Propagation of Reaction Front in Porous Media under Natural Convection

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Abstract: This paper examines the influence of convective instability on the reaction front propagation in porous media. The model includes heat and concentration equations and motion equations under Boussinesq approximation. The non-stationary Darcy equation is adopted and the fluid is supposed to be incompressible. Numerical results are performed via the dispersion relation. The simulations show that the propogating reaction front loses stability as Vadasz number increase, it shows also more stability is gained when Zeldovich number increased.

1 INTRODUCTION

The reaction front propagation in porous media presents much important interest in many technological and scientifical fields like biology, ecology, groundwater, hydrology, oil recovery to name just a few (see for instance (Murphy, 2000; Linninger, 2008; Stevens, 2009; Nield, 2006; Gaikwad, 2015; Faradonbeh, 2015); and references therein). In order to describe the flow dynamics in porous media, one can use the quasi-stationary or the non-stationary Darcy equation combined to reaction-diffusion equations. In porous media, the reaction front propagation under convective effect has been studied by (Allali et al., 2007; Allali, 2014); the authors have used the reaction-diffusion equations coupled with a quasi-stationary Darcy equation. In an other work, the authors in (Khalfi. O, 2016) use non-sationary Darcy equation and they study the influence of the reaction front by depicting the evolution of Zeldovich number versus wave number and versus Vadasz number highlight the influence of Vasdasz number on convective instability. In this work, an extension of this last study will be done to the same model wich describ the reaction front propagation with non-stationary Darcy equation; that model will be considered and delivred to linear stability analysis. For this objectif, the porous media is considerd to be filled by an incompressible fluid, morover the reaction front propagates in opposite sense of gravity. To perform the linear stability analysis, the Zeldovich-Frank-Kamenetskii method is used. The interface problem is obtained and the stability boundary is found from the derived dispersion relation.

This paper is organized as follows. The next section will be devoted to the governing equations, followed in Section 3 by the linear stability analysis. Numerical simulations are performed in Section 4. last section concludes the work.

2 GOVERNING EQUATIONS

The propagating reaction front in porous media can be modelled as follows:

$$\frac{\partial T}{\partial t} + v.\nabla T = \kappa \Delta T + qK(T)\phi(\alpha), \quad (1)$$

$$\frac{\partial \alpha}{\partial t} + v.\nabla \alpha = d\Delta\alpha + K(T)\phi(\alpha), \quad (2)$$

$$\frac{\partial v}{\partial t} + \frac{\mu}{K} v + \nabla p = g\beta\rho(T - T_0)\gamma, \quad (3)$$

$$\nabla . v = 0, \quad (4)$$

under the following boundary conditions:

$$T = T_i,\ \alpha = 1 \text{ and } v = 0 \text{ when } y \to +\infty, \quad (5)$$

$$T = T_b,\ \alpha = 0 \text{ and } v = 0 \text{ when } y \to -\infty, \quad (6)$$

here β denotes the coefficient of thermal expansion, μ the viscosity, K_p the permeability of the porous media, γ the upward unit vector in the vertical direction, T the temperature, α the depth of conversion, v

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the velocity, p the pressure, κ the coefficient of thermal diffusivity, d the diffusion coefficient, q the adiabatic heat release, g the gravity acceleration, ρ the density. Finally, ∇ and Δ denote the standard differential operators,

in addition T_0 is the mean value of the temperature, T_i is an initial temperature while T_b is the temperature of the reacted mixture given by $T_b = T_i + q$. The function $K(T)\phi(\alpha)$ is the reaction rate where the temperature dependence is given by the Arrhenius exponential:

$$K(T) = k_0 \exp\Big(-\frac{E}{R_0 T}\Big), \quad (7)$$

E means the activation energy, R_0 the universal gas constant and k_0 the pre-exponential factor.

2.1 The Dimensionless Model

For the dimensionless model, we introduce the spatial variables $y' = \frac{yc}{\kappa}$, $x' = \frac{xc}{\kappa}$, time $t' = \frac{tc^2}{\kappa}$, velocity $\mathbf{v} = \frac{v}{c}$ and the pressure $p' = \frac{p\mu\kappa}{K}$ are introduced. Denoting $\theta = \frac{T - T_b}{q}$ and omitting the primes of the new variables, the model becomes:

$$\frac{\partial\theta}{\partial t} + \mathbf{v}.\nabla\theta = \Delta\theta + W(\theta)\phi(\alpha), \quad (8)$$

$$\frac{\partial\alpha}{\partial t} + \mathbf{v}.\nabla\alpha = \Lambda\Delta\alpha + W(\theta)\phi(\alpha), \quad (9)$$

$$\frac{1}{V_a}\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v} + \nabla p = R_p(\theta + \theta_0)\gamma, \quad (10)$$

$$\nabla.\mathbf{v} = 0, \quad (11)$$

which is supllemented by the following conditions in infinity:

$$\theta = -1,\ \alpha = 1 \text{ and } \mathbf{v} = 0 \text{ when } y \to +\infty, \quad (12)$$

$$\theta = 0,\ \alpha = 0 \text{ and } \mathbf{v} = 0 \text{ when } y \to -\infty. \quad (13)$$

