An Interest Rate Decision Method for Risk-averse Portfolio Optimization using Loan

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Abstract: Portfolio optimization using loan is formulated as a chance constrained problem in which the borrowing money from loan can be invested in risk assets. The chance constrained problem is proven to a convex optimization problem. The low interest rate of loan benefits borrowers. On the other hand, the high interest rate of loan doesn't benefits lenders because such a loan is not often used. For deciding a proper interest rate of loan that benefits both borrowers and lenders, a new method is proposed. Experimental results show that the loan is used completely to improve the efficient frontier if the interest rate is decided by the proposed method.

1 INTRODUCTION

Portfolio optimization is the process of determining the best proportion of investment in different assets according to some objective. The objective typically maximizes factors such as expected return, and minimizes costs like financial risk. Portfolio optimization is one of the most challenging problems in the field of finance. Therefore, a large number of works about portfolio optimization have been reported (Mokhtar et al., 2014; Mansini et al., 2014). In these works, portfolio optimization has been discussed both in a deterministic and in a stochastic domain, either in a single period or in a multi-period framework.

In our previous work (Tagawa, 2019), portfolio optimization using bank deposit and loan has been formulated as a chance constrained problem in which a non-risk asset called bank deposit is included in a portfolio and the borrowing money from loan can be invested in risk assets. It has been also proven that the chance constrained problem is a multimodal optimization problem having multiple optimal solutions. Therefore, for solving the optimization problem, an optimization method based on Differential Evolution (DE) (Price et al., 2005) has been proposed.

The effect of the loan on portfolio optimization has been also studied independently (Tagawa, 2020). Portfolio optimization using only loan has been formulated as a chance constrained problem. It has been also proven that the chance constrained problem is a convex optimization problem. Therefore, for solving the convex optimization problem, an interior point method (Horst and Pardalos, 1995) has been used. Experimental results show that the efficient frontier is improved if the loan is used. Consequently, the low interest rate of loan benefits borrowers. On the other hand, the high interest rate of loan does not benefits lenders because such a loan is not often used.

In this paper, portfolio optimization using loan is studied more intensively. Specifically, a proper interest rate of loan that benefits both borrowers and lenders is considered. Then, a new method to decide a proper interest rate of loan from an acceptable risk is proposed. Experimental results show that the loan is used completely to improve the efficient frontier if the interest rate is decided by the proposed method.

The remainder of this paper is organized as follows. Section 2 explains conventional models for portfolio optimization. Section 3 formulates a new portfolio optimization problem using loan. Section 4 proposes a new method to decide a proper interest rate of loan. Section 5 shows the results of numerical experiments and discusses about them. Finally, Section 6 concludes this paper and mentions future work.

2 RELATED WORK

2.1 Definition of Portfolio

Let $x_i \in \Re$, $i = 1, \cdots, n$ be the proportion of i-asset normalized by owned capital invested in n assets. A portfolio is defined as $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Re^n$. Since

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we consider a long-only portfolio in a single period, the portfolio $\boldsymbol{x} \in \Re^n$ is constrained as

$$x_1 + x_2 + \cdots + x_n = 1 \tag{1}$$

where $0 \leq x_i$, $i = 1, \cdots, n$.

The unit investment in the i-asset provides return $\xi_i \in \Re$ over a single period operation. Each of asset returns $\xi_i \in \Re$, $i = 1, \cdots, n$ is modeled by a random variable following a normal distribution as

$$\xi_i \sim \mathrm{Normal}(\mu_i, \sigma_i^2). \tag{2}$$

Incidentally, it is known that Normal distribution can represent a fairly accurate model of asset returns for portfolio optimization (Ruppert, 2011).

Let ρ_{ij} be the correlation coefficient between ξ_i and ξ_j, $i \neq j$. As well as the mean μ_i and the standard deviation σ_i in (2), ρ_{ij} is estimated statistically from historical data (Rubio et al., 2012). In recent years, an Artificial Intelligence (AI) method based on deep learning is also reported to predict the future returns of assets from market data (Obeidat et al., 2018).

The vector $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$ of random returns in (2) follows a multivariable normal distribution as

$$\boldsymbol{\xi} \sim \mathrm{Normal}(\boldsymbol{\mu}, \boldsymbol{C}) \tag{3}$$

where the mean is given as $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n) \in \Re^n$.

In order to derive the covariance matrix $\boldsymbol{C}$ in (3), a matrix $\boldsymbol{D}$ is defined by using σ_i in (2) as

$$\boldsymbol{D} = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix}. \tag{4}$$

From the correlation coefficient ρ_{ij} between ξ_i and ξ_j, a coefficient matrix $\boldsymbol{R}$ is also defined as

$$\boldsymbol{R} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{pmatrix}. \tag{5}$$

From $\boldsymbol{D}$ in (4) and $\boldsymbol{R}$ in (5), $\boldsymbol{C}$ is obtained as

$$\boldsymbol{C} = \boldsymbol{D}\boldsymbol{R}\boldsymbol{D}. \tag{6}$$

The return of a portfolio $\boldsymbol{x} \in \Re^n$ is defined as

$$r(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=1}^{n} \xi_i x_i = \boldsymbol{\xi}\boldsymbol{x}^T. \tag{7}$$

According to the reproductive property of normal distribution (Ash, 2008), the return in (7) also follows a normal distribution as

$$r(\boldsymbol{x}, \boldsymbol{\xi}) \sim \mathrm{Normal}(\mu_r(\boldsymbol{x}), \sigma^2(\boldsymbol{x})) \tag{8}$$

where the mean and the variance are given as

$$\mu_r(\boldsymbol{x}) = \sum_{i=1}^{n} \mu_i x_i = \boldsymbol{\mu}\boldsymbol{x}^T \tag{9}$$

$$\sigma^2(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{C}\boldsymbol{x}^T. \tag{10}$$

2.2 Portfolio Optimization

By using the portfolio stated above, we explain basic models used to formulate portfolio optimization.

