Local Antimagic Vertex Coloring of Wheel Graph and Helm Graph

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Abstract: Let $\chi(G)$ be a chromatic number of vertex coloring of a graph G. A bijection $f: E \rightarrow \{1,2,3, \ldots, |E(G)|\}$ is called local antimagic vertex coloring if for any adjacent vertices do not share the same weight, where the weight of a vertex in G is the sum of the label of edges incident to it. We denote the minimum number of distinct weight of vertices in G so that the graph G admits a local antimagic vertex coloring as $\chi_{la}(G)$. In this study, we established the missing value of χ_{la} for a case in wheel graph and χ_{la} for helm graph.

1 INTRODUCTION

Suppose $G(V, E)$ be a connected simple graph such that $v, u \in V(G)$. We define local antimagic vertex coloring of G as a bijection $f: E \rightarrow \{1,2,3, \ldots, |E(G)|\}$ such that for any adjacent vertices v and u, $w(v) \neq w(u)$, which $w(v) = \sum_{e \in E(G)} f(e)$ for every edge e incident to v. We are able to distinguish weights of vertices by assigning distinct colors for every distinct weights. Using a well-known notation, $\chi(G)$ denoted as the chromatic number of G. The local antimagic vertex chromatic number $\chi_{la}(G)$ is the minimum number of colors for vertices taken over all colorings induced by local antimagic vertex coloring of G. A remark written by Arumugam et al. (2017) tells us that for any graph G, $\chi_{la}(G) \geq \chi(G)$.

Hartsfield & Ringel (1990) introduced the term of antimagic labeling of a graph. We can see many variations of this antimagic labeling. One of many variations is a concept of local antimagic vertex coloring introduced by Arumugam et al. (2017). They also give the exact values for χ_{la} for wheel W_n when $n \not\equiv 0 (\bmod 4)$. For $n \equiv 0 (\bmod 4)$, they found only the interval.

Arumugam et al. (2018) found the exact value $\chi_{la}(G)$ for some corona product graphs. Nazula et al. (2018) established the exact value of $\chi_{la}(G)$ for certain unicyclic graphs, which are kite graphs and cycle graphs with two neighbour pendants. Lau et al. (2018) showed further results of local antimagic vertex coloring for some graphs and established a sharp lower bound for graphs which we use in our proof. Haslegrave (2018) proved a conjecture proposed by Arumugam et al. whether any connected graph other than K_2 admits a local antimagic vertex coloring, by using probabilistic method.

In this paper, we study local antimagic vertex coloring for wheel graphs and helm graphs. We establish an exact value of χ_{la} for a case in wheel graph, that has not been proved yet by Arumugam et al. Also, we have exact values of χ_{la} for helm graphs. Silaban et al. (2009) gave an efficient way of labeling by defining some conditional function which we use much in our paper.

2 SUPPLEMENTARY PROPERTIES

For convenience, we would like to introduce some simpler notations that we use in this paper. Firstly, we denote $i \in [a, b]$ as i being an integer greater or equal to a, while lower or equal to b. Next, we add additional index of e or o, as in $i \in [a, b]_e$ that has additional information of i an even integer, while using o simply means i an odd integer.

Silaban et al. (2009) introduced a function which checks a condition of certain value and returns according to whether the condition is satisfied. One of the example is the $\text{odd}(x)$ function which defined as follows

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$$\text{odd}(i) = \begin{cases} 1, & \text{if } i \equiv 1 (\text{mod } 2), \\ 0, & \text{otherwise.} \end{cases}$$

We will use this convenient function in our proofs. Other than that, we would like to introduce another function called modulo congruency check. Modulo congruency check $m(x,t)$ is a function that values to 1 if x is equivalent t by mod 4, while otherwise 0. Formally, we write as follows

$$m(i,t) = \begin{cases} 1, & \text{if } i \equiv t (\text{mod } 4), \\ 0, & \text{otherwise.} \end{cases}$$

We use a definition of the wheel graph W_n of order $n+1$ with the vertex set

$$V(W_n) = \{c, v_i | i \in [1,n]\}$$

and the edge set

$$E(W_n) = \{v_n v_1, cv_n, v_1 v_{i+1}, cv_i | i \in [1, n-1]\}.$$

Arumugam et al. (2017) proved the exact value of χ_{la} in many cases of wheel graphs as follows

Theorem 1 (Arumugam et al., 2017). For the wheel W_n of order $n+1$, we have

$$\chi_{la}(W_n) = \begin{cases} 4, & \text{if } n \equiv 1,3 (\text{mod } 4), \\ 3, & \text{if } n \equiv 2 (\text{mod } 4). \end{cases}$$

For $n \equiv 0 (\text{mod } 4)$, the authors found only the interval $3 \leq \chi_{la}(W_n) \leq 5$. They also found a sharp lower bounds for arbitrary tree graph. Lau et al. (2018) generalizes this theorem as follows

Theorem 2 (Lau et al., 2018). Let G be a graph having k pendants. If G is not K_2, then $\chi_{la}(G) \geq k+1$ and the bound is sharp.