Here $W(\theta) = Z\exp\Big(\frac{\theta}{Z^{-1} + \delta\theta}\Big)$, where $Z = \frac{qE}{R_0 T_b^2}$ is the Zeldovich number, $\delta = \frac{R_0 T_b}{E}$,
$\theta_0 = \frac{T_b - T_0}{q}$, $\Lambda = \frac{d}{\kappa}$ is the inverse of Lewis number, $R_p = \frac{K_p c^2 P_r^2 R}{\mu^2}$ where R is the Rayleigh number defined by $R = \frac{g\beta q\kappa^2}{\mu c^3}$, P_r is the Prandtl number defined by $P_r = \frac{\mu}{\kappa}$, $V_a = \frac{\kappa\mu}{K_p c^2}$ is the Vadasz number $\Big($called also Darcy-Prandtl number; V_a wich is given by $V_a = \frac{D_a}{P_r}$, where D_a is the Darcy number defined by $D_a = \frac{c^2 K_p}{\kappa^2}\Big)$. In what follows, $\Lambda = 0$ will be considered, wich corresponds to liquid mixture.

3 LINEAR STABILITY ANALYSIS

3.1 Approximation of Infinitely Narrow Reaction Zone

By using an analytical study which reduces the problem (8)-(13) to a singular perturbation one, where the reaction zone is assumed to be narrow (which is named the reaction front), the linear stability analysis is performed. The reaction source term is neglected ahead of the reaction front (because the temperature is not sufficiently high) and behind the reaction front (since in this region there are no fresh reactant left). This approach, called Zeldovich-Frank-Kamenetskii approximation, leads to an interface problem by applying a formal asymptotic analysis for large Zeldovich number. In other words, for this asymptotic analysis treatment, let considers $\varepsilon = \frac{1}{Z}$ as a small parameter. And let assum by $\zeta(t,x)$ the location of the reaction zone, and consider the new independent variable is

$$y_1 = y - \zeta(t,x). \quad (14)$$

Taking into account those new statements, the new functions θ_1, α_1, $\mathbf{v}_1$, p_1 can be written as:

$$\theta(t,x,y) = \theta_1(t,x,y_1),\ \alpha(t,x,y) = \alpha_1(t,x,y_1),$$
$$\mathbf{v}(t,x,y) = \mathbf{v}_1(t,x,y_1),\ p(t,x,y) = p_1(t,x,y_1). \quad (15)$$

By the way, the equations will be re-written in the following form (the index 1 for the dependent variables is omitted):

$$\frac{\partial\theta}{\partial t} - \frac{\partial\theta}{\partial y_1}\frac{\partial\zeta}{\partial t} + \mathbf{v}\cdot\widetilde{\nabla}\theta = \widetilde{\Delta}\theta + W(\theta)\phi(\alpha), \quad (16)$$

$$\frac{\partial\alpha}{\partial t} - \frac{\partial\alpha}{\partial y_1}\frac{\partial\zeta}{\partial t} + \mathbf{v}\cdot\widetilde{\nabla}\alpha = W(\theta)\phi(\alpha), \quad (17)$$

$$\frac{1}{V_a}\Big(\frac{\partial\mathbf{v}}{\partial t} - \frac{\partial\mathbf{v}}{\partial y_1}\frac{\partial\zeta}{\partial t}\Big) + \mathbf{v} + \tilde{\nabla}p = R_p(\theta + \theta_0)\gamma, \quad (18)$$

$$\frac{\partial v_x}{\partial x} - \frac{\partial v_x}{\partial y_1}\frac{\partial\zeta}{\partial x} + \frac{\partial v_y}{\partial y_1} = 0, \quad (19)$$

where

$$\widetilde{\Delta} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y_1^2} - 2\frac{\partial\zeta}{\partial x}\frac{\partial^2}{\partial x\partial y_1} + \Big(\frac{\partial\zeta}{\partial x}\Big)^2\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial}{\partial y_1}, \quad (20)$$

$$\widetilde{\nabla} = \Big(\frac{\partial}{\partial x} - \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_1}\Big). \qquad (21)$$

To find the jump conditions approximation and solution of the interface problem, the matched asymptotic expansions method is used. To this aim, the outer solution of the problem is sought in the form:

$$\theta = \theta^{\check{0}} + \varepsilon\theta^{\check{I}} + ..., \quad \alpha = \alpha^{\check{0}} + \varepsilon\alpha^{\check{I}} + ...,$$
$$\mathbf{v} = \mathbf{v}^{\check{0}} + \varepsilon\mathbf{v}^{\check{I}} + ..., \quad p = p^{\check{0}} + \varepsilon p^{\check{I}} + ... \qquad (22)$$

For the inner solution, the stretching coordinate $\eta = y_1/\varepsilon$ is introduced and the inner solution is sought in the following form:

$$\theta = \varepsilon\tilde{\theta}^{\check{I}} + ..., \quad \alpha = \tilde{\alpha}^{\check{0}} + \varepsilon\tilde{\alpha}^{\check{I}} + ...,$$
$$\mathbf{v} = \tilde{\mathbf{v}}^{\check{0}} + \varepsilon\tilde{\mathbf{v}}^{\check{I}} + ..., \quad p = \tilde{p}^{\check{0}} + \varepsilon\tilde{p}^{\check{I}} + ..., \qquad (23)$$
$$\zeta = \tilde{\zeta}^{\check{0}} + \varepsilon\tilde{\zeta}^{\check{I}} + ...$$