2.2.1 Markowitz's Model

In Markowitz's model (Markowitz, 1952), the risk of a portfolio $\boldsymbol{x} \in \Re^n$ is evaluated by the variance $\sigma^2(\boldsymbol{x})$ shown in (10). Then, the risk is minimized keeping an expected return $\mu_r(\boldsymbol{x})$ larger than $\gamma \in \Re$ as

$$\left[\begin{array}{ll} \text{min} & \sigma^2(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{C}\boldsymbol{x}^T \\ \text{sub. to} & \mu_r(\boldsymbol{x}) = \boldsymbol{\mu}\boldsymbol{x}^T \geq \gamma, \\ & x_1 + x_2 + \cdots + x_n = 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \tag{11}$$

2.2.2 Roy's Model

In Roy's model (Roy, 1952), the risk of portfolio is evaluated by the probability that the return $r(\boldsymbol{x}, \boldsymbol{\xi})$ in (7) falls bellow a desired value $\gamma \in \Re$. In order to minimize the risk $\alpha \in (0, 1)$, portfolio optimization is formulated as a chance constrained problem:

$$\left[\begin{array}{ll} \text{min} & \alpha \\ \text{sub. to} & \Pr(r(\boldsymbol{x}, \boldsymbol{\xi}) \leq \gamma) \leq \alpha, \\ & x_1 + x_2 + \cdots + x_n = 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \tag{12}$$

where $\Pr(\mathcal{A})$ is the probability that event $\mathcal{A}$ occurs.

2.2.3 Kataoka's Model

Contrary to Roy's model in (12), Kataoka's model (Kataoka, 1963) maximizes the desired value $\gamma \in \Re$ of the return $r(\boldsymbol{x}, \boldsymbol{\xi})$ for an acceptable risk $\alpha \in (0, 0.5)$ given by a probability. Then, portfolio optimization is also formulated as a chance constrained problem:

$$\left[\begin{array}{ll} \text{max} & \gamma \\ \text{sub. to} & \Pr(r(\boldsymbol{x}, \boldsymbol{\xi}) \leq \gamma) \leq \alpha, \\ & x_1 + x_2 + \cdots + x_n = 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \tag{13}$$

2.3 Extended Models

There is a trade-off relationship between return and risk. Therefore, by using a risk aversion indicator $\lambda \in [0, 1]$, Efficient Frontier model (Chang et al., 2000) modifies Markowitz's model defined in (11) as

$$\left[\begin{array}{ll} \text{min} & \lambda\sigma^2(\boldsymbol{x}) - (1-\lambda)\mu_r(\boldsymbol{x}) \\ \text{sub. to} & x_1 + x_2 + \cdots + x_n = 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \tag{14}$$

By changing the value of $\lambda \in [0, 1]$ in (14), we can obtain the efficient frontier. The efficient frontier is a continuous curve illustrating the trade-off between the expected return (mean) and the risk (variance).

Genetic Algorithm (GA), Tabu Search (TS), and Simulated Annealing (SA) have been applied to a portfolio optimization problem based on the efficient frontier model (Chang et al., 2000). Artificial Bee Colony (ABC) algorithm has been also proposed for solving a portfolio optimization problem based on the efficient frontier model (Strumberger et al., 2018).

Portfolio optimization can be also formulated as a multi-objective optimization problem as

$$\left[\begin{array}{ll} \min & \sigma^2(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{C}\boldsymbol{x}^T \\ \max & \mu_r(\boldsymbol{x}) = \boldsymbol{\mu}\boldsymbol{x}^T \\ \text{sub. to} & x_1 + x_2 + \cdots + x_n = 1, \\ & 0 \le x_i,\ i = 1, \cdots, n. \end{array}\right. \tag{15}$$

In order to obtain the efficient frontier for multi-objective portfolio optimization problems, several Multi-Objective Evolutionary Algorithms (MOEAs) have been used successfully (Anagnostopoulos and Mamanis, 2010; Ponsich et al., 2013).

Cardinality constraints restrict a portfolio to have a specified number of assets. Specifically, several assets to be invested in have to be selected from a list of many assets. Thus, portfolio optimization including cardinality constraints is usually formulated as a mixed integer problem (Konno and Yamamoto, 2005). Furthermore, the number of assets has been minimized by a multi-objective portfolio optimization problem (Anagnostopoulos and Mamanis, 2010).

Portfolio optimization is often formulated based on multiple periods. In order to evaluate the total return over multiple periods, a risk function called Mean Absolute Deviation (MAD) has been proposed (Konno and Yamazaki, 1995). Conditional Value-at-Risk (CVaR) has been also used to formulate portfolio optimization considering the return expected through multiple periods (Angelelli et al., 2008).

In the multi-period framework, transaction costs have to be paid for any assets. Therefore, a portfolio optimization problem considering the costs for selling and buying assets to change the structure of portfolio between periods has been formulated and solved by using an extended GA (Aranha and Iba, 2007).

Currently, portfolio optimization is extended in various ways. For example, Goal Programming (GP) models have been proposed to compose a portfolio of international mutual funds (Tamiz et al., 2013). A deep learning network has been used to predict the composite index of stock market (Pang et al., 2018). The latest technology of AI has been also introduced into portfolio optimization (Obeidat et al., 2018).

3 PROBLEM FORMULATION

Portfolio Optimization Problem using Loan (POPL) is an extended version of Kataoka's Model in (13). The loan can be introduced into any models shown in (11) to (13). Actually, Markowitz's Model in (11) is the most popular one. However, by using Kataoka's Model, we can decide the interest rate of loan from an acceptable risk $\alpha \in (0, 0.5)$ given in advance.

3.1 Portfolio Including Loan

The borrowing money from loan is invested in risk assets. Let $x_0 \in \Re$ be the proportion of loan used for a portfolio $\boldsymbol{x} \in \Re^n$. Let $M \in \Re$, $M > 0$ be the upper limit of the loan, which is specified by a multiple of owned capital. If the loan is not used, the proportion of loan is $x_0 = 0$. On the other hand, if the loan is used up to the limit, the proportion of loan is $x_0 = -M$. Therefore, the constraints of POPL are

$$\left\{\begin{array}{l} x_0 + x_1 + x_2 + \cdots + x_n = 1, \\ -M \le x_0 \le 0,\ 0 \le x_i,\ i = 1, \cdots, n. \end{array}\right. \tag{16}$$

From the first constraint in (16), the proportion of loan $x_0 \in \Re$ used for a portfolio $\boldsymbol{x} \in \Re^n$ is

$$x_0 = 1 - \mathbb{1}\boldsymbol{x}^T \tag{17}$$

where $\mathbb{1} \in \Re^n$ is a vector defined as $\mathbb{1} = (1, \cdots, 1)$.

