An Inverse Method of the Natural Setting for Integer, Half-integer and Rational "Perfect" Hypocycloids

Zarema S. Seidametova[1][a] and Valerii A. Temnenko[2][b]

[1] *Crimean Engineering and Pedagogical University, 8 Uchebnyi Ln., Simferopol, 95015, Ukraine*

[2] *Independent researcher, Simferopol, Ukraine*

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Abstract: The paper describes a family of remarkable curves (integer and half-integer hypocycloids and rational perfect hypocycloids) given in an inverse-natural form using a simple trigonometric relation $s = s(\chi)$, where s is the arc coordinate and χ is the angle defining the direction of the tangent. In the paper we presented all perfect hypocycloids with the number of cusps $\nu \leq 10$. From designing the hypocycloid using inverse natural setting easy to determine the number of cusps and find the values of the λ_m parameter, corresponding to perfect hypocycloids.

1 INTRODUCTION

Many remarkable curves have emerged in mathematics over the past centuries. The study of these curves is a very effective tool in the teaching of calculus, differential geometry and computer science. Many great curves are described in the classical book "A Catalog of Special Plane Curves" (Lawrence, 2014) that featured more than 60 special curves. The other work on plane curves is "A handbook on curves and their properties" (Yates, 2012). This handbook contains curves constructions, equations, physical and mathematical properties, and connections to each other.

Wang et al. (Wang et al., 2019) explored hypocycloid's parametric equation and discussed the application of the astroid on the bus door for saving space. For simulating its dynamic opening process, they used MATLAB. There are a lot of examples of the using curves and surfaces innovation in the architectural designs of modern buildings (Biran, 2018).

Almost all curves can be represented mathematically and on a computer. The mathematical study of curves and surfaces in space is called "differential geometry". There are a lot of mathematical tools available to the computer scientist. The combination of these tools depends on what and how curves need to be represented.

There are different types of curves using in the design of geometric data structures. For example, Space-Filling Curves described in the papers (Asano et al., 1997; Rad and Karimipour, 2019).

There are a lot of ways to define curves. One of the most convenient ways to describe a plane curve is the "Euler" or "natural" way of locally defining the curve. In this method, the angle of inclination of the tangent is set as a function of the length of the arc along the curve.

In some situations, the "reverse" method of "natural" curve definition is convenient, in which the arc length is set as a function of the angle of inclination of the tangent. We will demonstrate in this article how convenient this "reverse" method is when describing some types of hypocycloids.

2 AN INVERSE METHOD OF THE NATURAL SETTING FOR PLANAR CURVES

One well known way to define flat curves is to describe them in the so-called natural form (or, another name is "Euler's form"):

$$\chi = \chi(s), \qquad (1)$$

where χ is an angle between some fixed direction – for example, the x-axis – and the direction of the tangent to the curve; s is the arc coordinate along the curve.

[a] https://orcid.org/0000-0001-7643-6386

[b] https://orcid.org/0000-0002-9355-9086

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If the natural equation of the curve (1) is known, then the equations of the corresponding curve in parametric form $x = x(s)$, $y = y(s)$ can be written in the following form:

$$x = \int_0^s \cos\chi(s)\mathrm{d}s + x_0,$$
$$y = \int_0^s \sin\chi(s)\mathrm{d}s + y_0, \tag{2}$$

where (x_0, y_0) is an arbitrarily chosen point (x, y) in the plane, corresponding on the curve to the origin of the arc coordinate $s = 0$.

Leonhard Euler studied a family of curves of the form (1) with a power-law dependence of χ on s ($\chi = \lambda s^p$, $\lambda = const$, $p = const$) (MacTutor History of Mathematics, 2020). Euler called these curves as “clothoids”. The most famous of these curves for p=2 is called the “Euler spiral” or “Cornu spiral”. Euler investigated this curve a century earlier than did Marie Alfred Cornu.

Instead of the equation (1), we can consider the *inverse* method of natural setting for the curve:

$$s = s(\chi). \tag{3}$$

This method is convenient if the function inverse to (3) is multivalued or does not have an explicit analytic expression.