Substituting these expansions into (16)-(19), it follows:

order ε^{-1}

$$\Big(1 + \Big(\frac{\partial\tilde{\zeta}^{\check{0}}}{\partial x}\Big)^2\Big)\frac{\partial^2\tilde{\theta}^{\check{I}}}{\partial\eta^2} + \exp\Big(\frac{\tilde{\theta}^{\check{I}}}{1+\delta\tilde{\theta}^{\check{I}}}\Big)\phi(\tilde{\alpha}^{\check{0}}) = 0, \qquad (24)$$

$$-\frac{\partial\tilde{\alpha}^{\check{0}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{0}}}{\partial t} - \frac{\partial\tilde{\alpha}^{\check{0}}}{\partial\eta}\Big(\tilde{v}_x^{\check{0}}\frac{\partial\tilde{\zeta}^{\check{0}}}{\partial x} - \tilde{v}_y^{\check{0}}\Big) = \exp\Big(\frac{\tilde{\theta}^{\check{I}}}{1+\delta\tilde{\theta}^{\check{I}}}\Big)\phi(\tilde{\alpha}^{\check{0}}), \qquad (25)$$

$$-\frac{1}{V_a}\frac{\partial\tilde{v}_x^{\check{0}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{0}}}{\partial t} - \frac{\partial\tilde{p}^{\check{0}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{0}}}{\partial x} = 0, \qquad (26)$$

$$-\frac{1}{V_a}\frac{\partial\tilde{v}_y^{\check{0}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{0}}}{\partial t} - \frac{\partial\tilde{p}^{\check{0}}}{\partial\eta} = 0, \qquad (27)$$

$$-\frac{\partial\tilde{v}_x^{\check{0}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{0}}}{\partial x} + \frac{\partial\tilde{v}_y^{\check{0}}}{\partial\eta} = 0, \qquad (28)$$

order ε^0

$$\frac{1}{V_a}\Big(\frac{\partial v_x^{\check{0}}}{\partial t} - \frac{\partial v_x^{\check{I}}}{\partial\eta}\frac{\partial\zeta^{\check{0}}}{\partial t} - \frac{\partial\zeta^{\check{I}}}{\partial t}\frac{\partial v_x^{\check{0}}}{\partial\eta}\Big) + \frac{\partial p^{\check{0}}}{\partial x} - \frac{\partial p^{\check{0}}}{\partial\eta}\frac{\partial\zeta^{\check{I}}}{\partial x} - \frac{\partial p^{\check{I}}}{\partial\eta}\frac{\partial\zeta^{\check{0}}}{\partial x} + v_x^{\check{0}} = R_p\theta_0, \qquad (29)$$

$$\frac{1}{V_a}\Big(\frac{\partial v_y^{\check{0}}}{\partial t} - \frac{\partial v_y^{\check{I}}}{\partial\eta}\frac{\partial\zeta^{\check{0}}}{\partial t} - \frac{\partial\zeta^{\check{I}}}{\partial t}\frac{\partial v_y^{\check{0}}}{\partial\eta}\Big) + \frac{\partial p^{\check{I}}}{\partial\eta} + v_y^{\check{0}} = R_p\theta_0, \qquad (30)$$

order ε^1

$$\frac{1}{V_a}\Big(\frac{\partial\tilde{v}_x^{\check{0}}}{\partial t} - \frac{\partial\tilde{v}_x^{\check{0}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{2}}}{\partial t} - \frac{\partial\zeta^{\check{I}}}{\partial\eta}\frac{\partial\tilde{v}_x^{\check{I}}}{\partial t} - \frac{\partial\tilde{\zeta}^{\check{2}}}{\partial\eta}\frac{\partial\tilde{v}_x^{\check{0}}}{\partial t}\Big) + \frac{\partial\tilde{p}^{\check{I}}}{\partial x} - \frac{\partial\tilde{p}^{\check{0}}}{\partial\eta}\frac{\partial\zeta^{\check{2}}}{\partial x} - \frac{\partial\tilde{p}^{\check{I}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{I}}}{\partial x} - \frac{\partial\tilde{p}^{\check{2}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{0}}}{\partial x} + v_x^{\check{I}} = R_p\tilde{\theta}^{\check{I}}, \qquad (31)$$

$$\frac{1}{V_a}\Big(\frac{\partial\tilde{v}_y^{\check{0}}}{\partial t} - \frac{\partial\tilde{v}_y^{\check{0}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{2}}}{\partial t} - \frac{\partial\zeta^{\check{I}}}{\partial\eta}\frac{\partial\tilde{v}_y^{\check{I}}}{\partial t} - \frac{\partial\tilde{\zeta}^{\check{2}}}{\partial\eta}\frac{\partial\tilde{v}_y^{\check{0}}}{\partial t}\Big) + \frac{\partial\tilde{p}^{\check{2}}}{\partial\eta} + v_y^{\check{I}} = R_p\tilde{\theta}^{\check{I}}, \qquad (32)$$

order ε^2

$$\frac{\partial\tilde{v}_x^{\check{2}}}{\partial x} - \frac{\partial\tilde{v}_x^{\check{3}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{0}}}{\partial x} - \frac{\partial\tilde{v}_x^{\check{2}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{I}}}{\partial x} - \frac{\partial\tilde{v}_x^{\check{I}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{2}}}{\partial x} - \frac{\partial\tilde{v}_x^{\check{0}}}{\partial\eta}\frac{\partial\tilde{\zeta}^{\check{3}}}{\partial x} + \frac{\partial\tilde{v}_y^{\check{3}}}{\partial\eta} = 0. \qquad (33)$$