Let $L \in \Re$ be the interest rate of loan. The interest rate $L \in \Re$, $L \ge 0$ is a constant value. Considering the proportion of loan $x_0 \le 0$ and $L \ge 0$, the return $r(\boldsymbol{x}, \boldsymbol{\xi})$ of a portfolio $\boldsymbol{x} \in \Re^n$ defined in (7) is revised as

$$\begin{array}{rcl} g(\boldsymbol{x}, \boldsymbol{\xi}) & = & r(\boldsymbol{x}, \boldsymbol{\xi}) + L x_0 \\ & = & \boldsymbol{\xi}\boldsymbol{x}^T + L(1 - \mathbb{1}\boldsymbol{x}^T) \\ & = & (\boldsymbol{\xi} - L\,\mathbb{1})\boldsymbol{x}^T + L. \end{array} \tag{18}$$

According to the reproductive property of normal distribution (Ash, 2008), the return of POPL in (18) also follows a normal distribution as

$$g(\boldsymbol{x}, \boldsymbol{\xi}) \sim \mathrm{Normal}(\mu_g(\boldsymbol{x}), \sigma^2(\boldsymbol{x})) \tag{19}$$

where the mean is given as

$$\mu_g(\boldsymbol{x}) = (\boldsymbol{\mu} - L\,\mathbb{1})\boldsymbol{x}^T + L. \tag{20}$$

The variance $\sigma^2(\boldsymbol{x})$ in (19) is given by (10).

3.2 Portfolio Optimization using Loan

As stated above, POPL is formulated as an extended version of Kataoka's Model in (13). An acceptable risk $\alpha \in (0, 0.5)$ is given in advance. Therefore, from

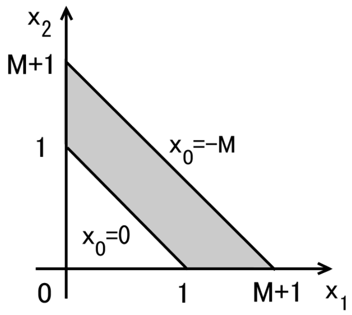

Figure 1: Feasible region of POPL.

(16) and (18), POPL is also formulated as a chance constrained problem:

$$\left[\begin{array}{ll} \max & \gamma \\ \text{sub. to} & \Pr(g(\boldsymbol{x}, \boldsymbol{\xi}) \leq \gamma) \leq \alpha, \\ & x_1 + x_2 + \cdots + x_n \leq M+1, \\ & x_1 + x_2 + \cdots + x_n \geq 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n \end{array}\right. \tag{21}$$

where the proportion of loan $x_0 \in [-M, 0]$ does not appear in (21) because it has been eliminated from the constraints in (16) by using the equation in (17).

The chance constrained problem is usually hard to solve directly (Prékopa, 1995). However, we can transform the above POPL in (21) into an equivalence problem. Since the return $g(\boldsymbol{x}, \boldsymbol{\xi})$ of POPL follows the normal distribution in (19), we can standardize the chance constraint of POPL in (21) as

$$\Pr\left(\frac{g(\boldsymbol{x}, \boldsymbol{\xi}) - \mu_g(\boldsymbol{x})}{\sigma(\boldsymbol{x})} \leq \frac{\gamma - \mu_g(\boldsymbol{x})}{\sigma(\boldsymbol{x})}\right) \leq \alpha. \tag{22}$$

Furthermore, the probability in (22) is written as

$$\Phi\left(\frac{\gamma - \mu_g(\boldsymbol{x})}{\sigma(\boldsymbol{x})}\right) \leq \alpha \tag{23}$$

where $\Phi : \Re \to [0, 1]$ is the Cumulative Distribution Function (CDF) of the standard normal distribution.

From (23), we can derive the equivalence problem of the chance constrained problem in (21) as

$$\left[\begin{array}{ll} \max & \gamma(\boldsymbol{x}) = \mu_g(\boldsymbol{x}) + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x}) \\ \text{sub. to} & x_1 + x_2 + \cdots + x_n \leq M+1, \\ & x_1 + x_2 + \cdots + x_n \geq 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \tag{24}$$

The equivalence problem in (24) is a deterministic one. Thus, we don't need to evaluate the probability that appears in (21). The deterministic optimization problem in (24) is also called POPL in this paper.

Figure 1 illustrates the feasible region of POPL for the case of $n = 2$. The feasible region is denoted by the gray area between two hyper-planes. If a portfolio $\boldsymbol{x} \in \Re^n$ doesn't use the loan ($x_0 = 0$), it exists on the lower plane: $\mathbb{1}\boldsymbol{x}^T = 1$. On the other hand, if a portfolio $\boldsymbol{x} \in \Re^n$ uses the loan up to the limit ($x_0 = -M$), it exists on the upper plane: $\mathbb{1}\boldsymbol{x}^T = M + 1$.

3.3 Solution of Problem

We consider the solution of POPL in (24).

Lemma 1. *The standard deviation $\sigma(\boldsymbol{x})$ defined by (10) is convex (Tagawa, 2019).*

Proof. Since the covariance matrix $\boldsymbol{C}$ in (6) is positive semi-definite, it can be decomposed as

$$\sigma(\boldsymbol{x}) = \sqrt{\boldsymbol{x}\boldsymbol{C}\boldsymbol{x}^T} = \sqrt{\boldsymbol{x}\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{x}^T} = \sqrt{\boldsymbol{y}\boldsymbol{y}^T} \tag{25}$$

where $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{A}^T$ and $\boldsymbol{y} = \boldsymbol{x}\boldsymbol{A} \in \Re^n$.