Equations (2) with this method for specifying the curve (3) become:

$$x(\chi) = \int_0^\chi \cos\chi \cdot \frac{\mathrm{d}s}{\mathrm{d}\chi}\mathrm{d}\chi + x_0,$$
$$y(\chi) = \int_0^\chi \sin\chi \cdot \frac{\mathrm{d}s}{\mathrm{d}\chi}\mathrm{d}\chi + y_0. \tag{4}$$

Equations (4) define a parametric description of the curve. In this specification parameter χ has clear geometric meaning: it is the angle between the axis x and the direction tangent to the curve.

3 INTEGER HYPOCYCLOIDS

We consider in this note a one-parameter family of curves of the form (3):

$$s = \frac{n^2-1}{n^2} \cdot \sin(n\chi), \tag{5}$$

in which $n \geq 2$ is an integer parameter. Let’s call the equation (5) the “trigonometric Euler relation”. This relation in local variables (s, λ) describes the classic family of curves: integer hypocycloids.

For an even value of n, the range of the function (5) is $0 \leq \chi \leq 2\pi$. For an odd value of n, the range of the function (5) is $0 \leq \chi \leq \pi$. On this interval the trigonometric Euler’s relation (5) defines a closed curve.

Assuming that $x_0 = 0$, $y_0 = 1/n$, and performing the integration in (4), we obtain the equations of the integer hypocycloids in parametric form:

$$x = \frac{1}{2n}\Big((n+1)\sin\big((n-1)\chi\big) + (n-1)\sin\big((n+1)\chi\big)\Big),$$
$$y = \frac{1}{2n}\Big((n+1)\cos\big((n-1)\chi\big) - (n-1)\cos\big((n+1)\chi\big)\Big). \tag{6}$$

4 CUSPS OF THE INTEGER HYPOCYCLOIDS

The curves (6) are smooth everywhere except the points $\chi_{n,k}$, in which the cusps of the curve (6) are located. The positions of the cusps’ vertices are determined by the points of a curvature singularity of the curve (6):

$$\chi'_s = \frac{1}{s'_\chi} = \frac{n}{(n^2-1)\cos(n\chi)}. \tag{7}$$

Respectively, the cusp-points are zeros of $\cos(n\chi)$:

$$\chi_{n,k} = \frac{\pi}{2n}(2k+1). \tag{8}$$

Let ν denote a number of cusps for the integer hypocycloids (6). In equation (8) k can take $2n$ values for even n $(0 \leq k \leq 2n-1)$ and n values for odd $(0 \leq k \leq n-1)$. Accordingly, the integer hypocycloids can have an odd number of ν cusps at $n = 2m+1$ or the it can have ν as a multiple of 4 ($\nu = 4m$ at $n = 2m$). There is no integer hypocycloids with $\nu = 4m+2$ cusps – for example, there is no six-pointed “Euler star”, but there is a five-pointed Euler star, eight-pointed and twelve-pointed Euler stars.

Substituting (8) into (6), we define the positions of the cusps’ vertices on the plane (x, y):

$$x_{n,k} = (-1)^k \cos\left(\frac{\pi}{2n}(2k+1)\right),$$
$$y_{n,k} = (-1)^k \sin\left(\frac{\pi}{2n}(2k+1)\right). \tag{9}$$

All the cusps’ vertices (9) lie on a circle of unit radius with the center at the origin.

5 APPEARANCE OF THE INTEGER HYPOCYCLOIDS

Figures 1-5 show an appearance of the integer hypocycloids with the number of rays ν, equal to 3, 4, 5, 8 and 12.

