Extending Narrowband Descriptions and Optimal Solutions to Broadband Sensor Arrays

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Abstract: This overview paper motivates the description of broadband sensor array problems by polynomial matrices, directly extending notation that is familiar from the characterisation of narrowband problems. To admit optimal solutions, the approach relies on extending the utility of the eigen- and singular value decompositions, by finding decompositions of such polynomial matrices. Particularly the factorisation of parahermitian polynomial matrices — including space-time covariance matrices that model the second order statistics of broadband sensor array data — is important. The paper summarises recent findings on the existence and uniqueness of the eigenvalue decomposition of such parahermitian polynomial matrices, demonstrates some algorithms that implement such factorisations, and highlights key applications where such techniques can provide advantages over state-of-the-art solutions.

1 INTRODUCTION

When processing signals obtained from an M-element sensor array in a data vector $\mathbf{x}[n]$, where n is the discrete time index, information on e.g. the angle of arrival of sources is contained in the delay with which different signals arrive at sensors. In the narrowband case, this delay is sufficiently expressed by a phase shift, information on which can be found in e.g. the instantaneous covariance matrix $\mathbf{R} = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n]\}$ of the sensor signals, where $\mathcal{E}\{\cdot\}$ is the expectation operator and $\{\cdot\}^{\mathrm{H}}$ the Hermitian transpose operator. Many narrowband array problems therefore are based on this covariance matrix $\mathbf{R}$, and optimum beamforming and direction finding methods are often subsequently based on factorisations — typically the eigenvalue decomposition (EVD) — of $\mathbf{R}$ (Schmidt, 1986) or equivalently the singular value decomposition (SVD) of the data matrix (Moonen and de Moor, 1995).

In the broadband case, explicit delays must be considered instead of phase shifts. These lags can be capture by the second order statistics via the space-time covariance matrix $\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\}$, which includes a discrete lag parameter τ. Since $\mathbf{R}[\tau]$ contains auto- and cross-correlation terms of $\mathbf{x}[n]$, it inherits the symmetry $\mathbf{R}[\tau] = \mathbf{R}^{\mathrm{H}}[-\tau]$. When taking the z-transform, the resulting cross spectral density (CSD) matrix $\boldsymbol{R}(z) = \sum_{\tau}\mathbf{R}[\tau]z^{-\tau}$ satisfies the parahermitian property $\boldsymbol{R}(z) = \boldsymbol{R}^{\mathrm{P}}(z)$, where the parahermitian operation $\boldsymbol{R}^{\mathrm{P}}(z) = \boldsymbol{R}^{\mathrm{H}}(1/z^*)$ involves Hermitian transposition and time reversal(Vaidyanathan, 1993).A matrix $\mathbf{R}(z)$ that satisfies the parahermitian property is called a parahermitian matrix.

While the polynomial matrix notation $\boldsymbol{R}(z)$ permits the formulation of broadband problems, the utility of the EVD does not naturally extend from the narrowband to the broadband case. If a constant similarity transform is applied to $\boldsymbol{R}(z)$ or $\mathbf{R}[\tau]$, the CSD or space-time covariance matrices can generally only be diagonalised for one single coefficient or lag. Therefore an extension of the EVD to polynomial matrices is required in order to provide solutions for broadband problem formulations. For this purpose, (McWhirter et al., 2007; McWhirter and Baxter, 2004) have defined a polynomial EVD that can approximately diagonalise $\boldsymbol{R}(z)$ for all its coefficients, with recently analysis providing the underpinning theory on the existence of polynomial eigenvalues and -vectors, and the ambiguity of the latter.

Over the past decade, a number of algorithms have emerged that implement a polynomial EVD (McWhirter et al., 2007; Redif et al., 2011; Tohidian et al., 2013; Corr et al., 2014c; Redif et al., 2015; Wang et al., 2015a), and also triggered a range of applications in the area of filter banks (Redif et al., 2011;

Weiss et al., 2006), beamforming (Redif et al., 2006; Koh et al., 2009; Alrmah et al., 2011; Weiss et al., 2013; Vouras and Tran, 2014; Weiss et al., 2015; Alzin et al., 2016), communications (Weiss et al., 2006; Davies et al., 2007; Ta and Weiss, 2007a; Sandmann et al., 2015; Ahrens et al., 2017), or generic theoretical problems such as blind source separation (Redif et al., 2017) or spectral factorisation (Wang et al., 2015b).