From (25), $\sigma(\boldsymbol{x})$ is a norm. The norm meets the triangle inequality for any $\theta \in [0, 1]$ as

$$\sigma(\theta\boldsymbol{x} + (1-\theta)\hat{\boldsymbol{x}}) \leq \sigma(\theta\boldsymbol{x}) + \sigma((1-\theta)\hat{\boldsymbol{x}}). \tag{26}$$

The right side of (26) can be transformed as

$$\sigma(\theta\boldsymbol{x}) = \sqrt{\theta\boldsymbol{y}\,(\theta\boldsymbol{y})^T} = \theta\sqrt{\boldsymbol{y}\boldsymbol{y}^T} = \theta\sigma(\boldsymbol{x}). \tag{27}$$

From (26) and (27), we have

$$\sigma(\theta\boldsymbol{x} + (1-\theta)\hat{\boldsymbol{x}}) \leq \theta\sigma(\boldsymbol{x}) + (1-\theta)\,\sigma(\hat{\boldsymbol{x}}). \tag{28}$$

From (28), $\sigma(\boldsymbol{x})$ in (10) is a convex function. □

Theorem 1. *The objective function $\gamma(\boldsymbol{x})$ of POPL in (24) is concave. In other words, $-\gamma(\boldsymbol{x})$ is convex.*

Proof. From (20) and $\gamma(\boldsymbol{x})$ in (24), we have

$$\begin{aligned} &\theta\gamma(\boldsymbol{x}) + (1-\theta)\,\gamma(\hat{\boldsymbol{x}}) - \gamma(\theta\boldsymbol{x} + (1-\theta)\,\hat{\boldsymbol{x}}) \\ &= \Phi^{-1}(\alpha) \times \\ &(\theta\sigma(\boldsymbol{x}) + (1-\theta)\,\sigma(\hat{\boldsymbol{x}}) - \sigma(\theta\boldsymbol{x} + (1-\theta)\,\hat{\boldsymbol{x}})). \end{aligned} \tag{29}$$

From Lemma 1 and $\Phi^{-1}(\alpha) < 0$ for $\alpha \in (0, 0.5)$, the right side of (29) is negative. Hence, we have

$$\gamma(\theta\boldsymbol{x} + (1-\theta)\,\hat{\boldsymbol{x}}) \geq \theta\gamma(\boldsymbol{x}) + (1-\theta)\,\gamma(\hat{\boldsymbol{x}}). \tag{30}$$

From (30), $\gamma(\boldsymbol{x})$ in (24) is a concave function. □

Since all constraints of POPL in (24) are linear, the feasible region of POPL is convex. Furthermore, from Theorem 1, POPL is a convex optimization problem. If $\boldsymbol{x}^\star \in \Re^n$ is a local optimal solution of a convex optimization problem, the solution $\boldsymbol{x}^\star \in \Re^n$ is guaranteed to be a global optimal one of the convex optimization problem (McCormick, 1983).

The gradient of $\gamma(\boldsymbol{x})$ in (24) can be derived as

$$\nabla\gamma(\boldsymbol{x}) = (\boldsymbol{\mu} - L\,\mathbb{1}) + \Phi^{-1}(\alpha)\frac{\boldsymbol{x}\boldsymbol{C}}{\sqrt{\boldsymbol{x}\boldsymbol{C}\boldsymbol{x}^T}}. \tag{31}$$

The global optimal solution $\boldsymbol{x}^\star \in \Re^n$ of POPL in (24) satisfies either of the following two conditions.

- $\nabla\gamma(\boldsymbol{x}^\star) = \boldsymbol{0}$ holds.
- Some constraints in (24) are active with $\boldsymbol{x}^\star \in \Re^n$.

4 INTEREST RATE OF LOAN

4.1 Proper Interest Rate

We think about a proper interest rate of loan $L \in \Re$ that benefits borrowers to get much return.

Let $\boldsymbol{x} \in \Re^n$ be a portfolio of POPL in which the loan is not used as $x_0 = 0$. Therefore, $\mathbb{1}\boldsymbol{x}^T = 1$ holds. For $\boldsymbol{x} \in \Re^n$, the objective function in (24) is

$$\begin{aligned}\gamma(\boldsymbol{x}) &= \mu_g(\boldsymbol{x}) + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x})\\ &= (\boldsymbol{\mu} - L\,\mathbb{1})\boldsymbol{x}^T + L + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x})\\ &= \boldsymbol{\mu}\boldsymbol{x}^T + L(1 - \mathbb{1}\boldsymbol{x}^T) + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x})\\ &= \boldsymbol{\mu}\boldsymbol{x}^T + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x})\\ &= \mu_r(\boldsymbol{x}) + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x}) = \gamma_0(\boldsymbol{x}).\end{aligned} \tag{32}$$

Theorem 2. *Let $\boldsymbol{x} \in \Re^n$ be a solution of POPL in which the loan is not used as $x_0 = 0$. The solution can be improved by borrowing money from the loan if the interest rate of loan $L \in \Re$ meets the condition:*

$$\gamma_0(\boldsymbol{x}) > L \tag{33}$$

where $\gamma_0(\boldsymbol{x}) = \mu_r(\boldsymbol{x}) + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x})$.

Proof. Let's consider a new portfolio $\hat{\boldsymbol{x}} = \kappa\boldsymbol{x}$, $\kappa > 1$. The new portfolio $\hat{\boldsymbol{x}} \in \Re^n$ borrows money as

$$\hat{x}_0 = 1 - \mathbb{1}\hat{\boldsymbol{x}}^T = 1 - \kappa\,\mathbb{1}\boldsymbol{x}^T = 1 - \kappa < 0 \tag{34}$$

where $\hat{x}_0 \in \Re$ is the proportion of loan for $\hat{\boldsymbol{x}} \in \Re^n$.

The objective function value of $\hat{\boldsymbol{x}} \in \Re^n$ is

$$\begin{aligned}\gamma(\hat{\boldsymbol{x}}) &= (\boldsymbol{\mu} - L\,\mathbb{1})\hat{\boldsymbol{x}}^T + L + \Phi^{-1}(\alpha)\,\sigma(\hat{\boldsymbol{x}})\\ &= \kappa(\boldsymbol{\mu} - L\,\mathbb{1})\boldsymbol{x}^T + L + \kappa\Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x})\\ &= \kappa\gamma_0(\boldsymbol{x}) + L(1 - \kappa\,\mathbb{1}\boldsymbol{x}^T).\end{aligned} \tag{35}$$

From (35) and $\mathbb{1}\boldsymbol{x}^T = 1$, the difference between the returns of $\hat{\boldsymbol{x}} \in \Re^n$ and $\boldsymbol{x} \in \Re^n$ is

$$\left(\begin{aligned} & \gamma(\hat{\boldsymbol{x}}) - \gamma_0(\boldsymbol{x})\\ =\ & (\kappa - 1)\,\gamma_0(\boldsymbol{x}) + L(1 - \kappa\,\mathbb{1}\boldsymbol{x}^T)\\ =\ & (\kappa - 1)\,(\gamma_0(\boldsymbol{x}) - L).\end{aligned}\right. \tag{36}$$

If the condition in (33) is satisfied, we have

$$\gamma(\hat{\boldsymbol{x}}) > \gamma_0(\boldsymbol{x}). \tag{37}$$

Therefore, $\hat{\boldsymbol{x}} \in \Re^n$ is better than $\boldsymbol{x} \in \Re^n$. □