The preceding theorem is useful for finding a lower bound of $\chi_{la}(H_n)$. We continue this reasoning to have sharp lower bounds for this particular helm graphs.

3 MAIN RESULTS

We start to establish our main theorem.

Theorem 3. Let $n \equiv 0 (\text{mod } 4)$. Then $\chi_{la}(W_n) = 3$.
Proof. It is known that

$$\chi(W_n) = \begin{cases} 4, & \text{if } n \text{ is odd,} \\ 3, & \text{if } n \text{ is even.} \end{cases} \quad (1)$$

Therefore, for $n \equiv 0 (\text{mod } 4)$, $\chi_{la}(W_n) \geq 3$. To show $\chi_{la}(W_n) \leq 3$ for $n \equiv 0 (\text{mod } 4)$, we will define $f: E(W_n) \rightarrow \{1,2,3,\ldots,|E(W_n)|\}$ that admits local antimagic vertex coloring of W_n.

Case 1: $n = 4$

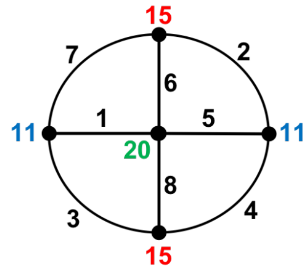

Figure 1: Local Antimagic Vertex Coloring of W4.

Label the edges of W_n isomorphic to the following figure. Therefore, for a small case of $n = 4$, $\chi_{la}(W_4) = 3$.

Case 2: $n \equiv 0 \mod 8$

Label the edges W_n as follows

$$f(cv_i) = \begin{cases} \frac{n-2i+10}{8} + m(i,3)\frac{n+4}{8}, & \text{if } i \in \left[1, \frac{n}{2}+1\right]_o, \\ \frac{3n-2i+12}{8} + m(i,0)\frac{n+4}{8}, & \text{if } i \in \left[2, \frac{n}{2}\right]_e, \\ \frac{3n+4}{4}, & \text{if } i = \frac{n}{2}+2, \\ \frac{3n+2i+6}{8} + m(i,3)\frac{n-4}{8}, & \text{if } i \in \left[\frac{n}{2}+3, n-1\right]_o, \\ \frac{5n+2i+8}{8} + m(i,2)\frac{n-4}{8}, & \text{if } i \in \left[\frac{n}{2}+4, n\right]_e. \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} \frac{15n-2i-2}{8} + m(i,1)\frac{n+4}{8}, & \text{if } i \in \left[1, \frac{n}{2}+3\right]_o \\ \frac{5n+2i}{4}, & \text{if } i \in \left[2, \frac{n}{2}\right]_e \\ \frac{7n+2i+4}{8} + m(i,0)\frac{n-4}{8}, & \text{if } i \in \left[\frac{n}{2}+2, n-2\right]_e \\ \frac{4n-i+1}{2}, & \text{if } i \in \left[\frac{n}{2}+5, n-1\right]_o \end{cases}$$

$$f(v_n v_1) = \frac{5n}{4}$$

The weights of vertices are

$$w(v_i) = \begin{cases} \frac{27n}{8} + 1, & \text{if } i \text{ is odd,} \\ \frac{29n}{8} + 2, & \text{if } i \text{ is even.} \end{cases}$$

$$w(c) = \frac{n(n+1)}{2}.$$

It is clear that these three weights are distinct. Therefore, $\chi_{la}(W_n) \leq 3$ for $n \equiv 0 \mod 8$. We conclude that $\chi_{la}(W_n) = 3$ for $n \equiv 0 \mod 8$.