The aim of this paper is to provide an overview over efforts in the area of polynomial matrix decompositions, and to offer some insight into the advantages that this may bring for two exemplified applications, Therefore, this paper is organised as followed. Sec. 2 defined the space-time covariance matrix and its parahermitian matrix factorisation and its polynomial approximation. Sec. 3 provides an overview over polynomial matrix EVD algorithms, which are then applied to two problems: Sec. 4 demonstrates the use of polynomial matrix techniques for angle of arrival estimation, while Sec. 5 discussed the applications in broadband beamforming. A conclusion and outlook over related fields is provided in Sec. 6.

2 PARAHERMITIAN MATRIX EVD

Based on a short discourse on space-time covariance and its properties in Sec. 2.1, we define a parahermitian matrix EVD in Sec. 2.2. Its polynomial approximation is discussed in Sec. 2.3.

2.1 Space-time Covariance and Cross-spectral Density Matrices

A scenario where L independent sources with non-negative, real-valued power spectral densities (PSD) $S_\ell(z)$, $\ell = 1 \ldots L$, contribute to M sensor measurements $x_m[n]$, $m = 1 \ldots M$, the space-time covariance matrix of the vector $\mathbf{x}[n] = [x_1[n] \ldots x_M[n]]^{\mathrm{T}}$ is

$$\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n - \tau]\} . \tag{1}$$

If the PSD of the ℓth source is generated by a stable and causal innovation filter $F_\ell(z)$ (Papoulis, 1991), and $H_{m\ell}(z)$ describes the transfer function of the causal and stable system between the ℓth source and the mth sensor, then

$$\mathbf{R}(z) = \mathbf{H}(z) \begin{bmatrix} S_1(z) & & \\ & \ddots & \\ & & S_L(z) \end{bmatrix} \mathbf{H}^{\mathrm{P}}(z) \tag{2}$$

with the element in the mth row and ℓth column of $\mathbf{H}(z) : \mathbb{C} \to \mathbb{C}^{M \times L}$ given by $H_{m\ell}(z)$, and $S_\ell(z) = F_\ell(z)F_\ell^{\mathrm{P}}(z)$ the ℓth element of the diagonal matrix of source PSDs.

The factorisation (2) can include the source model matrix $\mathbf{F}(z) = \mathrm{diag}\{F_1(z), \ldots, F_L(z)\} : \mathbb{C} \to \mathbb{C}^{L \times L}$, such that

$$\mathbf{R}(z) = \mathbf{H}(z)\mathbf{F}(z)\mathbf{F}^{\mathrm{P}}(z)\mathbf{H}^{\mathrm{P}}(z) . \tag{3}$$

The components of $\mathbf{H}(z)$ and the source model $\mathbf{F}(z)$ are assumed to be causal and stable, and their entries can be either polynomials or rational functions in z. In the most general latter case, the CSD matrix $\mathbf{R}(z)$ in (3) can be represented as a Laurent series that is absolutely convergent and therefore analytic within an annulus containing the unit circle (Girod et al., 2001). Further, since the PSDs satisfy $S_\ell(z) = S_\ell^{\mathrm{P}}(z)$, it is evident from both (2) and (3) that $\mathbf{R}(z) = \mathbf{R}^{\mathrm{P}}(z)$ and so is parahermitian.

2.2 Parahermitian Matrix EVD

For an analytic $\mathbf{R}(z)$, the factorisation

$$\mathbf{R}(z) = \mathbf{Q}(z)\mathbf{\Lambda}(z)\mathbf{Q}^{\mathrm{P}}(z) \tag{4}$$

is called the parahermitian matrix EVD (Weiss et al., 2018). If evaluated on the unit circle, the EVD at every frequency Ω, $\mathbf{R}(e^{\mathrm{j}\Omega}) = \mathbf{Q}(e^{\mathrm{j}\Omega})\mathbf{\Lambda}(e^{\mathrm{j}\Omega})\mathbf{Q}^{\mathrm{H}}(e^{\mathrm{j}\Omega})$ can exist with analytic factors $\mathbf{Q}(e^{\mathrm{j}\Omega})$ and $\mathbf{\Lambda}(e^{\mathrm{j}\Omega})$ (Rellich, 1937). The reparameterisation $z = e^{\mathrm{j}\Omega}$ can lead to analytic factors $\mathbf{Q}(z)$ and $\mathbf{\Lambda}(z)$ provided that the eigenvalues are selected appropriately. This selection will be motivated by an example.