Theorem 3. *Let $\boldsymbol{x} \in \Re^n$ be a solution of POPL that uses the loan. The portfolio $\boldsymbol{x} \in \Re^n$ borrows money from the loan up to the limit such as $x_0 = -M$ if*

$$\gamma(\boldsymbol{x}) > L. \tag{38}$$

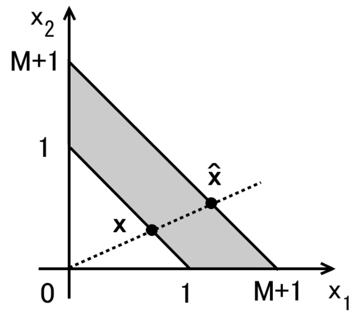

Figure 2: Portfolio $\hat{\boldsymbol{x}} \in \Re^2$ is better than $\boldsymbol{x} \in \Re^2$.

Proof. Let's consider a new portfolio $\hat{\boldsymbol{x}} = \kappa\boldsymbol{x}$, $\kappa > 1$. The new portfolio $\hat{\boldsymbol{x}} \in \Re^n$ borrows much money than the current one $\boldsymbol{x} \in \Re^n$ as

$$\begin{aligned}\hat{x}_0 &= 1 - \mathbb{1}\hat{\boldsymbol{x}}^T = 1 - \kappa\,\mathbb{1}\boldsymbol{x}^T\\ &< 1 - \mathbb{1}\boldsymbol{x}^T = x_0 \le 0\end{aligned} \tag{39}$$

where $\hat{x}_0 \in \Re$ is the proportion of loan for the new portfolio $\hat{\boldsymbol{x}} \in \Re^n$, while $x_0 \in \Re$ is the proportion of loan for the current portfolio $\boldsymbol{x} \in \Re^n$.

The objective function value of $\boldsymbol{x} \in \Re^n$ is

$$\begin{aligned}\gamma(\boldsymbol{x}) &= \mu_g(\boldsymbol{x}) + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x})\\ &= (\boldsymbol{\mu} - L\,\mathbb{1})\boldsymbol{x}^T + L + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x}).\end{aligned} \tag{40}$$

The objective function value of $\hat{\boldsymbol{x}} \in \Re^n$ is

$$\begin{aligned}\gamma(\hat{\boldsymbol{x}}) &= \mu_g(\hat{\boldsymbol{x}}) + \Phi^{-1}(\alpha)\,\sigma(\hat{\boldsymbol{x}})\\ &= \kappa(\boldsymbol{\mu} - L\,\mathbb{1})\boldsymbol{x}^T + L + \kappa\Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x}).\end{aligned} \tag{41}$$

From (40) and (41), the gap between them is

$$\gamma(\hat{\boldsymbol{x}}) - \gamma(\boldsymbol{x}) = (\kappa - 1)\,(\gamma(\boldsymbol{x}) - L). \tag{42}$$

From (42) and $\kappa > 1$, if the condition in (38) is satisfied, $\hat{\boldsymbol{x}} \in \Re^n$ is better than $\boldsymbol{x} \in \Re^n$ as

$$\gamma(\hat{\boldsymbol{x}}) > \gamma(\boldsymbol{x}). \tag{43}$$

Consequently, every portfolio $\boldsymbol{x} \in \Re^n$ of POPL that satisfies the condition in (38) can be improved proportionally to the amount of debt. □

Please notice that if the condition shown in (33) is satisfied by an interest rate $L \in \Re$ and a portfolio $\boldsymbol{x} \in \Re^n$ ($x_0 = 0$), the condition in (38) is also satisfied by the interest rate $L \in \Re$ and a new portfolio $\hat{\boldsymbol{x}} \in \Re^n$ ($\hat{x}_0 < 0$) generated as $\hat{\boldsymbol{x}} = \kappa\boldsymbol{x}$, $\kappa > 0$. That is because the relation $\gamma(\hat{\boldsymbol{x}}) > \gamma_0(\boldsymbol{x}) > L$ holds. Besides, from Theorem 3, the portfolio $\hat{\boldsymbol{x}} \in \Re^n$ exists on the upper plane such as $\mathbb{1}\hat{\boldsymbol{x}}^T = M + 1$. Figure 2 illustrates the above $\boldsymbol{x} \in \Re^n$ and $\hat{\boldsymbol{x}} \in \Re^n$ for the case of $n = 2$.

4.2 Interest Rate Decision Method

The low interest rate of loan benefits borrowers. On the other hand, the high interest rate doesn't benefit lenders because such a loan is not often used. Hence, we propose a method to decide an interest rate of loan that benefits both borrowers and lenders.

We formulate a sub-problem of POPL in the case that the loan is not used. From (32) and $\mathbb{1}\boldsymbol{x}^T = 1$, the sub-problem of POPL can be formulated as

$$\left[\begin{array}{ll} \max & \gamma_0(\boldsymbol{x}) = \mu_r(\boldsymbol{x}) + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x}) \\ \text{sub. to} & x_1 + x_2 + \cdots + x_n = 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \tag{44}$$

In the same way with Theorem 1, we can prove that the sub-problem of POPL in (44) is also a convex optimization problem. Furthermore, the gradient of $\gamma_0(\boldsymbol{x})$ in (44) can be derived as

$$\nabla\gamma_0(\boldsymbol{x}) = \boldsymbol{\mu} + \Phi^{-1}(\alpha)\frac{\boldsymbol{x}\boldsymbol{C}}{\sqrt{\boldsymbol{x}\boldsymbol{C}\boldsymbol{x}^T}}. \tag{45}$$

From Theorem 2, the procedure of the proposed interest rate decision method is stated as follows:

Step 1: Give an acceptable risk $\alpha \in (0, 0, 5)$.

Step 2: By solving the sub-problem of POPL in (44) with the above risk α, get a solution $\boldsymbol{x}^\star \in \Re^n$.

Step 3: Choose a proper value for the interest rate of loan $L \in \Re$ in the range from 0 to $\gamma_0(\boldsymbol{x}^\star)$.

If $\gamma_0(\boldsymbol{x}^\star) \leq 0$ holds in Step 2, we should give up the investment in assets. That is because POPL doesn't have any solutions that generate profits. Otherwise, we need to increase the acceptable risk $\alpha \in (0, 0.5)$ in Step 1. Then, we look for another $L \in \Re$ again.

