(i_1,\ldots,i_k) \in \mathcal{K}$, the state's balance equation is given by

$$
\begin{aligned}
&\pi_{i_1,\ldots,i_K}\Big(\sum_{k=1}^{K}\lambda_l I_{(i_1,\ldots,i_k+1,\ldots,i_K)} \\
&\qquad + \sum_{k=1}^{K} i_l\mu_l I_{(i_1,\ldots,i_k-1,\ldots,i_K))}\Big) \\
&= \sum_{k=1}^{K}(i_k+1)\mu_{i_k+1}\pi_{i_1,\ldots,i_k+1,\ldots,i_K} I_{(i_1,\ldots,i_k+1,\ldots,i_K)} \\
&\qquad + \sum_{k=1}^{K}\lambda_k\pi_{i_1,\ldots,i_k-1,\ldots,i_K} I_{(i_1,\ldots,i_k-1,\ldots,i_K)},
\end{aligned}
\tag{9}
$$

and the normalizing condition is

$$
\sum_{(i_1,\ldots,i_K)\in\mathcal{K}} \pi_{i_1,\ldots,i_K} = 1. \tag{10}
$$

In (9), $I_{(i_1,\ldots,i_k)}$ is one if $(i_1,\ldots,i_k) \in \mathcal{K}$ and zero otherwise.

We show that the results of Section 3.1 extend to any number of customer types.

Proposition 2. *Let $a_k := \frac{\lambda_k}{\mu_k}$. The unique solution to (9-10) is*

$$
\begin{aligned}
\pi_{i_1,\ldots,i_k} &= \prod_{k=1}^{K}\frac{a_k^{i_k}}{i_k!}\pi_{0,\ldots,0} \quad (i_1,\ldots,i_k),\in\mathcal{K} \qquad (11)\\
\pi_{0,\ldots,0} &= \left(\sum_{(i_1,\ldots,i_k),\in\mathcal{K}}\prod_{k=1}^{K}\frac{a_k^{i_k}}{i_k!}\right)^{-1}
\end{aligned}
$$

Proof: The proof is similar to the proof of Proposition 1 and is described here briefly. Consider the following system of equations:

$$
\begin{aligned}
&\forall(i_1,\ldots,i_K)\in\mathcal{K} \\
&\qquad \pi_{i_1,\ldots,i_k-1,\ldots,i_K}\lambda_k I_{i_1,\ldots,i_k-1,\ldots,i_K} \\
&\qquad\quad = \quad i_k\mu_k\pi_{i_1,\ldots,i_k,\ldots,i_K} I_{i_1,\ldots,i_k-1,\ldots,i_K} \\
&\qquad\qquad\qquad k=1,\ldots,K.
\end{aligned}
\tag{12}
$$

$\vec{\pi}$ is a solution to this system and therefore solve (9), too. □

Next, we need to show that the relationship (7) holds here, too. Let r_k is the probability that a type-k customer is rejected and L_k is the expected number of type-k customers in the yard.

Proposition 3. $a_k(1-r_k) = L_k$ *for each* $k = 1,\ldots,K$.

Proof: By definition,

$$
\begin{aligned}
L_k &= \sum_{i_K=0}^{N_K(S)}\cdots\sum_{i_k=0}^{N_k(S)}\cdots\sum_{i_1=0}^{N_1(S)} i_k\cdot\pi_{i_1,\ldots,i_K} \\
&= \pi_{0,\ldots,0}\sum_{i_K=0}^{N_K(S)}\cdots\sum_{i_k=0}^{N_k(S)}\cdots\sum_{i_1=0}^{N_1(S)} i_k\prod_{j=1}^{K}\frac{a_j^{i_j}}{i_j!}
\end{aligned}
\tag{13}
$$

where, recall, $N_k(S)$ is a shorthand notation for:

$$
N_k(S) \equiv \frac{S-\sum\limits_{j=k+1}^{K} j\cdot i_j}{k},
$$

and, in particular,

$$
N_1(S) \equiv S-Ki_K-(K-1)i_{K-1}\cdots-2i_2. \tag{14}
$$

The expression $1-r_k$ denotes the probability that a customer type k is accepted to the yard and is the sum of the probabilities of all the states with up to and including $S-k$ spots in use. Thus, $1-r_k$ is given by

$$
\sum_{i_K=0}^{N_K(S-k)}\cdots\sum_{i_k=0}^{N_k(S-k)}\cdots\sum_{i_1=0}^{N_1(S-k)}\pi_{0,\ldots,0}\prod_{j=1}^{K}\frac{a_j^{i_j}}{i_j!}. \tag{15}
$$