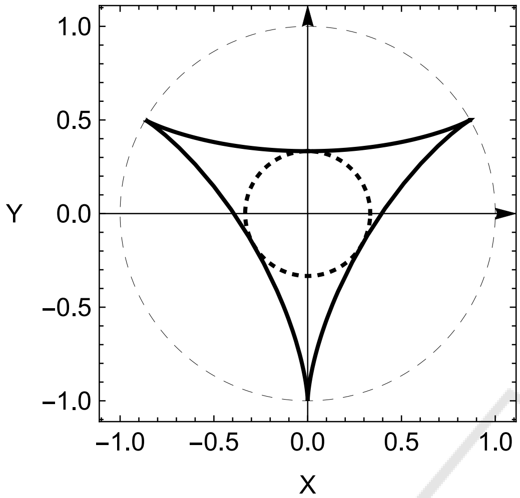

Figure 1: The tricuspidate hypocycloid (deltoid) ($n = 3$, $\nu = 3$).

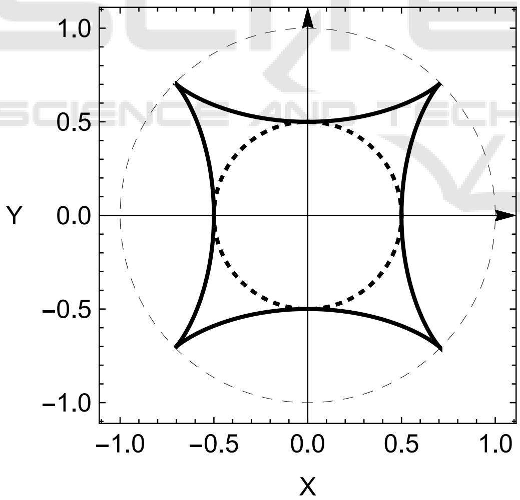

Figure 2: The tetracuspidate hypocycloid (astroid) ($n = 2$, $\nu = 4$).

6 HALF-INTEGER HYPOCYCLOIDS

Consider the half-integer hypocycloid, assuming that in equations (5) and (6) the integer parameter n is replaced by a half-integer $n \to n + \dfrac{1}{2}$ ($n \geq 1$).

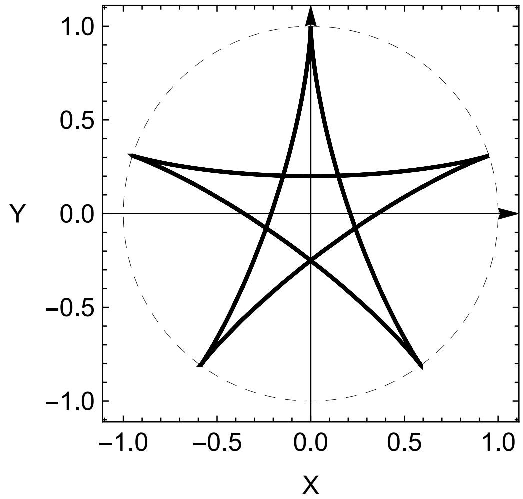

Figure 3: The pentacuspidate hypocycloid (the integer hypocycloid with $n = 5$, $\nu = 5$).

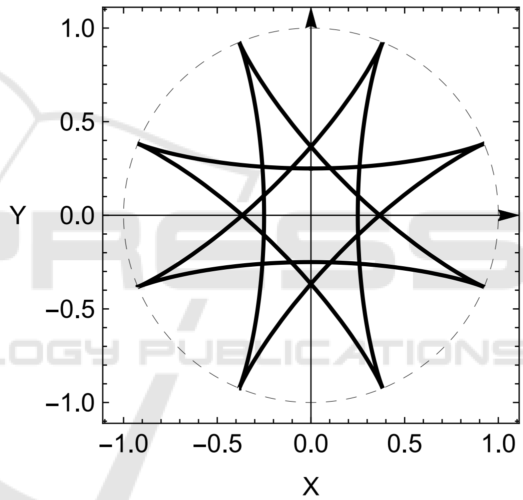

Figure 4: The octacuspidate hypocycloid (the integer hypocycloid with $n = 4$, $\nu = 8$).