From Theorem 3, if the interest rate of loan $L \in \Re$ is decided by the proposed method, POPL in (24) can be written by a rather simple form as

$$\left[\begin{array}{ll} \max & \gamma(\boldsymbol{x}) = \mu_g(\boldsymbol{x}) + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x}) \\ \text{sub. to} & x_1 + x_2 + \cdots + x_n = M + 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \tag{46}$$

Please notice that the portfolio $\hat{\boldsymbol{x}} = \kappa\boldsymbol{x}^\star$ generated by the optimal solution $\boldsymbol{x}^\star \in \Re^n$ of the sub-problem of POPL in (44) is not guaranteed to be an optimal solution of POPL in (46). Therefore, we have to solve POPL in (46) seriously under the interest rate of loan $L \in \Re$ decided by using the proposed method.

5 NUMERICAL EXPERIMENT

For solving convex optimization problems shown in (24), (44), and (46), an interior point method provided by MATLAB (López, 2014) is employed. In order to enhance the performance of the interior point method, the gradients of objective functions, namely $\nabla\gamma(\boldsymbol{x})$ in (31) and $\nabla\gamma_0(\boldsymbol{x})$ in (45), are used explicitly. As stated above, the optimality of the solutions $\boldsymbol{x} \in \Re^n$ obtained the interior point method have been also verified.

Table 1: Mean and variance of asset return by port0.

ξ_i	ξ_1	ξ_2	ξ_3	ξ_4
μ_i	0.05	0.06	0.07	0.08
σ_i^2	0.10^2	0.20^2	0.15^2	0.25^2

Table 2: Correlation coefficient by port0.

ρ_{ij}	ξ_1	ξ_2	ξ_3	ξ_4
ξ_1	1.0	−0.7	0.1	−0.4
ξ_2	−0.7	1.0	−0.5	0.2
ξ_3	0.1	−0.5	0.1	−0.3
ξ_4	−0.4	0.2	−0.3	1.0

Table 3: Index of data set and number of assets.

Data set	Index	n
port1	Hang Seng	31
port2	DAX	85

5.1 Problem Instances

Instances of POPL are defined by using three data sets of assets, which are named port0, port1, and port2.

The data set called port0 is given by Table 1 and Table 2. The port0 consists of $n = 4$ assets. Table 1 shows the mean μ_i and variance σ_i^2 of asset returns $\xi_i \in \Re$, $n = 1, \cdots, n$. Table 2 shows the correlation coefficient ρ_{ij} between asset returns ξ_i and ξ_j.

The data sets called port1 and port2 are provided by OR-Library (Beasley, 1990). The data set contains means, variances, and a coefficient matrix of n asset returns. Table 3 shows the capital market indices of those data sets and the numbers of their assets.

5.2 Fixed Interest Rate

From each of the data sets, POPL is formulated as shown in (24). By changing the value of the risk $\alpha \in (0, 0.5)$, POPL in (24) is solved repeatedly. A constant value is used for the interest rate of loan $L \in \Re$ regardless of the value of $\alpha \in (0, 0.5)$.

Figure 3 shows the efficient frontier evaluated for POPL of port0, where the upper limit of loan is given as $M = 2$. Three different interest rates, $L = 0.03$, 0.04, and 0.05, are compared in Figure 3. "None" is the efficient frontier when the loan is not used.

Figure 4 shows the proportion of loan $x_0 \leq 0$ for each portfolio shown in Figure 3. Since "None"

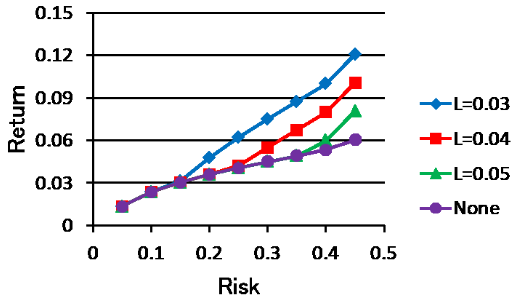

Figure 3: Efficient frontier for port0 with $M = 2$.

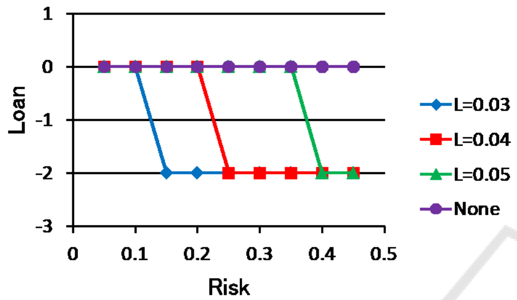

Figure 4: Proportion of loan x_0 for port0 with $M = 2$.

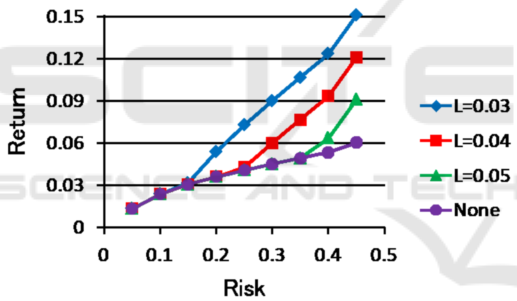

Figure 5: Efficient frontier for port0 with $M = 3$.

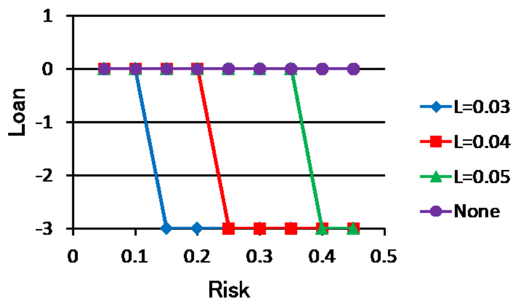

Figure 6: Proportion of loan x_0 for port0 with $M = 3$.

doesn't use the loan, it keeps $x_0 = 0$ in Figure 4.

Figure 5 shows the efficient frontier evaluated for port0 with $M = 3$. Figure 6 shows the proportion of loan $x_0 \in [-M, 0]$ for each portfolio in Figure 5.