The matching conditions as $\eta \to \pm\infty$ are

$$\tilde{v}^{\check{0}} \sim v^{\check{0}}\Big|_{y_1=\pm 0}, \qquad (34)$$

$$\tilde{v}^{\check{I}} \sim \Big(\frac{\partial v^{\check{0}}}{\partial y_1}\Big|_{y_1=\pm 0}\Big)\eta + v^{\check{I}}\Big|_{y_1=\pm 0}, \qquad (35)$$

$$\tilde{v}^{\check{2}} \sim \frac{1}{2}\Big(\frac{\partial^2 v^{\check{0}}}{\partial y_1^2}\Big|_{y_1=\pm 0}\Big)\eta^2 + \Big(\frac{\partial v^{\check{I}}}{\partial y_1}\Big|_{y_1=\pm 0}\Big)\eta + v^{\check{2}}\Big|_{y_1=\pm 0}, \qquad (36)$$

$$\tilde{v}^{\check{3}} \sim \frac{1}{6}\Big(\frac{\partial^3 v^{\check{0}}}{\partial y_1^3}|_{y_1=\pm 0}\Big)\eta^3 + \frac{1}{2}\Big(\frac{\partial^2 v^{\check{I}}}{\partial y_1^2}\Big|_{y_1=\pm 0}\Big)\eta^2 + \Big(\frac{\partial v^{\check{2}}}{\partial y_1}\Big|_{y_1=\pm 0}\Big)\eta + v^{\check{3}}\Big|_{y_1=\pm 0}. \qquad (37)$$

Let us consider both the outer and the inner expansion of the temperature (the same technique for the variables α and $\mathbf{v}$ is used) and recall that $\eta = y_1/\varepsilon$:

$$\theta(x, y_1) = \theta^{\check{0}}(x, y_1) + \varepsilon\theta^{\check{I}}(x, y_1) + \varepsilon^2\theta^{\check{2}}(x, y_1) + ...,$$
$$\theta(x, \varepsilon\eta) = \varepsilon\tilde{\theta}^{\check{I}}(x, \eta) + \varepsilon^2\tilde{\theta}^{\check{2}}(x, \eta) + ...$$

Writing the outer solution in terms of the inner variable η and using the Taylor expansion:

$$\theta(x, y_1) = \theta^{\check{0}}(x, 0) + \varepsilon\Big(\frac{\partial\theta^{\check{0}}}{\partial\eta}(x, 0)\eta + \theta^{\check{I}}(x, 0)\Big)$$

$$+\varepsilon^2\Big(\frac{1}{2}\frac{\partial^2\theta^{\breve{o}}}{\partial\eta^2}(x,0)\eta^2+\frac{\partial\theta^{\breve{i}}}{\partial\eta}(x,0)\eta+\theta^{\breve{2}}(x,0)\Big)+....$$

The zero order terms with respect to ε correspond to the stationary solution. Taking into account that $\frac{\partial\theta^{\breve{o}}}{\partial\eta}(x,0^-)=0$ and using the matching principle (Nayfeh, 2008), it follows the matching conditions:

$$\eta\to+\infty:\tilde{\theta}^{\breve{i}}\sim\theta^{\breve{i}}\Big|_{y_1=0^+}+\eta\frac{\partial\theta^{\breve{o}}}{\partial y_1}\Big|_{y_1=0^+},$$

$$\tilde{\alpha}^{\breve{o}}\to 0,\ \tilde{\mathbf{v}}^{\breve{o}}\to\mathbf{v}^{\breve{o}}\Big|_{y_1=0^+},\quad(38)$$

$$\eta\to-\infty:\ \tilde{\theta}^{\breve{i}}\to\theta^{\breve{i}}\Big|_{y_1=0^-},\ \tilde{\alpha}^{\breve{o}}\to 1,$$

$$\tilde{\mathbf{v}}^{\breve{o}}\to\mathbf{v}^{\breve{o}}\Big|_{y_1=0^-}.\quad(39)$$

Denoting by s the quantity

$$s=\tilde{v}_x^{\breve{o}}\frac{\partial\tilde{\zeta}^{\breve{o}}}{\partial x}-\tilde{v}_y^{\breve{o}},\quad(40)$$

Relation (28) gives that the parameter s does not depend on η.

The jump conditions are derived from (24), in the same way as it is usually done for combustion problems. From (25), it follows that $\tilde{\alpha}^{\breve{o}}$ is a monotone function of η and $0<\tilde{\alpha}^{\breve{o}}<1$. Since zero-order reaction is considered, it results $\phi(\tilde{\alpha}^{\breve{o}})\equiv 1$. Using (24), one concludes that $\frac{\partial^2\tilde{\theta}^{\breve{i}}}{\partial\eta^2}\le 0$. when $\eta\to-\infty$ and taking into account (39), one has $\frac{\partial\tilde{\theta}^{\breve{i}}}{\partial\eta}=0$. Then $\frac{\partial\tilde{\theta}^{\breve{i}}}{\partial\eta}\le 0$ and $\tilde{\theta}^{\breve{i}}$ is a monotone function.