Example for eigenvalues. Inspected on the unit circle, consider the eigenvalues $\lambda_1(e^{\mathrm{j}\Omega}) = 1$ and $\lambda_2(e^{\mathrm{j}\Omega}) = 1 + \cos\Omega$. Potentially, both functions can be permuted at any frequency, and still form valid eigenvalues as long as they retain a 2π-periodicity. Besides the analytic selection $\lambda_1(e^{\mathrm{j}\Omega})$ and $\lambda_2(e^{\mathrm{j}\Omega})$ shown in Fig. 1(a), an important alternative are spectrally majorised eigenvalues $\lambda_1'(e^{\mathrm{j}\Omega})$ and $\lambda_2'(e^{\mathrm{j}\Omega})$ in Fig. 1(b), where spectral majorisation implies that $\lambda_1'(e^{\mathrm{j}\Omega}) \geq \lambda_2'(e^{\mathrm{j}\Omega}) \, \forall \Omega$ (Vaidyanathan, 1998).

If on the unit circle eigenvalues have algebraic multiplicities greater than one, as in Fig. 1 for $\Omega = \frac{\pi}{2}$ and $\Omega = \frac{3\pi}{2}$, then only the analytic selection can lead to analytic eigenvalues in $\mathbf{\Lambda}(z)$. In the case of spectral majorisation, the region for absolute convergence is restricted to the unit circle itself.

For the eigenvectors, the representation on the unit circle can have an arbitrary phase response. Only if both the eigenvalues in $\mathbf{\Lambda}(z)$ and the arbitrary phase responses are selected as analytic, it is be guaranteed
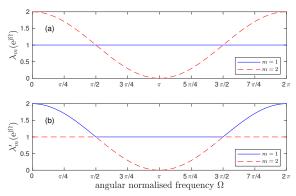
Figure 1: (a) Analytic vs (b) spectrally majorised selection of eigenvalues.

that $\boldsymbol{Q}(z)$ is analytic as well. If enforcing spectral majorisation violates the analyticity of the eigenvalues, then no analytic solution exists for the eigenvectors in $\boldsymbol{Q}(z)$.

2.3 Polynomial Approximation

Analyticity is important when trying to design realisable filters. Specifically, while the factors in (4) are analytic and therefore absolutely convergent, they generally form algebraic or even transcendental functions, i.e. are infinite in length and do not have a rational representation (Weiss et al., 2018). Due to the absolute convergence of these analytic functions, an arbitrarily close approximation can be achieved by truncating the Laurent series to sufficiently long Laurent polynomials, whereby the term 'polynomial' implies finite length.

The truncation of (4) leads to the polynomial EVD or McWhirter decomposition

$$\boldsymbol{R}(z) \approx \hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{\Lambda}}(z)\hat{\boldsymbol{Q}}^{\mathrm{P}}(z) , \qquad (5)$$

which was postulated in (McWhirter et al., 2007), based on a paraunitary factor $\hat{\boldsymbol{Q}}(z)$, and a diagonal parahermitian $\hat{\boldsymbol{\Lambda}}(z)$. All matrices — $\boldsymbol{R}(z)$, $\hat{\boldsymbol{Q}}(z)$, and $\hat{\boldsymbol{\Lambda}}(z)$ — are Laurent polynomials, ambiguity in the ordering of the eigenvalues had been suppressed by demanding spectral majorisation for $\hat{\boldsymbol{\Lambda}}(z)$.

3 ALGORITHMS FOR POLYNOMIAL MATRIX EVD

Even though eigenvalues and particularly eigenvectors are not guaranteed to exist as analytic functions in case of spectral majorisation, a number of algorithms targeting the McWhirter decomposition (5) have been created over the past decade (McWhirter

and Baxter, 2004; McWhirter et al., 2007; Tkacenko and Vaidyanathan, 2006; Tkacenko, 2010; Redif et al., 2011; Tohidian et al., 2013; Corr et al., 2014c; Redif et al., 2015; Wang et al., 2015a). These all share the restriction of considering the EVD of a parahermitian matrix $\boldsymbol{R}(z)$ whose elements are Laurent polynomials, which may be enforced by estimating or approximating $\mathbf{R}[\tau]$ over a finite lag windwo (Redif et al., 2011).