With half-integer parameter, the hypocycloid equations (5) and (6) take the following form:

$$s = \frac{(2n-1)\,(2n+3)}{(2n+1)^2} \sin\Big((2n+1)\,\frac{\chi}{2} \Big), \tag{10}$$

$$\begin{aligned} x &= \frac{1}{2\,(2n+1)} \Big((2n+3) \sin\Big((2n-1)\,\frac{\chi}{2} \Big) + \\ &\qquad + (2n-1) \sin\Big((2n+3)\,\frac{\chi}{2} \Big) \Big), \\ y &= \frac{1}{2\,(2n+1)} \Big((2n+3) \cos\Big((2n-1)\,\frac{\chi}{2} \Big) - \\ &\qquad - (2n-1) \cos\Big((2n+3)\,\frac{\chi}{2} \Big) \Big). \end{aligned} \tag{11}$$

The functions $x(\chi)$ and $y(\chi)$ (11) are periodic in the argument χ with a period $\mathcal{P} = 4\pi$.

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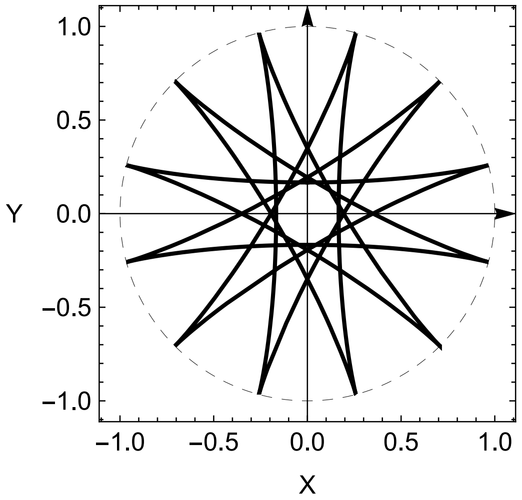

Figure 5: The dodecacuspidate hypocycloid (the integer hypocycloid with $n = 6$, $\nu = 12$).

The positions of the cusps of the half-integer hypocycloid (11) are determined by the condition:

$$\frac{ds}{d\chi} = 0, \tag{12}$$

or

$$\chi_k = \pi \frac{2k+1}{2n+1}; \quad 0 \le k \le k_{max} = 4n+1. \tag{13}$$

The number of cusps ν is determined by the condition

$$\nu = 1 + k_{max} = 4n+2. \tag{14}$$

In accordance with (14), half-integer hypocycloids together with integer hypocycloids make it possible to obtain an hypocycloid with any number of rays. In particular, for $n = 1$, equation (1) describes a six-beam astroid.

Figure 6 and figure 7 show half-integer hypocycloids at $n = 1$ (figure 6) and $n = 2$ (figure 7).

A half-integer hypocycloid with $n = 1$ has no self-intersection points (like two integer hypocycloids of the lowest index 1, even and odd). The remaining half-integer hypocycloids with $n \ge 2$ (and integer hypocycloids with index $n \ge 2$) have self-intersection points. The half-integer hypocycloids are located in the ring between $R_{min} = \dfrac{2}{2n+1}$ and $R_{max} = 1$. It is easy to show that these curves touch a circle of radius R_{min} in $\nu = 4n+2$ points for $\chi_{t,k}$:

$$\chi_{t,k} = \frac{2\pi k}{2n+1}; \quad 0 \le k \le k_{max} = 4n+1. \tag{15}$$

The totality of integer and half-integer hypocycloids forms the set of figures, called in (Seidametova and Temnenko, 2019) “The Euler Insignia”.

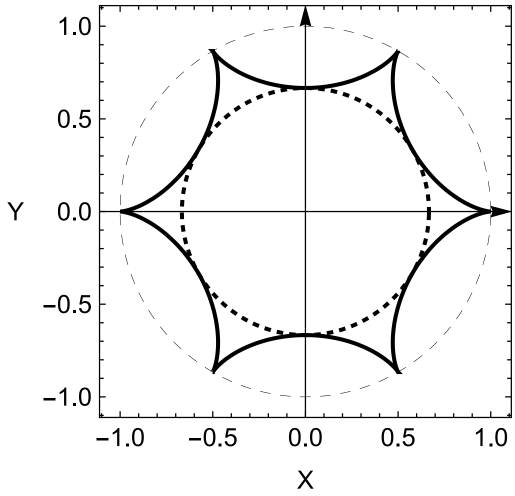

Figure 6: The half-integer hypocycloid at $n = 1$(the six-pointed star).