Multiplying (24) by $\frac{\partial\tilde{\theta}^{\breve{i}}}{\partial\eta}$ and integrating with respect to η from $-\infty$ to $+\infty$ taking into account (38), (39) ($\tilde{\theta}^{\breve{i}}$ changes from $\theta^{\breve{i}}|_{y_1=0^-}$ to $-\infty$ when η changes from $-\infty$ to $+\infty$):

$$\left(\frac{\partial\tilde{\theta}^{\breve{i}}}{\partial\eta}\right)^2\Big|_{\eta=+\infty}-\left(\frac{\partial\tilde{\theta}^{\breve{i}}}{\partial\eta}\right)^2\Big|_{\eta=-\infty}=$$

$$\frac{2}{A}\int_{-\infty}^{\theta^{\breve{i}}|_{y_1=0^-}}\exp\Big(\frac{\tau}{1+\delta\tau}\Big)d\tau,\quad(41)$$

where

$$A=1+\Big(\frac{\partial\tilde{\zeta}^{\breve{o}}}{\partial x}\Big)^2.\quad(42)$$

By adding (24) and (25) and integrating, one obtains:

$$\frac{\partial\tilde{\theta}^{\breve{i}}}{\partial\eta}\Big|_{\eta=+\infty}-\frac{\partial\tilde{\theta}^{\breve{i}}}{\partial\eta}\Big|_{\eta=-\infty}=-\frac{1}{A}\Big(\frac{\partial\tilde{\zeta}^{\breve{o}}}{\partial t}+s\Big).\quad(43)$$

Truncating the outer expansion (22), it follows that $\theta\approx\theta^{\breve{o}}$, $\zeta^{\breve{o}}\approx\zeta$, $\mathbf{v}\approx\mathbf{v}^{\breve{o}}$. From the inner expansion (23), one obtains $Z\theta\approx\tilde{\theta}^{\breve{i}}$, and from the matching condition (39), $\tilde{\theta}^{\breve{i}}\Big|_{\eta=-\infty}\approx\theta^{\breve{i}}\Big|_{y_1=0^-}\approx Z\theta|_{y_1=0^-}$. Thus,

$$\theta^{\breve{o}}\approx\theta,\ \theta^{\breve{i}}\Big|_{y_1=0^-}\approx Z\theta\Big|_{y_1=0},\ \zeta^{\breve{o}}\approx\zeta,\ \mathbf{v}\approx\mathbf{v}^{\breve{o}},\quad(44)$$

The jump conditions are obtained (with the change of variables $\tau\to Z\tau$ under the integral)

$$\left(\frac{\partial\theta}{\partial y_1}\right)^2\Bigg|_{y_1=0^+}-\left(\frac{\partial\theta}{\partial y_1}\right)^2\Bigg|_{y_1=0^-}=2Z\left(1+\left(\frac{\partial\zeta}{\partial x}\right)^2\right)^{-1}$$

$$\times\int_{-\infty}^{\theta|_{y_1=0}}\exp\left(\frac{\tau}{Z^{-1}+\delta\tau}\right)d\tau,\quad(45)$$

$$\frac{\partial\theta}{\partial y_1}\Big|_{y_1=0^+}-\frac{\partial\theta}{\partial y_1}\Big|_{y_1=0^-}=-\Big(1+\Big(\frac{\partial\zeta}{\partial x}\Big)^2\Big)^{-1}$$

$$\times\Big(\frac{\partial\zeta}{\partial t}+\Big(v_x\frac{\partial\zeta}{\partial x}-v_y\Big)\Big|_{y_1=0}\Big),\quad(46)$$

$$v_z\Big|_{z_1=0^+}=v_z\Big|_{z_1=0^-},\quad(47)$$

$$\frac{\partial v_y}{\partial y_1}\Big|_{y_1=0^+}=\frac{\partial v_y}{\partial y_1}\Big|_{y_1=0^-},\quad(48)$$

$$\frac{\partial^2 v_y}{\partial y_1^2}\Big|_{y_1=0^+}=\frac{\partial^2 v_y}{\partial y_1^2}\Big|_{y_1=0^-},\quad(49)$$

$$\frac{\partial^3 v_y}{\partial y_1^3}\Big|_{y_1=0^-}-\frac{\partial^3 v_y}{\partial y_1^3}\Big|_{y_1=0^+}=$$

$$-R_p\left(1+\left(\frac{\partial\zeta}{\partial x}\right)^2\right)^{-1}\times$$

$$\left(\left(1+\left(\frac{\partial\zeta}{\partial x}\right)^2\right)^{-1}-1\right)\frac{\partial\theta}{\partial y}\Big|_{y_1=0^+}.\quad(50)$$

Here only the component v_y of the velocity is used.

3.2 Formulation of the Interface Problem

The interface problem will be written in two system of equations one for the media before reaction, and one for the media after reaction; and for the jump conditions they will be extracted from the free boundary problem seen previously.