The approximation sign in the McWhirter decomposition (5), highlighting the approximation by polynomials, has been included in all subsequent algorithm designs over the past decade. Even though many algorithms can be proven to converge, in the sense that they reduce off-diagonal energy of $\boldsymbol{\Gamma}(z)$ at each iteration, see e.g. (McWhirter et al., 2007; Redif et al., 2011; Corr et al., 2014c; Redif et al., 2015; Wang et al., 2015a), there is no practical experience yet where these algorithms could not find a practicable factorisation.

Enforcing spectral majorisation in the case of an algebraic multiplicity greater than one as shown in Fig. 1 leads to eigenvalues that are not infinitely differentiable and to eigenvectors with discontinuities (Weiss et al., 2018). Since current PEVD algorithms can be shown to either favour or can even be proven to yield spectral majorisation (McWhirter and Wang, 2016), they result in matrix factors with high polynomial order to approximate the factors in (5). Therefore, some mechanisms to curb the order of these polynomial (Foster et al., 2006) and specifically the paraunitary factors (Ta and Weiss, 2007b; McWhirter et al., 2007; Corr et al., 2015c; Corr et al., 2015d) have been suggested, which are generally based on a truncation with limited error impact, and in some cases judiciously exploit the arbitrary phase response of the eigenvectors.

Current efforts in terms of algorithmic research have targetted numerical efficiencies to enhance the convergence speed of PEVD algorithms; these e.g have exploited search space reductions (Corr et al., 2014b; Corr et al., 2015b; Coutts et al., 2016c; Coutts et al., 2017a), approximate EVD algorithms (Corr et al., 2014a; Corr et al., 2015b; Corr et al., 2015a; Coutts et al., 2016b), and matrix partitioning (Coutts et al., 2016c; Coutts et al., 2017a). Also, (Tohidian et al., 2013) have presented a frequency domain algorithm which can favour analytic over spectrally majorised solutions (Coutts et al., 2017b; Coutts et al., 2018). A further route of investigation is the impact which estimation errors in the space-time covariance matrix have on the accuracy of the factorisation (Delaosa et al., 2018).

4 APPLICATION I: ANGLE OF ARRIVAL ESTIMATION

As a first application example, this section visits angle of arrival estimation. Sec. 4.1 first defines steering vectors, which together with the instantaneous covariance matrix are exploited in the multiple signal classification (MUSIC) algorithm (Schmidt, 1986) in Sec. 4.2. Broadband angle of arrival estimation techniques are briefly touched in on Sec. 4.3, with the polynomial broadband generalisation of narrowband MUSIC outlined in Sec. 4.2.

4.1 Steering Vector

If a source illuminates an M-element array from an elevation ϑ and azimuth angle φ, we assume that different delays τ_m, $m = 1 \ldots M$, are experienced as the wavefront travels across the array. To describe these sensor signals, a vector

$$\mathbf{s}_{\vartheta,\varphi}[n] = \frac{1}{\sqrt{M}} \begin{bmatrix} f[n-\tau_1] \\ f[n-\tau_2] \\ \vdots \\ f[n-\tau_M] \end{bmatrix}, \qquad (6)$$

contains an ideal fractional delay filter $f[n-\tau]$, creating a delay of $\tau \in \mathbb{R}$ samples (Laakso et al., 1996), with $n \in \mathbb{Z}$ the discrete time index. Thus, given a source signal $u[n]$ and neglecting attenuation, its contribution to the sensor signal vector $\mathbf{x}[n]$ is

$$\mathbf{x}[n] = \mathbf{s}_{\vartheta,\varphi}[n] * u[n] . \qquad (7)$$

The lag values τ_m on the r.h.s. of (6) depend on the elevation ϑ and azimuth φ of the source via $tau_m = \mathbf{k}_{\vartheta,\varphi}^{\mathrm{T}} \mathbf{r}_m$, where $\mathbf{k}_{\vartheta,\varphi}$ is the source's slowness vector pointing in the direction of propagation, and $\mathbf{r}_m$ is the position vector of the mth sensor.

The z-transform of $\mathbf{s}_{\vartheta,\varphi}[n]$,

$$s_{\vartheta,\varphi}(z) = \sum_{n=-\infty}^{\infty} \mathbf{s}_{\vartheta,\varphi}[n] z^{-n} , \qquad (8)$$

is here called a broadband steering vector. By evaluating the broadband steering vector $s_{\vartheta,\varphi}(z) : \mathbb{C} \to \mathbb{C}^M$ on the unit circle, $z = \mathrm{e}^{\mathrm{j}\Omega}$, and for a particular frequency Ω_0 we can also derive a narrowband steering vector $\mathbf{s}_{\vartheta,\varphi,\Omega_0} = s_{\vartheta,\varphi}(z)\big|_{z=\mathrm{e}^{\mathrm{j}\Omega_0}}$.