From Figure 3 and Figure 5, we can confirm that the efficient frontier is further improved by borrowing

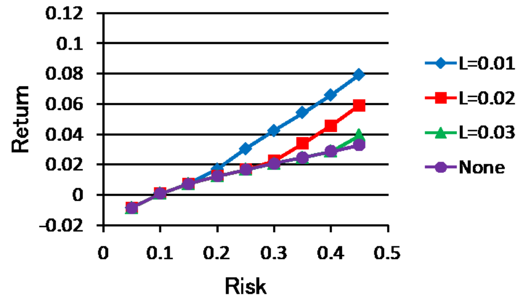

Figure 7: Efficient frontier for port1 with $M = 2$.

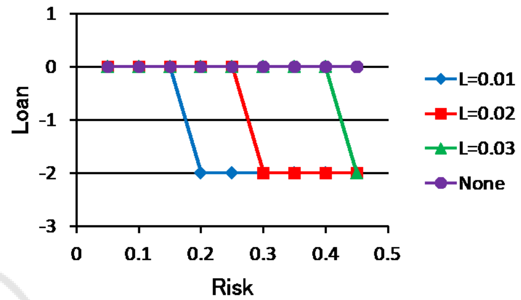

Figure 8: Proportion of loan x_0 for port1 with $M = 2$.

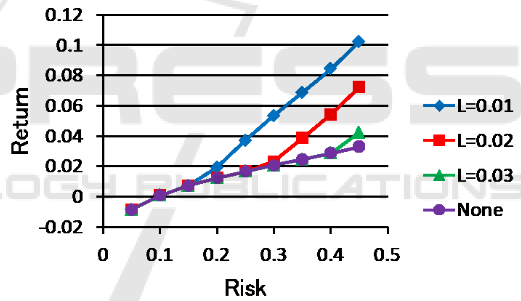

Figure 9: Efficient frontier for port1 with $M = 3$.

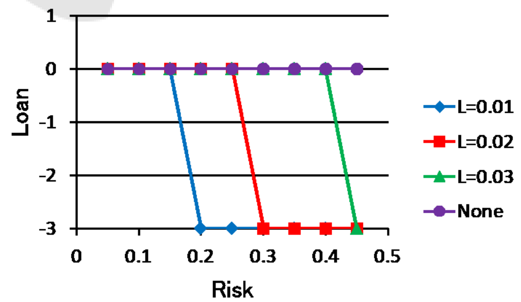

Figure 10: Proportion of loan x_0 for port1 with $M = 3$.

much more money. Furthermore, from Figure 4 and Figure 6, we can see that the loan is always used up to the limit ($x_0 = -M$) regardless of its value M.

Figure 7 shows the efficient frontier evaluated for port1 with $M = 2$. Figure 8 shows the proportion of loan $x_0 \in [-M, 0]$ for each portfolio in Figure 7.

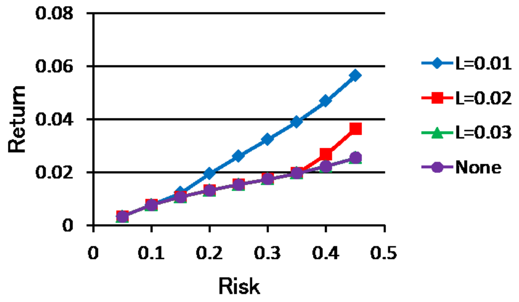

Figure 11: Efficient frontier for port2 with $M = 2$.

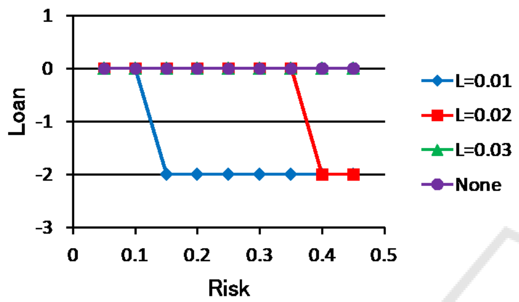

Figure 12: Proportion of loan x_0 for port2 with $M = 2$.

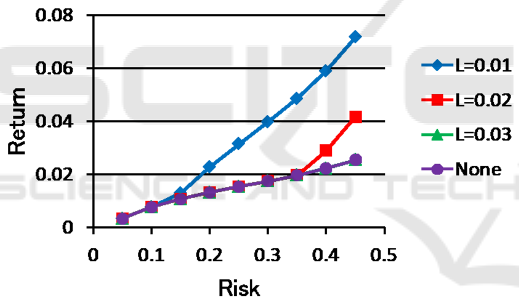

Figure 13: Efficient frontier for port2 with $M = 3$.

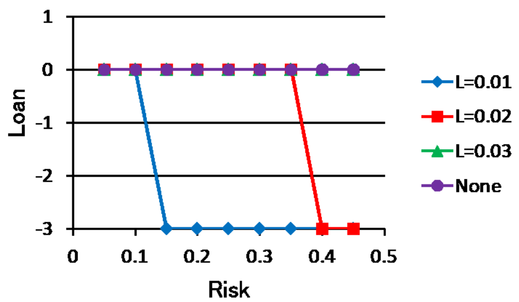

Figure 14: Proportion of loan x_0 for port2 with $M = 3$.

Figure 9 shows the efficient frontier evaluated for port1 with $M = 3$. Figure 10 shows the proportion of loan $x_0 \in [-M, 0]$ for each portfolio in Figure 9.

Figure 11 shows the efficient frontier evaluated for port2 with $M = 2$. Figure 12 shows the proportion of loan $x_0 \in [-M, 0]$ for each portfolio in Figure 11.

Table 4: Interest rate of loan by proposed method.

α	0.05	0.10	0.15	0.20	0.25
port0	0.005	0.010	0.015	0.020	0.020
port1	—	—	0.001	0.001	0.005
port2	0.001	0.002	0.004	0.006	0.008

α	0.30	0.35	0.40	0.45
port0	0.025	0.030	0.035	0.040
port1	0.005	0.010	0.015	0.020
port2	0.010	0.010	0.012	0.015

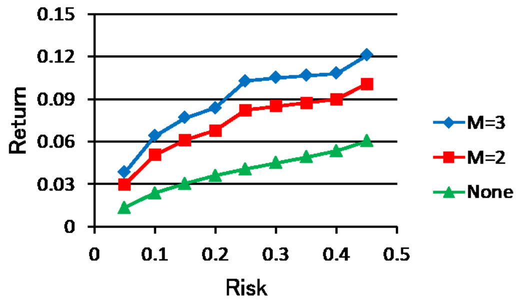

Figure 15: Efficient frontier for port0.