In the above, we can rewrite the upper limit of the variables i_x, $x>k$ as $N_x(S)$ instead of $N_x(S-k)$ since it does not add any more probabilities to the summation. Thus,

$$a_k(1-r_k) = \pi_{0,\ldots,0} a_k \left[\sum_{i_K=0}^{N_K(S)} \cdots \sum_{i_{k+1}=0}^{N_{k+1}(S)} \sum_{i_k=0}^{N_k(S-k)} \sum_{i_{k-1}=0}^{N_{k-1}(S-k)} \cdots \sum_{i_1=0}^{N_1(S-k)} \prod_{j=1}^{K} \frac{a_j^{i_j}}{i_j!} \right]$$

$$= \pi_{0,\ldots,0} \left[\sum_{i_K=0}^{N_K(S)} \frac{a_K^{i_K}}{i_K!} \cdots \sum_{i_k=0}^{N_{k+1}(S)} \frac{a_{k+1}^{i_{k+1}}}{i_{k+1}!} \sum_{i_k=0}^{N_k(S-k)} \frac{a_k^{i_k+1}}{i_k!} \sum_{i_{k-1}=0}^{N_{k-1}(S-k)} \frac{a_{k+1}^{i_{k+1}}}{i_{k+1}!} \sum_{i_{k-1}=0}^{N_{k-1}(S-k)} \cdots \sum_{i_1=0}^{N_1(S-k)} \frac{a_1^{i_1}}{i_1!} \right]$$

By definition, $N_k(S-k) = N_k(S) - 1$ and therefore

$$a_k(1-r_k) = \pi_{0,\ldots,0} \left[\sum_{i_K=0}^{N_K(S)} \frac{a_K^{i_K}}{i_K!} \cdots \sum_{i_k=0}^{N_{k+1}(S)} \frac{a_{k+1}^{i_{k+1}}}{i_{k+1}!} \sum_{i_k=0}^{N_k(S)-1} \frac{a_k^{i_k+1}}{(i_k+1)!}(i_k+1) \sum_{i_1=0}^{\hat{N}_{k-1}(S)} \frac{a_{k-1}^{i_{k-1}}}{i_{k-1}!} \cdots \sum_{i_1=0}^{\hat{N}_1(S)} \frac{a_1^{i_1}}{i_1!} \right]$$

where for $x < k$

$$\hat{N}_x(S) \equiv \frac{S - k(i_k+1) - \sum_{j=x+1, j\neq k}^{K} j \cdot i_j}{x},$$

Shifting the i_k variable by one we obtain

$$a_k(1-r_k) = \pi_{0,\ldots,0} \left[\sum_{i_K=0}^{N_K(S)} \frac{a_K^{i_K}}{i_K!} \cdots \sum_{i_k=0}^{N_{k+1}(S)} \frac{a_{k+1}^{i_{k+1}}}{i_{k+1}!} \sum_{i_k=1}^{N_k(S)} \frac{a_k^{i_k}}{(i_k)!} i_k \sum_{i_1=0}^{N_{k-1}(S)} \frac{a_{k-1}^{i_{k-1}}}{i_{k-1}!} \cdots \sum_{i_1=0}^{N_1(S)} \frac{a_1^{i_1}}{i_1!} \right] \quad (16)$$

Changing the bottom limit of the variable i_k variable to zero does not change the value of (16). Thus, $a_k(1-r_k) = L_k$ given in (13), which establishes the relationship between the proportional and the one-time pricing schemes. □

3.3 Pricing Schemes

The yard provides two different service schemes, each with it own pricing fee.