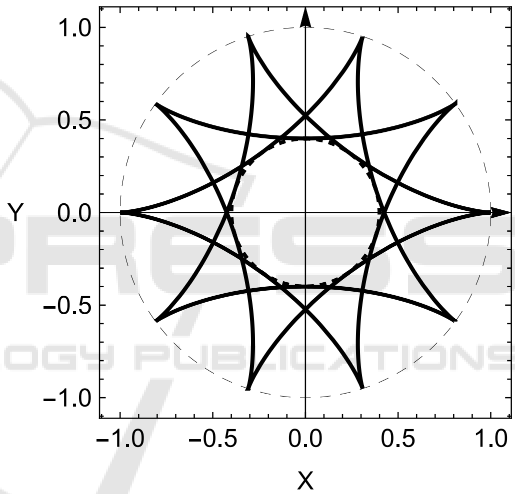

Figure 7: The half-integer hypocycloid at $n = 2$ (the ten-pointed star).

7 THE PERFECT HYPOCYCLOIDS

Let’s call a hypocycloid “perfect” if it has no self-intersection points. An example of a perfect hypocycloid is the deltoid (an odd integer hypocycloid with $n = 3$ and $\nu = 3$, figure 1), the astroid (an even integer hypocycloid, $n = 2$, $\nu = 4$, figure 2) and the six-point star (the half-integer hypocycloid, $n = 1$, $\nu = 6$, figure 6). All other integer and half-integer hypocycloids, in particular, shown in figures 3, 4, 5, 7, are not perfect.

Perfect hypocycloids are described by the trigonometric Euler relation (5), in which an integer n is replaced by some rational number λ_m of a certain type. The parameter λ_m is an irreducible fraction of one of three possible types:

$$\lambda_m = \frac{2m+1}{2m-1}; \quad m \geq 1. \tag{16}$$

$$\lambda_m = \frac{2m}{2m-1}; \quad m \geq 1. \tag{17}$$

$$\lambda_m = \frac{2m+1}{2m}; \quad m \geq 1. \tag{18}$$

Let call perfect hypocycloids of the type (16) the Odd-Odd perfect hypocycloids. Let call perfect hypocycloids of the type (17) the Even-Odd perfect hypocycloids. Let call perfect hypocycloids of the type (18) the Odd- Even perfect hypocycloids. For $m = 1$ a perfect hypocycloid of the type (16) is an integer hypocycloid with three cusps (the deltoid, figure 1), a perfect hypocycloid of the type (17) is an integer hypocycloid with four cusps (the astroid, figure 2), a perfect hypocycloid of the type (18) is a half-integral six-pointed star (figure 6).

Figure 8 shows the Odd-Odd perfect hypocycloid with $m = 2$ (the "five-pointed star of Euler"). In accordance with relations (5) and (6) and the value $\lambda_m = 5/3$, the equations of this perfect hypocycloid have the form:

$$s = \left(\frac{4}{5}\right)^2 \sin\left(\frac{5\chi}{3}\right), \tag{19}$$

$$\begin{aligned} x &= \frac{1}{5}\left(4\sin\left(\frac{2\chi}{3}\right) + \sin\left(\frac{8\chi}{3}\right)\right), \\ y &= \frac{1}{5}\left(4\cos\left(\frac{2\chi}{3}\right) - \cos\left(\frac{8\chi}{3}\right)\right). \end{aligned} \tag{20}$$