4.2 Narrowband MUSIC

A classic angle of arrival estimation techniques is the multiple signal classification (MUSIC) algorithm. It builds on the instantaneous covariance matrix $\mathbf{R} =$

$\mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n]\}$, provided that $\mathbf{x}[n]$ contains narrowband data. By means of an EVD, $\mathbf{R}$ is separated into a signal plus noise subspace, characterised by large eigenvalues in $\Lambda_s \in \mathbb{R}^{R \times R}$, and a noise only subspace, characterised by small remaining eigenvalues in $\Lambda_n \in \mathbb{R}^{(M-R) \times (M-R)}$:

$$\mathbf{R} = \begin{bmatrix} \mathbf{Q}_s & \mathbf{Q}_s^{\perp} \end{bmatrix} \begin{bmatrix} \Lambda_s & \mathbf{0} \\ \mathbf{0} & \Lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{Q}_s^{\mathrm{H}} \\ \mathbf{Q}_s^{\perp,\mathrm{H}} . \end{bmatrix} \qquad (9)$$

The matrix $\mathbf{Q}_s$ spans the signal plus noise subspace, which is an orthogonalisation of R contributing, linearly independent sources. The columns of its complement, $\mathbf{Q}_s^{\perp}$, span the noise only subspace.

The fact that the steering vector of any of the R linearly independent sources must be orthogonal to the noise subspace spanned by $\mathbf{Q}_s^{\perp}$ is exploited in the MUSIC algorithm by probing the noise subspace with steering vectors, such that

$$\rho(\vartheta,\varphi) = \|\mathbf{Q}_s^{\perp,\mathrm{H}} \mathbf{s}_{\vartheta,\varphi,\Omega_s}\|_2^{-1} \qquad (10)$$

$$= \mathbf{s}_{\vartheta,\varphi,\Omega_s}^{\mathrm{H}} \mathbf{Q}_s^{\perp} \mathbf{Q}_s^{\perp,\mathrm{H}} \mathbf{s}_{\vartheta,\varphi,\Omega_s} , \qquad (11)$$

where Ω_s is the narrowband frequency. The product under the norm in (11) take on very small values if the steering vector $\mathbf{s}_{\vartheta,\varphi,\Omega_s}$ belongs to a valid source and therefore is orthogonal to $\mathbf{Q}_s^{\perp}$. The MUSIC spectrum ρ is the reciprocal of this value, i.e. returns large values if $\mathbf{s}_{\vartheta,\varphi,\Omega_s}$ matches the steering vector of a source.

4.3 Broadband Approaches

Angle of arrival estimation techniques have been generalised to broadband signals. Recent works such as (Souden et al., 2010) are restricted to single-source scenarios. Early successful approaches have used the coherent signal subspace approach (Wang and Kaveh, 1985; Wang and Kaveh, 1987; Hung and Kaveh, 1988), where effectively an array is pre-steered such that the source appears at broadside, and can be treated as a narrowband signal as all contributions are aligned. This however requires approximate knowledge from which direction a source illuminates an array before the precise angle of arrival can be estimated.

4.4 Polynomial MUSIC

Using the polynomial broadband approach, the subspace decomposition in (9) can be applied to the polynomial EVD, and leads to a partitioning of the polynomial modal matrix,

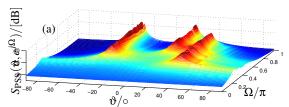$$Q(z) = \begin{bmatrix} Q_s(z) & Q_s^{\perp}(z) \end{bmatrix} , \qquad (12)$$

Figure 2: Polynomial MUSIC result for an 8 element linear array illuminated by 3 broadband sources (Weiss et al., 2013).

where the $R \leq M$ columns of $\boldsymbol{Q}_\mathrm{s}(z)$ contain the eigenvectors spanning the signal plus noise subspace, and $\boldsymbol{Q}_\mathrm{s}^\perp(z)$ its complement.