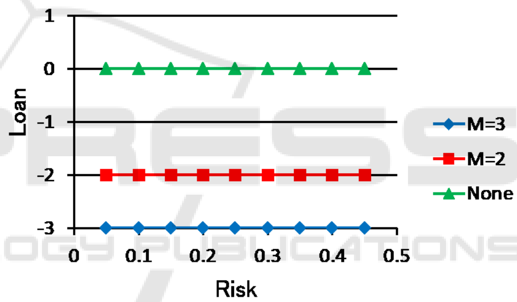

Figure 16: Proportion of loan x_0 for port0.

Figure 13 shows the efficient frontier evaluated for port2 with $M = 3$. Figure 14 shows the proportion of loan $x_0 \in [-M, 0]$ for each portfolio in Figure 13.

From Figure 12 and Figure 14, the loan is not used at all when the interest rate is high ($L = 0.03$).

From Figure 3 to Figure 14, we can confirm that the use of loan works well for improving the efficient frontier. Besides, the loan is always used up to the limit as $x_0 = -M$. The lower interest rate provides higher return and benefits borrowers. On the other hand, the high interest rate of loan doesn't benefit lenders because such a loan is not often used. The high interest rate of loan doesn't benefit borrowers, either. That is because the efficient frontier can't be improved for borrowers without using the loan.

5.3 Variable Interest Rate

From each of the data sets, POPL is formulated as shown in (46). According to the proposed method,

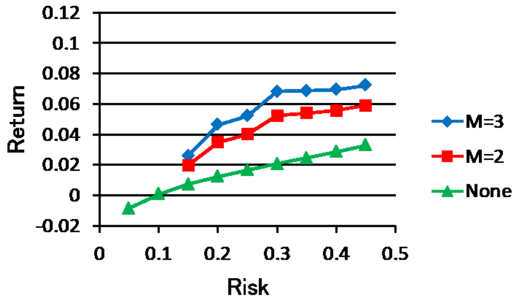

Figure 17: Efficient frontier for port1.

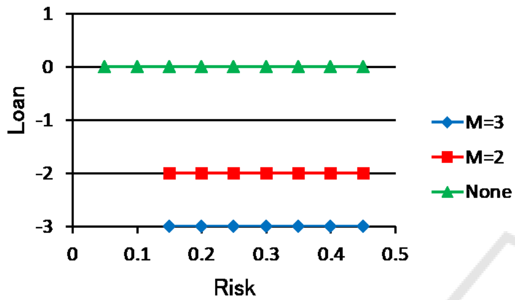

Figure 18: Proportion of loan x_0 for port1.

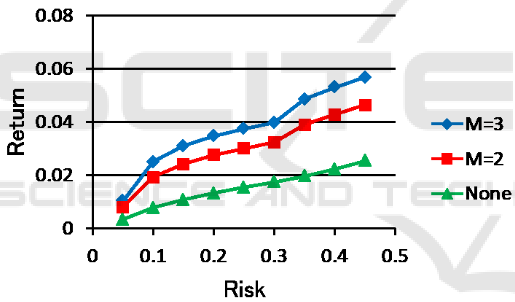

Figure 19: Efficient frontier for port2.

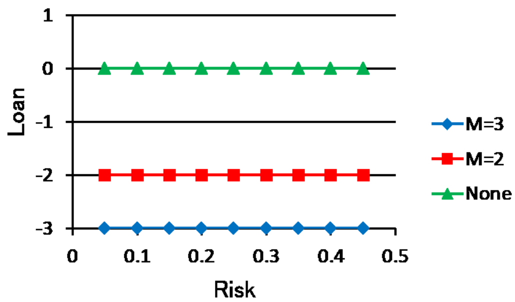

Figure 20: Proportion of loan x_0 for port2.

a proper interest rate of loan $L \in \Re$ is decided for each of the risk $\alpha \in (0, 0.5)$ as shown in Table 4. The loan can't be used for port1 when the risk is too low ($\alpha \leq 0.1$). That is because the optimal solution $\boldsymbol{x}^\star \in \Re^n$ of the sub-problem of POPL in (44) doesn't meet $\gamma_0(\boldsymbol{x}^\star) \geq 0$. For each pair of α and L shown in Table 4, POPL in (46) is solved repeatedly.

Figure 15 shows the efficient frontier evaluated for port0, where the proper interest rate $L \in \Re$ in Table 4 is used for each risk $\alpha \in (0, 0.5)$. Two different upper limits, $M = 2$ and $M = 3$, are compared in Figure 15.

Figure 16 shows the proportion of loan for each portfolio shown in Figure 15. From Figure 16, we can confirm that every portfolio except "None" borrows money from the loan up to the limit as $x_0 = -M$.

Figure 17 shows the efficient frontier evaluated for port1 with the interest rate of loan $L \in \Re$ in Table 4. The loan is not used for the low risks, $\alpha = 0.05$ and $\alpha = 0.1$, in Figure 17. Figure 18 shows the proportion of loan for each portfolio shown in Figure 17.

Figure 19 shows the efficient frontier evaluated for port2 with $L \in \Re$ in Table 4. Figure 20 shows the proportion of loan for each portfolio in Figure 19.

From Figure 15 to Figure 20, we can confirm that the loan with the proper interest rate $L \in \Re$ is always used for improving the efficient frontier regardless of the acceptable risk $\alpha \in (0, 0.5)$. Furthermore, we can see that the return $\gamma(\boldsymbol{x})$ of the optimal solution $\boldsymbol{x} \in \Re^n$ for POPL depends not only on the risk α but also on the upper limit of loan M. Specifically, we can get more return by borrowing more money.

6 CONCLUSION

Portfolio optimization using loan has been formulated as POPL and solved in this paper. The emphasis of our work is on the proposal of an interest rate decision method for POPL. From an acceptable risk α, the proposed method can derive a proper interest rate of loan L that benefits both borrowers and lenders. Thereby, POPL in (24) can be also written by a rather simple form in (46). Finally, from the result of the numerical experiment, we have confirmed that the efficient frontier is improved by using the loan completely.

As mentioned above, the proposed method in this paper benefits both borrowers and lenders. Therefore, we can expect that the proposed method contributes to economic revitalization through active investment using loan. On the other hand, we have to choose the acceptable risk $\alpha \in (0, 0.5)$ carefully to use the proposed method safely and effectively.

For future work, we will extend POPL based on a multi-period framework. Furthermore, we would like to include cardinality constraints into POPL.

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