To find the jump conditions approximation and solution of the interface problem, the matched asymptotic expansions method is used, this leads to:

For unburnt medium ($y > \zeta$):

$$\frac{\partial\theta}{\partial t} + \mathbf{v}\cdot\nabla\theta = \Delta\theta, \tag{51}$$

$$\alpha \equiv 0, \tag{52}$$

$$\frac{1}{V_a}\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} + \nabla p = R_p(\theta+\theta_0)\gamma, \tag{53}$$

$$\nabla\cdot\mathbf{v} = 0. \tag{54}$$

And for the burnt medium ($y < \zeta$):

$$\frac{\partial\theta}{\partial t} + \mathbf{v}\cdot\nabla\theta = \Delta\theta, \tag{55}$$

$$\alpha \equiv 1, \tag{56}$$

$$\frac{1}{V_a}\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} + \nabla p = R_p(\theta+\theta_0)\gamma, \tag{57}$$

$$\nabla\cdot\mathbf{v} = 0. \tag{58}$$

These systems is completed by the following jump conditions at the interface $y = \zeta$:

$$[\theta] = 0, \ \left[\frac{\partial\theta}{\partial y}\right] = \frac{\frac{\partial\zeta}{\partial t}}{1+\left(\frac{\partial\zeta}{\partial x}\right)^2}, \tag{59}$$

$$\left[\left(\frac{\partial\theta}{\partial y}\right)^2\right] = -\frac{2Z}{1+(\frac{\partial\zeta}{\partial x})^2}\int_{-\infty}^{\theta(t,x,\zeta)} \exp\left(\frac{s}{1/Z+\delta s}\right)ds, \tag{60}$$

$$[\mathbf{v}] = 0, \tag{61}$$

$$\left[\frac{\partial v}{\partial y}\right] = 0, \tag{62}$$

$$\left[\frac{\partial^2 v}{\partial y^2}\right] = 0, \tag{63}$$

with [ ] denotes the jump at the interface,

$$[f] = f|_{\zeta_{0+}} - f|_{\zeta_{0-}}.$$

This free boundary problem is completed by the following conditions:

$$\theta = -1 \text{ and } \mathbf{v} = 0 \text{ when } y \to +\infty, \tag{64}$$

$$\theta = 0 \text{ and } \mathbf{v} = 0 \text{ when } y \to -\infty. \tag{65}$$

3.3 Travelling Wave Solution

In this subsection, the linear stability analysis of the steady-state solution for the interface problem is performed. Indeed, this interface problem has a travelling wave solution $(\theta, \alpha, \mathbf{v})$ such as:

$$\theta(t,x,y) = \theta_s(y-ut), \ \alpha(t,x,y) = \alpha_s(y-ut), \quad \mathbf{v} = 0, \zeta = ut, \tag{66}$$

with

$$\theta_s(y_1) = \begin{cases} 0 & \text{if } y_1 < 0 \\ e^{-uy_1} - 1 & \text{if } y_1 > 0 \end{cases} \tag{67}$$

and

$$\alpha_s(y_1) = \begin{cases} 1 & \text{if } y_1 < 0 \\ 0 & \text{if } y_1 > 0 \end{cases}, \tag{68}$$

here $y_1 = y - ut$ denotes the moving coordinate frame, where u is the wave speed. With these coordinates, the travelling wave is a stationary solution of the problem:

$$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial y_1} + \mathbf{v}\cdot\nabla\theta = \Delta\theta, \tag{69}$$

$$\frac{1}{V_a}\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} + \nabla p = R_p(\theta+\theta_0)\gamma, \tag{70}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{71}$$

supplemented with the jump conditions (59)-(63).

Consider now a small time-dependent perturbation of the stationary solution of the form. To be done, let's choose the perturbation of the temperature and the velocity in the following form:

$$\zeta(t,x) = ut + \xi(t,x), \tag{72}$$

$$\begin{aligned} &\theta(t,x,y) = \theta_s(y-ut) + \theta_j(z)e^{\omega t + ikx}, \ j = 1,2, \\ &v_y(t,x,y) = v_j(z)e^{\omega t + ikx}, \end{aligned} \tag{73}$$

where

$$z = y - \zeta(t,x) = y - ut - \xi(t,x), \ \xi(t,x) = \varepsilon_1 e^{\omega t + ikx}, \tag{74}$$

$j = 1$ corresponds to $z < 0$ and $j = 2$ to $z > 0$.

Now, the system (69)-(71) needs to be linearized around the stationary solution. For simplicity, the pressure p and the component v_x of the velocity will be eliminated from the interface problem by applying two times the operator *curl*. Hence, this expression is obtained by applying this transformation to the equation (70):

$$\frac{1}{V_a}\frac{\partial}{\partial t}(\nabla\times(\nabla\times\mathbf{v})) + \nabla\times(\nabla\times\mathbf{v}) = R_p\nabla\times(\nabla\times(\theta+\theta_0)\gamma),$$

thus

$$\frac{1}{V_a}\frac{\partial}{\partial t}(\nabla(\nabla.\mathbf{v}) - \triangle\mathbf{v}) + \nabla(\nabla.\mathbf{v}) - \triangle\mathbf{v}$$
$$= R_p(\nabla(\nabla.(\theta+\theta_0)\gamma) - \triangle(\theta+\theta_0)\gamma),$$

due to the incompressibility of the fluid, it deduced:

$$\frac{1}{V_a}\frac{\partial}{\partial t}(-\triangle\mathbf{v}) + (-\triangle\mathbf{v}) = -R_p(\nabla(\nabla.(\theta\gamma)) - \triangle\theta\gamma),$$

which drive to

$$\frac{-1}{V_a}\frac{\partial}{\partial t}(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}) - (\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}) = R_p(\frac{\partial^2\theta}{\partial y^2} - (\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2})),$$

by replacing v_y by its value (73), it result

$$\frac{1}{V_a}\frac{\partial}{\partial t}(-k^2 v_j e^{\omega t+ikx} + v_j^{"} e^{\omega t+ikx}) + (-k^2 v_j e^{\omega t+ikx} + v_j^{"} e^{\omega t+ikx}) = R_p(-k^2\theta_j)e^{\omega t+ikx},$$

so

$$\frac{\omega}{V_a}(-k^2 v_j + v_j^{"}) + (-k^2 v_j + v_j^{"}) = R_p(-k^2\theta_j).$$