Based on this subspace decomposition of $\boldsymbol{R}(z)$, the polynomial MUSIC algorithm in (Alrmah et al., 2011; Alrmah et al., 2012; Weiss et al., 2013; Alrmah et al., 2014) provide a simple generalisation of (11) to polynomial matrices, such that

$$\rho(\vartheta, \varphi, z) = \boldsymbol{s}_{\vartheta,\varphi}^\mathrm{P}(z)\boldsymbol{Q}_\mathrm{s}^\perp(z)\boldsymbol{Q}_\mathrm{s}^{\perp,\mathrm{P}}(z)\boldsymbol{s}_{\vartheta,\varphi}(z) . \quad (13)$$

The implementation of broadband steering vectors can be achieved with filters of reasonable order if windowing (Selva, 2008) or other schemes such as in (Alrmah and Weiss, 2013; Alrmah et al., 2013) are employed. The result of the polynomial MUSIC algorithm in (13) is a power spectral density-type term, which can either be evaluated in terms of its total energy, thus depending on the angle of arrival only, or additionally resolve frequency.

Example. An example for an $M = 8$ element linear array illuminated by a mixture of three mutually uncorrelated Gaussian sources of equal power,

- $\vartheta_1 = -30°$, active over range $\Omega \in \left[\frac{3\pi}{8}; \pi\right]$,

- $\vartheta_2 = 40°$, active over range $\Omega \in \left[\frac{\pi}{2}; \pi\right]$, and

- $\vartheta_3 = 20°$, active over range $\Omega \in \left[\frac{2\pi}{8}; \frac{7\pi}{8}\right]$,

is shown in Fig. 2. When using PEVD algorithms, the accuracy of the result depends on the accuracy of the PEVD decomposition, with enhanced diagonalisation leading to improved results (Alrmah et al., 2012; Coutts et al., 2017c).

5 APPLICATION II: BROADBAND BEAMFORMING

As an example for beamforming, we review the narrowband definition of the minimum variance distortionless response (MVDR) beamformer in Sec. 5.1 and standard broadband extensions in Sec. 5.2, with its generalised polynomial formulation for the broadband case in Sec. 5.3. The polynomial approach is

then demonstrated to generalise the Capon beamformer as well as a generalised sidelobe canceller (GSC) in Secs. 5.4 and 5.5.

5.1 Narrowband MVDR

In beamforming, the aim is to isolate signals emitted by spatially separated sources by spatial filtering. This is achieved by creating constructive and destructive interference based on measurements obtained from M sensors, gathered in a data vector $\mathbf{x}[n] \in \mathbb{C}^M$. In the narrowband case, recalling the steering vector definition from Sec. 4.1, the alignment can be achieved by complex multipliers, since only the phase requires to be adjusted. The output of a narrowband beamformer therefore consists of a weighted sum of the sensor contributions, $e[n] = \mathbf{w}^\mathrm{H}\mathbf{x}[n]$, where $\mathbf{w} \in \mathbb{C}^M$ contains the weights of the beamformer.

In the presence of interference, the aim of a minimum variance beamformer is to minimise the output power $\sigma_e^2 = \mathcal{E}\{y[n]y^*[n]\} = \mathbf{w}^\mathrm{H}\mathbf{R}\mathbf{w}$,

$$\min_{\mathbf{w}} \mathbf{w}^\mathrm{H}\mathbf{R}\mathbf{w} \quad (14)$$

$$\text{s.t.} \quad \mathbf{s}_{\vartheta_\mathrm{s}, \varphi_\mathrm{s}, \Omega_\mathrm{s}}^\mathrm{H}\mathbf{w} = f , \quad (15)$$

where $\mathbf{R} = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^\mathrm{H}[n]\}$ is the instantaneous covariance matrix and the trivial solution is discouraged by imposing a gain constraint f in look direction $(\vartheta_\mathrm{s}, \varphi_\mathrm{s})$ at the narrowband operating frequency Ω_s.

Direct constrained optimisation of the MVDR problem via Lagrange multipliers leads to the Capon beamformer, see e.g. (Stoica et al., 2003; Lorenz and Boyd, 2005). Alternatively, the generalised sidelobe canceller projects that data onto an unconstrained subspace, where standard unconstrained optimisation techniques such the least mean squares or recursive least squares algorithms can then solve the MVDR problem (Widrow and Stearns, 1985; Haykin, 2002).

5.2 Broadband MVDR

In order to spatially filter broadband signals, explicit delays must be resolved, such that each sensor has to be followed by a tap delay line or finite impulse response filter in order to be able to constructively or destructively align signals. If a filter with temporal length L is employed, then the data vector needs to be extended to dimension ML, and include both spatial and temporal samples (Buckley, 1987; Van Veen and Buckley, 1988; Liu and Weiss, 2010). Subsequently, with a space-time covariance matrix of dimension $ML \times ML$, the output power of the MVDR problem can be defined.