Case 3: $n \equiv 4 \mod 8$ and $n \geq 12$

Label the edges of W_n as follows

$$f(cv_i) = \begin{cases} \frac{n}{2} + m(i,3)\frac{n}{2}, & \text{if } i \in [1,3]_o \\ \frac{i+2}{4} + m(i,0)\frac{n}{8}, & \text{if } i \in \left[2, \frac{n}{2}\right]_e \\ \frac{2n+2i-2}{8} + m(i,3)\frac{n}{8}, & \text{if } i \in \left[5, \frac{n}{2}-1\right]_o \\ \frac{3n+4}{8} + m(i,1)\frac{n-2}{2}, & \text{if } i \in \left[\frac{n}{2}+1, \frac{n}{2}+3\right]_o \\ \frac{3n-i+4}{4} + m(i,2)\frac{n}{8}, & \text{if } i \in \left[\frac{n}{2}+2, n\right]_e \\ \frac{4n-i+1}{4} + m(i,3)\frac{n}{8}, & \text{if } i \in \left[\frac{n}{2}+5, n-1\right]_o \end{cases}$$

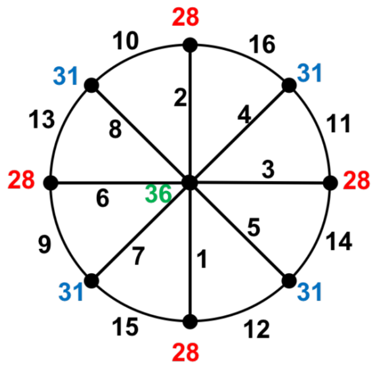

Figure 2: Local Antimagic Vertex Coloring of W_8.

$$f(v_iv_{i+1}) = \begin{cases} \frac{11n+4}{8} + m(i,1)\frac{5n-4}{8}, & \text{if } i \in [1,2], \\ \frac{5n+i+1}{4} + m(i,1)\frac{n}{8}, & \text{if } i \in \left[3, \frac{n}{2}-1\right]_o \\ \frac{4n-i+2}{2}, & \text{if } i \in \left[4, \frac{n}{2}+2\right]_e \\ n+1, & \text{if } i = \frac{n}{2}+1, \\ \frac{5n+2i-2}{4}, & \text{if } i \in \left[\frac{n}{2}+3, n-1\right]_o \\ \frac{5n-i+6}{4} + m(i,0)\frac{n}{8}, & \text{if } i \in \left[\frac{n}{2}+4, n-2\right]_e \end{cases}$$

The weights of vertices are

$$w(v_i) = \begin{cases} \frac{29n+12}{8}, & \text{if } i \text{ is odd}, \\ \frac{29n+12}{8} - \frac{n}{4}, & \text{if } i \text{ is even}. \end{cases}$$

$$w(c) = \frac{n(n+1)}{2}.$$

It is clear that these three weights are distinct. Hence, $\chi_{la}(W_n) \leq 3$ for $n \equiv 4 \mod 8$. We conclude that $\chi_{la}(W_n) = 3$ for $n \equiv 4 \mod 8$.

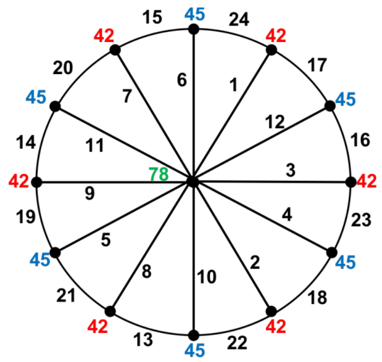

Figure 3: Local Antimagic Vertex Coloring of W_{12}.

Therefore, f is a local antimagic vertex coloring for W_n with $\chi_{la}(W_n) = 3$.

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Helm graph is acquired by attaching a pendant to every vertices in the wheel graph except the center. Helm graph H_n is formally defined with the vertex set

$$V(H_n) = \{c, v_i, x_i | i \in [1, n]\}$$

and the edge set

$$E(H_n) = \{v_nv_1, cv_n, v_iv_{i+1}, cv_i, x_iv_i, x_nv_n | i \in [1, n-1]\}$$

We start to call vertex c as a center, and vertices x_i as pendants. In preceding theorem, we use the chromatic number of the graph to prove the lower bound. Next, we try to use reasoning similar with Theorem 2 to establish the lower bound.

The center c incident to n number of edges. This results the weight of center is at least the sum of natural integers up to n. Meanwhile, the weight of pendants is at most $3n$ since pendants incident to only one edge. Vertices other than those are incident to four edges, the weight is at least the sum of four smallest available labels. By having assumptions and showing contradictions, this reasoning effectively adjusts the lower bound to be equal to the upper bound, giving an exact value for helm graphs.