From below, the following differential systems of equations are obtained:

For $z<0$:

$$\theta_1'' + u\theta_1' - (\omega+k^2)\theta_1 = 0, \quad (75)$$

$$v_1^{"} - k^2 v_1 = -\frac{V_a}{\omega+V_a}R_p k^2\theta_1. \quad (76)$$

For $z>0$:

$$\theta_2^{"} + u\theta_2^{'} - (\omega+k^2)\theta_2 = -v_2 u e^{-uz}, \quad (77)$$

$$v_2^{"} - k^2 v_2 = -\frac{V_a}{\omega+V_a}R_p k^2\theta_2. \quad (78)$$

These systems is supplemented with the following linearized jump conditions:

$$\theta_2(0) - \theta_1(0) = u\varepsilon_1, \quad (79)$$

$$\theta_2'(0) - \theta_1'(0) = -\varepsilon_1(u^2+\omega), \quad (80)$$

$$-u(u^2\varepsilon_1 + \theta_2'(0)) = Z\theta_1(0), \quad (81)$$

$$v_1^{(i)}(0) = v_2^{(i)}(0) \quad \text{for } i=0,1, \quad (82)$$

where

$$v^{(i)} = \frac{\partial^i v}{\partial z^i},\ \theta' = \frac{\partial\theta}{\partial y_1},$$

for $z<0$, the resolution of (75) and (76) gives:

$$\theta_1(z) = a_1 e^{\mu z},$$
$$v_1(z) = a_1\frac{R_p k^2 V_a}{(k^2-\mu^2)(\omega+V_a)}e^{\mu z} + a_2 e^{kz}$$
$$\text{with } \mu = \frac{-u+\sqrt{u^2+4(\omega+k^2)}}{2} \quad (83)$$

and for $z>0$, the resolution of (77) and (78) gives:

$$v_2(z) = b_1 w_1 + b_2 w_2,\ \theta_2(z) = b_1 s_1 + b_2 s_2,$$

with

$$w_i(z) = \sum_{j\ge1} a_{i,j}e^{\sigma_{i,j}z} \text{ and } s_i(z) = \sum_{j\ge1} c_{i,j}e^{\sigma_{i,j}z} \quad \text{for } i=1,2 \quad (84)$$

here a_1, b_1, a_2, and b_2 are arbitrary constants and the coefficients $a_{i,j}$, $\sigma_{i,j}$ are given by:

$$\sigma_{1,1} = \frac{-u-\sqrt{u^2+4(\omega+k^2)}}{2},\ \sigma_{2,1} = -k,$$

$$c_{1,1}=1,\ c_{2,1}=0,$$
$$a_{1,1}=0\ a_{2,1}=1,$$

$$\forall j\ge1: ;\sigma_{i,j+1} = \sigma_{i,j} - u \ \text{ for } i=1,2,$$
$$\forall j\ge1: ;c_{i,j+1} = \frac{-u}{\sigma_{i,j+1}^2 + u\sigma_{i,j+1}^2 - (\omega+k^2)}a_{i,j}$$

for $i=1,2$,

with respect of the conditions:

$$(V_a+\omega)\neq0,$$
$$\forall j\ge1: \sigma_{i,j+1}^2 + u\sigma_{i,j+1}^2 - (\omega+k^2)\neq0$$
for $i=1,2$,
$$\forall j\ge1: (\sigma_{i,j}^2 - k^2)\neq0$$
for $i=1,2$,

$$\forall j\ge1:\quad a_{i,j} = \frac{-R_p k^2 V_a}{(V_a+\omega)(\sigma_{i,j}^2-k^2)}c_{i,j} \quad (85)$$

for $i=1,2$ and $(i,j)\neq(2,1)$.

The constants are sought from the jump conditions

$$\begin{cases} b_1 s_1(0) + b_2 s_2(0) - a_1 = u\varepsilon_1 \\ b_1 s_1'(0) + b_2 s_2'(0) - \mu a_1 = -\varepsilon_1 (u^2 + \omega) \\ b_1 s_1'(0) + b_2 s_2'(0) + a_1 \dfrac{Z}{u} = -u^2 \varepsilon_1 \\ b_1 w_1(0) + b_2 w_2(0) + \dfrac{V_a R_p k^2 a_1}{(\omega + V_a)(\mu^2 - k^2)} - a_2 = 0 \\ b_1 w_1'(0) + b_2 w_2'(0) + \dfrac{V_a R_p k^2 \mu a_1}{(\omega + V_a)(\mu^2 - k^2)} - k a_2 = 0 \end{cases} \quad (86)$$