The constraint equation can be straightforwardly extended to the broadband case if the look direction

is towards broadside for a linear array. If the look direction is off-broadside, or the array elements are not arranged in a line, then either correction by pre-steering is required to create a virtual linear array with broadside look direction, or more complicated constraint formulations are required (Godara and Sayyah Jahromi, 2007; Somasundaram, 2013).

5.3 Polynomial MVDR Formulation

If the *M*-element vector $\mathbf{w}[n]$ contains the *M* filters following each sensor, then its *z*-transform $w(z) \bullet\!\!-\!\!\circ \mathbf{w}[n]$ enables to formulate the broadband MVDR problem as (Weiss et al., 2015)

$$\min_{\mathbf{w}(z)} \oint_{|z|=1} \boldsymbol{w}^{\mathrm{P}}(z)\boldsymbol{R}(z)\boldsymbol{w}(z)\frac{dz}{z} \qquad (16)$$

$$\text{s.t.} \quad \boldsymbol{s}^{\mathrm{P}}(\vartheta_{\mathrm{s}},\varphi_{\mathrm{s}},z)\boldsymbol{w}(z) = F(z) , \qquad (17)$$

where $\boldsymbol{s}(\vartheta_{\mathrm{s}},\varphi_{\mathrm{s}},z)$ is the broadband steering vector discussed in Sec. 4.1 that defines the beamformer's look direction. In the following sections, both the Capon and GSC polynomial formulations will be defined.

5.4 Polynomial Capon Beamformer

If we extend the constraint equation in (17) to include *N* known interferers at angles of arrival $(\vartheta_{\mathrm{i},n},\varphi_{\mathrm{i},n}, n = 1\ldots N)$, then

$$C(z)w(z) = \boldsymbol{f}(z) , \qquad (18)$$

with

$$C(z) = \begin{bmatrix} \boldsymbol{s}^{\mathrm{P}}(\vartheta_{\mathrm{s}},\varphi_{\mathrm{s}},z) \\ \boldsymbol{s}^{\mathrm{P}}(\vartheta_{\mathrm{i},1},\varphi_{\mathrm{i},1},z) \\ \vdots \\ \boldsymbol{s}^{\mathrm{P}}(\vartheta_{\mathrm{i},N},\varphi_{\mathrm{i},N},z) \end{bmatrix} \qquad (19)$$

$$\boldsymbol{f}(z) = \begin{bmatrix} F(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} , \qquad (20)$$

then in a first step a polynomial Capon beamformer requires a pseudo-inverse of the polynomial constraint matrix $C(z)$ to yield $v(z) = C^{\dagger}(z)\boldsymbol{f}(z)$. The inversion of such a polynomial pseudo-inverse is e.g. addressed in (Nagy and Weiss, 2017; Nagy and Weiss, 2018).

With this extended constraint equation, the Capon beamformer is given by (Alzin et al., 2016)

$$\mathbf{w}_{\mathrm{opt}}(z) = \frac{\boldsymbol{R}^{-1}(z)\boldsymbol{v}(z)}{\tilde{\boldsymbol{v}}(z)\boldsymbol{R}^{-1}(z)\boldsymbol{v}(z)} . \qquad (21)$$
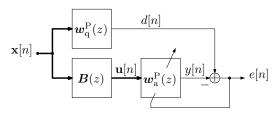

Figure 3: Polynomial generalised sidelobe canceller with quiescent beamformer $\boldsymbol{w}_{\mathrm{q}}(z)$, blocking matrix $\boldsymbol{B}(z)$, and adaptive multichannel filter $\boldsymbol{w}_{\mathrm{a}}(z)$.

This formulation is a direct polynomial extension of the narrowband formulation. The inversion of the cross spectral density matrix $\boldsymbol{R}(z)$ can be accomplished via a polynomial EVD and the inversion of the polynomial eigenvalues as discussed in (Weiss et al., 2010).

5.5 Polynomial Generalised Sidelobe Canceller

The GSC addresses the MVDR problem by forming a beam in look direction irrespective of any unknown structured interference. This quiescent beamformer $\boldsymbol{w}_{\mathrm{q}}(z)$ is the solution to the constraint equation — either (17), or, in the case of known interferers, (18). In order to remove the remaining interference, a blocking matrix $\boldsymbol{B}(z)$ passes all signal components orthogonal to $\boldsymbol{w}_{\mathrm{q}}(z)$, and therefore contains the remaining interference only in its output $\mathbf{u}[n]$ in Fig. 3. Thereafter, an adaptive noise canceller (Widrow and Stearns, 1985; Haykin, 2002) can remove the remaining interference from the quiescent beamformer output $d[n]$, thereby minimising the output power $\mathcal{E}\{e[n]e^*[n]\}$.