- One-time fee: The yard owners are functioning as shipping brokers and use the yard for temporary storage of their customers' containers. In this case the customers are charged a one-time storage fee, since the customer has no control on how long the container happens to be stored in the yard. Therefore, the revenue to the yard from handling a container is independent of the actual storage time. We let p_F^O and p_T^O denote the storage fee per FEU and TEU, respectively.
- Proportional fee: The yard owners provide storage services to their customers who decide when to bring it in and when to take it out. Now, it is more reasonable to charge storage costs that are proportional to the storage duration. For this revenue scheme p_F^P and p_T^P denote the storage fee *per time unit* per FEU and TEU, respectively.

As for the yard's costs, these comprise two components. The first is the cost of holding S storage sites. We assume a per unit cost c_s. The second cost arises when the yard if forced to reject containers due to lack of available storage. In this case, they incur a reputation cost s_F and s_T for each rejected FEU and TEU, respectively. Accordingly, depending on revenue scheme, the yard's expected profit as a function of the number of slots it leases, $F(S)$, is given by

- One-time fee: $F_O(S) = (1-r_F)p_F^O\lambda_F + (1-r_T)p_T^O\lambda_T - r_F s_F \lambda_F - r_T s_T \lambda_T - c_s S$ if $S > 0$ and zero, otherwise.
- Proportional fee: $F_P(S) = L_F p_F^O + L_T p_T^O - r_F s_F \lambda_F - r_T s_T \lambda_T - c_s S$ if $S > 0$ and zero, otherwise.

While the two revenue schemes seem to be inherently different, in reality they behave in a similar manner. To obtain the one-time fee profit function from the proportional fee profit function one needs to scale the storage fees (p_F, p_T) by the departure rates (μ_F, μ_T) as described in the following proposition.

Proposition 4. *If $p_F^O = \frac{p_F^P}{\mu_F}$ and $p_T^O = \frac{p_T^P}{\mu_T}$ then the two pricing schemes are equivalent.*

Proof: Substituting p_F^O and p_T^O with $\frac{p_F^P}{\mu_F}$ and $\frac{p_T^P}{\mu_T}$, respectively, into the definition of $F_O(S)$ results in

$$F_O(S) = a_F(1-r_F)p_F^P + a_F(1-r_T)p_T^P\lambda_T - r_F s_F \lambda_F - r_T s_T \lambda_T - c_s S.$$

By Proposition 3, $a_k(1-r_k) = L_k$ for each k resulting in $F_O(S) = F_P(S)$. □

The importance of Proposition 4 is in that it implies a very simple relationship between the two pricing schemes. Consider a yard operator that charges a proportional fee of p_F and p_T for FEU's and TEU's, respectively. If the operator wants to expand services to storage according to a one-time fee, the comparable one-time storage charge should be set to p_F/μ and p_T/μ for FEU's and TEU's, respectively.

4 NUMERICAL EXAMPLE AND DISCUSSION

In this numerical example we illustrate the effect of yard size and container arrival on the profits of a small container yard. This follows the author's experience with a small privately-owned container yard outside a shipping port in Israel. We assume that FEU-related prices are double the TEU-related prices and set $p_F = 50, p_T = 25, s_F = 10, s_T = 5$ and set the cost of each storage location to be $c_S = 20$. We normalize the departure rates $\mu_F = \mu_T = 1$ to make the pricing schemes equivalent. Let $\lambda := \lambda_T + 2\lambda_F$. We will use λ to measure the total demand for storage spots.

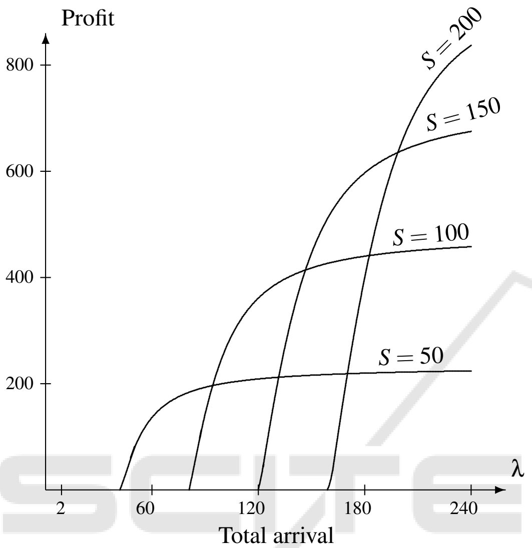


Figure 3: The profit as a function of the total arrival for different yard sizes.