To find the dispersion relation, it is convenient ot subtracts the third equation from second equation of the system (86), wich gives:

$$a_1 = \frac{u\omega}{u\mu + Z}\varepsilon_1. \quad (87)$$

Now by using the right linear combinaison of the first and second lines in (86), the coefficients b_1 and b_2 are obtained

$$b_1 = \frac{(a_1 + u\varepsilon_1)s_2'(0) + (\varepsilon_1(u^2 + \omega) - \mu a_1)s_2(0)}{s_2'(0)s_1(0) - s_2(0)s_1'(0)}, \quad (88)$$

$$= \frac{(us_2'(0) + (u^2 + \omega)s_2(0))\varepsilon_1 + (s_2'(0) - \mu s_2(0))a_1}{s_2'(0)s_1(0) - s_2(0)s_1'(0)},$$

$$b_2 = \frac{(a_1 + u\varepsilon_1)s_1'(0) + (\varepsilon_1(u^2 + \omega) - \mu a_1)s_1(0)}{s_1'(0)s_2(0) - s_1(0)s_2'(0)}, \quad (89)$$

$$= \frac{(us_1'(0) + (u^2 + \omega)s_1(0))\varepsilon_1 + (s_1'(0) - \mu s_1(0))a_1}{s_1'(0)s_2(0) - s_1(0)s_2'(0)},$$

to obtain a_2, the fourth line in the system (86) will be used. That's leads:

$$a_2 = b_1 w_1(0) + b_2 w_2(0) + \frac{V_a R_p k^2}{(\omega + V_a)(\mu^2 - k^2)} a_1. \quad (90)$$

At end from the fifth equation of the system (86), the following dispersion relation is obtained:

$$\frac{(kw_1(0) - w_1'(0))(Bu\omega + F(u\mu + Z))}{(u\mu + Z)D} + \frac{(Au\omega + E(u\mu + Z))(kw_2(0) - w_2'(0))}{(u\mu + Z)C} + \frac{V_a R_p k^2 u\omega}{(\omega + V_a)(\mu + k)(u\mu + Z)} = 0, \quad (91)$$

with

$$A = s_1'(0) - \mu s_1(0), \; B = s_2'(0) - \mu s_2(0),$$
$$C = -s_1'(0)s_2(0) + s_1(0)s_2'(0),$$
$$D = -s_2'(0)s_1(0) + s_2(0)s_1'(0),$$
$$E = us_1'(0) + (u^2 + \omega)s_1(0),$$
$$F = us_2'(0) + (u^2 + \omega)s_2(0).$$

4 NUMERICAL SIMULATIONS

The dispersion relation (91) is solved numerically by using Maple software and the convergence of the series (84) is tested within small tolerance limit. This gives the stability threshold of the reaction front. From (83), (85) and (91), it's seen that when $\omega = 0$, Vadasz number will not affect the stability thershold. For this reason, those numerical simulations will check the oscillatory stability threshold.

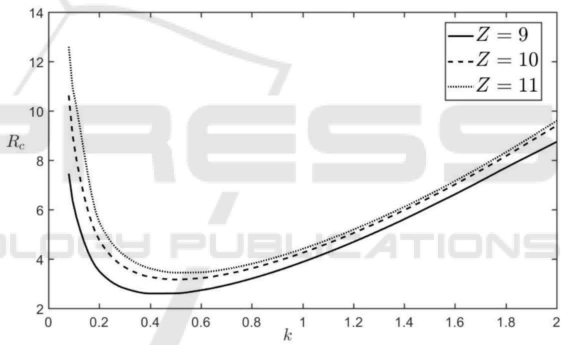

Figure 1: Critical Railey R versus wave number k for different values of Zeldovich number Z and for $V_a = 40$.

Following the same steps as in (Khalfi. O, 2016), the interface problem is sloved by introducing the travelling wave solution and reducing the system to a dispersion relation. Figure 1 shows the evolution of critical Raylaigh number R_c as function of wave number k for different values of Zeldovich number with $u = \sqrt{2}$. Here, it is remarqued that the reaction front gains stability when Zeldovich number increases. It can also be seen that for small values of k ($k < 0.4$) the number critical Rayleig becomes very large and R_c increases with the wave number k when ($k < 0.4$).

The critical Rayleigh number R_c versus Vadasz number for different values of Zeldovich number with $u = \sqrt{2}$ and $V_a = 40$ is shown in figure 2. Here, the lost of stablility is visible when Vadasz number V_a increases, it is also remarqued that the loss of stability is more improtant for a bigger values of Zeldovich number Z.

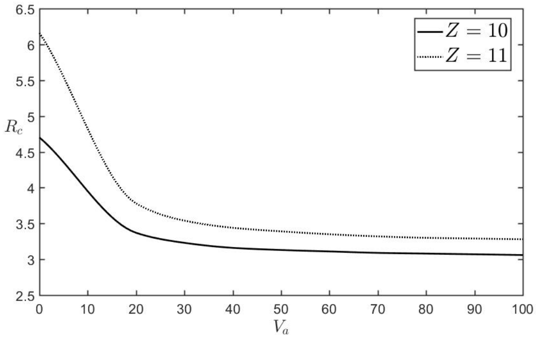

Figure 2: Critical Railey number R versus Vadasz number V_a for different values of Zeldovich number Z and for $k = 0.5$.

5 CONCLUSIONS

In this paper, we have studied a problem decribing the reaction front propagation in porous media. The model consists of two reaction diffusion equations of heat and concentration coupled of those of hydrodynamics under Darcy law.

It was observed that the key parameters of the problem play an essential role in the convective stability of the propagating reaction front. Indeed, it was observed that the reaction front gains stability when Zeldovich number increases, also the lost of stablility is remarked when Vadasz number increases.

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