The construction of $\boldsymbol{w}_{\mathrm{q}}(z)$ is such that its order (and therefore computational complexity) is determined by the accuracy that is required of the fractional delay filters (Laakso et al., 1996; Selva, 2008). The blocking matrix can then be determined by polynomial matrix completion from a polynomial EVD of $\boldsymbol{w}_{\mathrm{q}}(z)\boldsymbol{w}_{\mathrm{q}}^{\mathrm{P}}(z)$ (Weiss et al., 2015). Its computational complexity is determined by the accuracy of the PEVD and the desired suppression of leakage of the signal of interest. In general, this order is significantly lower (by at least a factor of *L*) compared to tap-delay-line implementation (Buckley, 1987; Van Veen and Buckley, 1988; Liu and Weiss, 2010) with off-broadside constraints (Godara and Sayyah Jahromi, 2007).

The computational advantage of the polynomial GSC is based on the fact that the complexities for $\boldsymbol{w}_{\mathrm{q}}(z)$, $\boldsymbol{B}(z)$ and $\boldsymbol{w}_{\mathrm{a}}(z)$ are decoupled, while in the case of a standard time domain broadband beamfor-
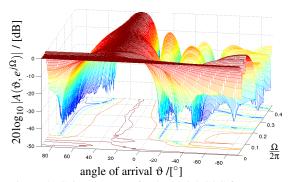
Figure 4: Gain response of polynomial GSC for $M = 8$ sensors in a linear array and look direction $\vartheta_s = 30°$, in dependency of the angle of arrival and normalised angular frequency.

mer (Buckley, 1987; Van Veen and Buckley, 1988; Liu and Weiss, 2010), all quantities are linked to the tap delay line length L. Additionally, for off-broadside look directions without pre-steering, constraints generally have to be defined in the frequency domain. As a result, the gain response is only tied down at isolated frequencies, while the broadband constraint in (17) preserved coherence across the spectrum.

Example. Fig. 4 shows the gain response of an adapted beamformer for a linear array with $M = 8$ sensors with look direction $\vartheta_s = 30°$, with interference by three broadband jammers. The gain in look direction is preserved, while spatial nulls are placed in the directions of the interfering sources over the frequency ranges of these jammers.

6 CONCLUSIONS

This paper has summarised some of the developments in the area of polynomial matrix factorisations and their application in particular to broadband array problems. Many of these problems can be straightforwardly formulated as a simple extension from the classical narrowband case to a broadband scenario when utilising polynomial matrix notation. The solution, in the narrowband case often reliant on decompositions such as the EVD or SVD, has its broadband equivalent in the parahermitian — or if approximated — the polynomial EVD, for which several mature algorithms exist (see e.g. pevd-toolbox.eee.strath.ac.uk for Matlab implementations and examples). Even though the focus of this paper has been on parahermitian or polynomial EVD, the polynomial approach can also be extended to other linear algebraic factorisations such as the SVD (Foster et al., 2010; McWhirter, 2010), the QR decomposition (Foster et al., 2010;

Coutts et al., 2016a) or the generalised EVD (Corr et al., 2016).

Generally, the advantage of polynomial matrix methods as opposed to DFT-based approaches is generally that they preserve coherence between frequency bins. This has lead to the exploration of a number of applications besides the angle of arrival and beamforming examples summarised on this paper. Successful applications have, for example, targeted for example in denoising-type (Redif et al., 2006) or decorrelating array pre-processors (Koh et al., 2009), transmit and receive beamforming across broadband MIMO channels (Davies et al., 2007; Ta and Weiss, 2007a; Sandmann et al., 2015; Ahrens et al., 2017), broadband angle of arrival estimation (Alrmah et al., 2011; Weiss et al., 2013), optimum subband partitioning of beamformers (Vouras and Tran, 2014), filter bank-based channel coding (Weiss et al., 2006) or broadband blind source separation (Redif et al., 2017). In some cases the polynomial approach can enable solutions that otherwise have been unobtainable: e.g. the design of optimal compaction filter banks beyond the two channel case (Redif et al., 2011).

It is hoped that this overview paper can inspire the use of these methods to a wider range of applications.

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