Theorem 4. For integer $n \geq 3$, helm graphs H_n have

$$\chi_{la}(H_n) = \begin{cases} n+3, & \text{if } n \neq 4, \\ 6, & \text{if } n = 4. \end{cases}$$

Proof. From the definition, helm graph H_n has n number of pendants. Using Theorem 2 directly, we are guaranteed to have $\chi_{la}(H_n) \geq n+1$.

Let f be a labeling of helm graph. We divide the problems into cases.

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Case 1: $n = 3,4,5$

To prove the upper bound, labels the edges of H_n isomorphic as the following figures.

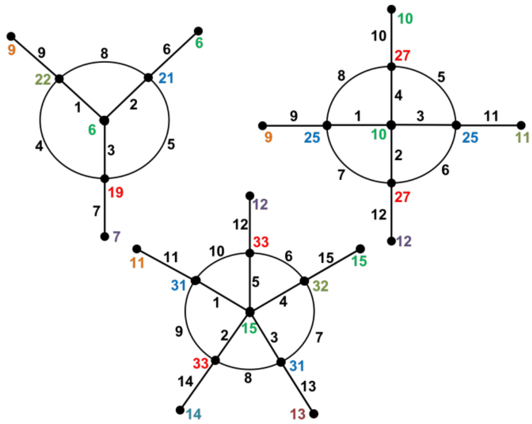

Figure 4: Local Antimagic Vertex Coloring of H_3, H_4, and H_5.

Hence, we have

$$\chi_{la}(H_n) \leq \begin{cases} 6, & \text{if } n = 3,4, \\ 8, & \text{if } n = 5. \end{cases}$$

Subcase 1.1: $n = 3$

Suppose $\chi_{la}(H_3) \leq n + 2 = 5$. Then, there exists $w(v_i)$ that equals $w(x_j)$ for some i, j. Notice that every v_i incident to four edges, which means $w(v_i) \geq 1 + 2 + 3 + 4 = 10$, if we chose smallest labels on edges incident to v_i. Meanwhile, $w(x_j) \leq 9$ because pendants only have one label. It contradicts the fact the existence of $w(v_i)$ that equals $w(x_j)$ for some i, j. Therefore, $\chi_{la}(H_3) \geq n + 3 = 6$. We conclude that $\chi_{la}(H_3) = 6$.

Subcase 1.2: $n = 4$

Suppose $\chi_{la} \leq n + 1$. Therefore, there exists two v_i such that each one $w(v_i)$ equals to $w(x_j)$ for some i, j. The sum of those $w(v_i) \geq \sum_{i=1}^{8} i = 36$. While the sum of weights from pendants $w(x_i) \leq 2(3n) = 6n < 36$. It contradicts the fact that each one $w(v_i)$ equals to $w(x_j)$ for some i, j. Hence, $\chi_{la}(H_4) \geq n + 2 = 6$. We conclude that $\chi_{la}(H_4) = 6$.

Subcase 1.3: $n = 5$.

A similar reasoning with previous case, that there is no two v_i such that each one $w(v_i)$ equals to $w(x_j)$ for some i, j which means $\chi_{la} \geq n + 2$. Suppose $\chi_{la} \leq n + 2$. Therefore, there exists at least one v_i and center c such that each one $w(v_i) = w(x_j)$ and $w(c) = w(xk)$ for some i, j, k. Notice that $w(x_j) \leq 15$ for any j. There is only one possibility to satisfy $w(c) = 15$, by giving cv_i labels from 1 to 5. Hence, we have a new least weight of $w(v_i) \geq 1 + 5 + 6 + 7 = 19$. It contradicts the fact that $w(v_i) = w(x_j) \leq 15$ for some i, j. Therefore, $\chi_{la}(H_5) \geq n + 3$. We conclude that $\chi_{la}(H_5) = n + 3 = 8$.

Case 2: $n \geq 6$.

Suppose $\chi_{la}(H_n) \leq n + 1$. Then, $w(c)$ will equal to $w(x_j)$ for some j. Notice that $w(c) \geq \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, while $w(x_j) \leq 3n$ for any j. It is clear that $\frac{n(n+1)}{2} > 3n$, if $n \geq 6$. Therefore, contradiction exists so that $\chi_{la}(H_n) \geq n + 2$.