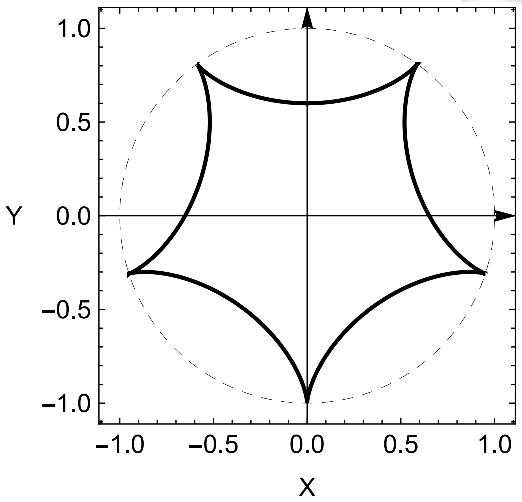

Figure 8: The Odd-Odd perfect hypocycloid with $m = 2$ ($\lambda_m = 5/3$, the five-pointed star of Euler).

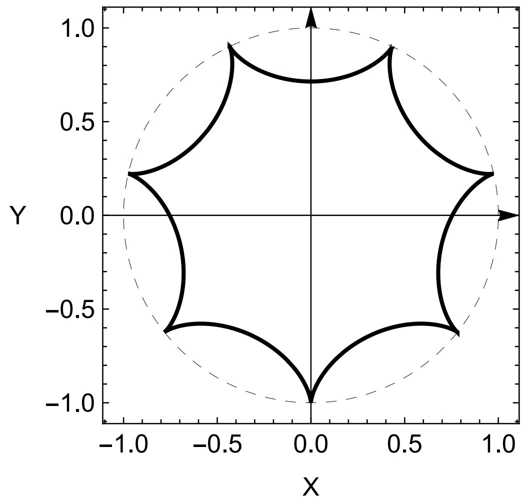

Figure 9: The Odd-Odd perfect hypocycloid with $m = 3$ ($\lambda_m = 7/5$, the seven-pointed star of Euler).

Figure 9 shows the Odd-Odd perfect hypocycloid with $m = 3$ ($\lambda_m = 7/5$, the "seven-pointed star of Euler"). The equations of this hypocycloid are following:

$$s = \frac{24}{49}\sin\left(\frac{7\chi}{5}\right), \tag{21}$$

$$\begin{aligned} x &= \frac{1}{7}\left(6\sin\left(\frac{2\chi}{5}\right) + \sin\left(\frac{12\chi}{5}\right)\right), \\ y &= \frac{1}{7}\left(6\cos\left(\frac{2\chi}{5}\right) - \cos\left(\frac{12\chi}{5}\right)\right). \end{aligned} \tag{22}$$

Figure 10 shows the Odd-Odd perfect hypocycloid with $m = 4$ ($\lambda_m = 9/7$). This is the "nine-pointed Euler star"). The equations of this curve are following:

$$s = \frac{32}{81}\sin\left(\frac{9\chi}{7}\right), \tag{23}$$

$$\begin{aligned} x &= \frac{1}{9}\left(8\sin\left(\frac{2\chi}{7}\right) + \sin\left(\frac{16\chi}{7}\right)\right), \\ y &= \frac{1}{9}\left(8\cos\left(\frac{2\chi}{7}\right) - \cos\left(\frac{16\chi}{7}\right)\right). \end{aligned} \tag{24}$$

Figure 11 shows the Even-Odd perfect hypocycloid with $m = 2$ ($\lambda_m = 4/3$). This is the "eight-pointed Euler star"). The equations of this curve are following:

$$s = \frac{7}{16}\sin\left(\frac{4\chi}{3}\right), \tag{25}$$

$$\begin{aligned} x &= \frac{1}{8}\left(7\sin\left(\frac{\chi}{3}\right) + \sin\left(\frac{7\chi}{3}\right)\right), \\ y &= \frac{1}{8}\left(7\cos\left(\frac{\chi}{3}\right) - \cos\left(\frac{7\chi}{3}\right)\right). \end{aligned} \tag{26}$$