We first demonstrate how the profit function changes with the total arrival λ where $\lambda_T = \lambda_F$ (and therefore $\lambda_T = 1/3\lambda$ and $\lambda_F = 2/3\lambda$). In Figure 3 we plot the profit function for different yard sizes. The price function is the fixed cost of the lease ($c_s S$) plus an income that is increasing and concave with the arrival rate. Therefore, while bigger yards provide considerably higher profits when the arrival rate is sufficiently large, they require a higher threshold arrival rate to provide be profitable in the first place.

In many cases the yard owner can expand or contract the yard size by leasing space from or to neighboring yards. In Figure 4 we plot the profit as a function of the yard size for different levels of arrival rates. The profit function is a sum of an increasing concave net income and a lease cost that is decreasing linearly with the lot size. Therefore, for a fixed arrival rate level, if the yard is very small then (assuming that the storage price is greater than the unit lease cost) increasing the lot size is profitable. However, for a sufficiently large lot, the rejection rate is sufficiently low that the decrease in container rejection due to an

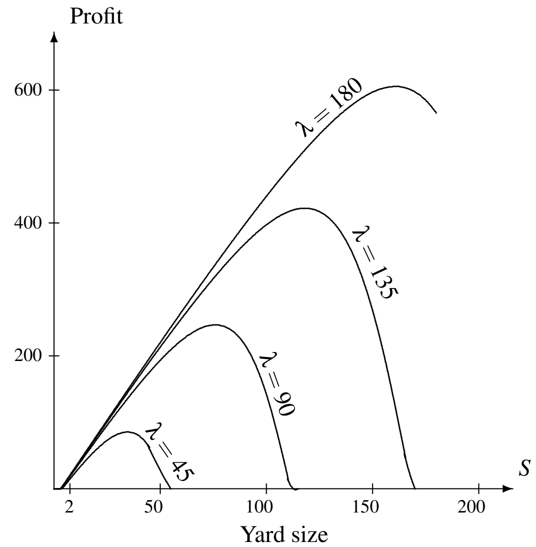


Figure 4: The profit as a function of the yard size for different total arrival levels.

additional spot does not offset its lease cost and therefore profits will decrease if the yard size is increased. In our example, when $\lambda = 45$ then the optimal lot size is very small, $S^* = 35$, whereas in a lot with $\lambda = 180$ then the optimal size is $S^* = 161$. In this analysis of the optimal yard size we do not take into account possible future growth in the customers' arrivals rate. See Giat (2018) for an analysis of such a situation.

5 CONCLUSION

This paper was motivated by a small storage yard that temporarily stores containers for a short period of time before they are transported to a neighboring marine port. The vast majority of containers that are stored in the yard are either the 20 feet TEU or its exactly double counterpart, the forty feet TEU. We describe the two storage services that the yard provides and show that if the proportional fee is equal to the one-time fee multiplied by the expected duration then the two schemes provide the same profits.

The numerical illustration shows the profit function's properties. When viewed as function of the yard size, it is a sum of a linearly increasing cost and an increasing concave revenue and therefore there is an optimal yard size. For the yard to be profitable, there is a minimal necessary arrival rate. This threshold is increasing with the yard size and therefore large yards are initially at a disadvantage. However, if yard owners are confident that there will be sufficiently many customers, then they the larger yards will secure them greater profits.

While the numerical analysis in this paper is limited to a two-type customer model, this paper also lays the foundation to analyzing more complex settings with multiple client types. Such an analysis can focus on questions about whether it is beneficial to preemptively reject smaller clients in order to secure space for larger (and perhaps more profitable) customers.

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