Suppose $\chi_{la}(H_n) \leq n + 2$. Since $w(c)$ is unique, then there exists $\left\lfloor \frac{n}{2} \right\rfloor$ number of v_i such that each one $w(v_i)$ equals to $w(x_j)$ for some i, j. The sum of those at least $w(v_i) \geq \sum_{i=1}^{4\left\lfloor \frac{n}{2} \right\rfloor} i = 2\left\lfloor \frac{n}{2} \right\rfloor \left(4\left\lfloor \frac{n}{2} \right\rfloor + 1\right)$. While the sum of weights from pendants $w(x_i) \leq \left\lfloor \frac{n}{2} \right\rfloor (3n)$. It is not hard to prove the inequality $2\left\lfloor \frac{n}{2} \right\rfloor \left(4\left\lfloor \frac{n}{2} \right\rfloor + 1\right) > \left\lfloor \frac{n}{2} \right\rfloor (3n)$ for $n \geq 6$, which contradicts the fact that each one $w(v_i)$ equals to $w(x_j)$ for some i, j. Hence, $\chi_{la}(H_n) \geq n + 3$.

Subcase 2.1: $n \geq 6$, n is even.

To prove the upper bound, labels the edges of H_n as follows

$$f(v_i v_{i+1}) = i, \text{ if } i \in [1, n-1],$$
$$f(v_n v_1) = n,$$
$$f(cv_i) = 2n + 1 - i, \text{ if } i \in [1, n],$$
$$f(x_i v_i) \leq \begin{cases} 2n + 2, & \text{if } i = 1, \\ 3n - i + 1 + 2\,\mathrm{odd}(i), & \text{if } i \in [2, n]. \end{cases}$$

The weights of the vertices are

$$w(x_i) = f(x_i v_i)$$
$$w(v_i) = \begin{cases} 5n + 1, & \text{if } i \text{ is even}, \\ 5n + 3, & \text{if } i \text{ is odd}. \end{cases}$$
$$w(c) = \frac{n(3n+1)}{2}$$

Therefore, $\chi_{la}(H_n) \leq n + 3$ for n is even. We conclude $\chi_{la}(H_n) = n + 3$ if n is even.

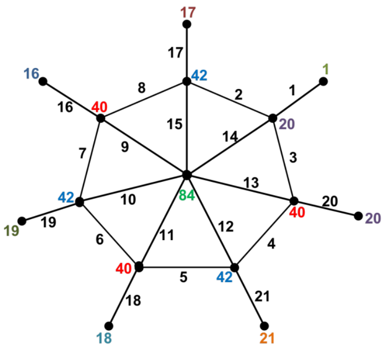

Figure 5: Local Antimagic Vertex Coloring of H_7.

Subcase 2.2: $n \geq 6$, n is odd.

To prove the upper bound, labels the edges of H_n as follows

$$f(v_i v_{i+1}) = i + 1, \text{ if } i \in [1, n-1],$$
$$f(v_n v_1) = n + 1,$$
$$f(cv_i) = 2n + 2 - i, \text{ if } i \in [1, n],$$
$$f(x_i v_i) = \begin{cases} 2n+3, & \text{if } i = 1, \\ 1, & \text{if } i = 2, \\ 3n - i + 2 + 2\,\text{odd}(i+1), & \text{if } i \in [3, n]. \end{cases}$$

The weights of the vertices are

$$w(x_i) = f(x_i v_i)$$
$$w(v_i) = \begin{cases} 2n+6, & \text{if } i = 2, \\ 5n+5, & \text{if } i \text{ is even and } i \neq 2, \\ 5n+7, & \text{if } i \text{ is odd.} \end{cases}$$
$$w(c) = \frac{n(3n+3)}{2}$$

Notice that $w(v_2) = w(x_{n-4})$. Hence, $\chi_{la}(H_n) \leq n + 3$ for n is odd. We conclude $\chi_{la}(H_n) = n + 3$ if n is odd.

Since every case is covered, then the theorem holds.

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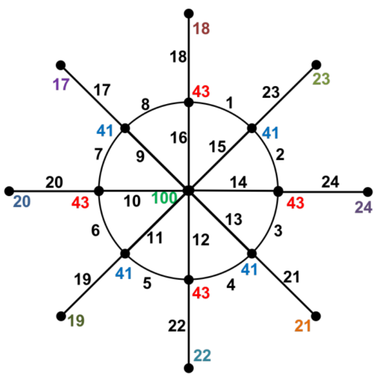

Figure 6: Local Antimagic Vertex Coloring of H_8.

4 CONCLUSIONS

With preceding researchs, we have completed all exact value of $\chi_{la}(W_n)$ and $\chi_{la}(H_n)$ for any integer n. Future researchers are recommended to study the value of χ_{la} for any other class of graphs.

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