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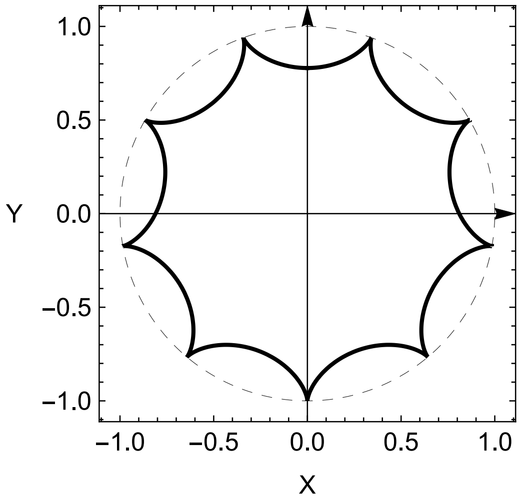

Figure 10: The Odd-Odd perfect hypocycloid with $m = 4$ ($\lambda_m = 9/7$, the nine-pointed Euler star).

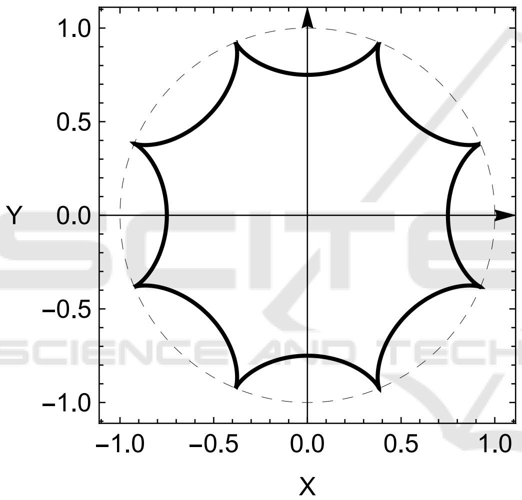

Figure 11: The Even-Odd perfect hypocycloid with $m = 2$ ($\lambda_m = 4/3$, the eight-pointed Euler star).

Figure 12 shows the Odd-Even perfect hypocycloid with $m = 2$ ($\lambda_m = 5/4$). This is the "ten-pointed Euler star"). The equations of this curve are following:

$$s = \left(\frac{3}{5}\right)^2 \sin\left(\frac{5\chi}{4}\right), \tag{27}$$

$$\begin{aligned} x &= \frac{1}{10}\left(9\sin\left(\frac{\chi}{4}\right) + \sin\left(\frac{9\chi}{4}\right)\right), \\ y &= \frac{1}{10}\left(9\cos\left(\frac{\chi}{4}\right) - \cos\left(\frac{9\chi}{4}\right)\right). \end{aligned} \tag{28}$$

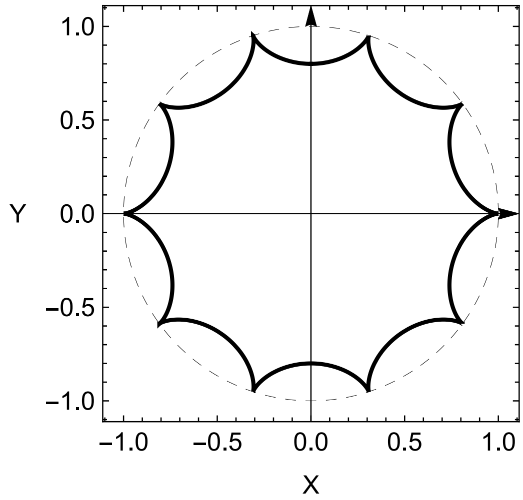

Figure 12: The Odd-Even perfect hypocycloid with $m = 2$ ($\lambda_m = 5/4$, the ten-pointed Euler star).

8 CONCLUSIONS

Figures 1, 2, 6, 8, 9, 10, 11, 12 presented in the paper demonstrate all perfect hypocycloids with the number of cusps $\nu \leq 10$.

Designing the hypocycloid by inverse natural setting makes it easy to determine the number of cusps and find the values of the λ_m parameter ((16), (17) and (18)), corresponding to perfect hypocycloids.

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