Robust Vertex-dependant H_∞ Filtering of Stochastic Discrete-time Systems with Delay

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Abstract: Linear, state delayed, discrete-time systems with stochastic uncertainties in their state-space model are considered. The problems of robust vertex-dependant polytopic H_∞ filtering is solved, for the stationary case, via an input-output approach by which the system is replaced by a nonretarded system with deterministic norm-bounded uncertainties. A vertex-dependent solution is obtained by applying a modified version of the Finsler lemma. In this problem, a cost function is defined which is the expected value of the standard H_∞ performance index with respect to the uncertain parameters.

1 INTRODUCTION

We address the problem of robust polytopic H_∞ filtering of state-delayed, discrete-time, state-multiplicative linear systems via the *input-output* approach based on the robust vertex-dependant stability and Bounded Real Lemma (BRL) of these systems, which are developed here. The multiplicative noise appears in the system model in both the delayed and the non delayed states of the system.

The stability and control of stochastic delayed systems of various types (i.e constant time-delay, slow and fast varying delay) have been a central issue in the theory of stochastic state-multiplicative systems over the last decade (Boukas and Liu, 2002), (Mao, 1996), (Verriest and Florchinger, 1995), (S.Xu and Chen, 2002), (Chen et al., 2005), (Gershon and Shaked, 2011). The results that have been obtained for the stability of deterministic retarded systems, since the 90's, have been extended also to the stochastic case, mainly for continuous-time systems. In the continuous-time stochastic setting, for example, the Lyapunov-Krasovskii (L-K) approach is applied in (Xu et al., 2005) and (Chen et al., 2005), to systems with constant delays, and stability criteria are derived for cases with norm-bounded uncertainties. The H_∞ state-feedback control for systems with time-varying delay is treated in (S.Xu and Chen, 2002) for restricted LKFs that provide delay-independent, rate dependent results. Also (Boukas and Liu, 2002) considers H_∞ control (both state and output feedback) and estimation of time delay systems.

In the discrete-time setting, the mean square exponential stability and the control and filtering problems of these systems were treated by several groups (Xu et al., 2004), (Gao and Chen, 2007), (Yue et al., 2009). In (Xu et al., 2004), the state-feedback control problem solution is solved for norm-bounded uncertain systems, for the restrictive case where the same multiplicative noise sequence multiplies both the states and the input of the system. The solution there is delay-dependent.

The filtering problem of the discrete-time retarded stochastic systems was already solved in (Gershon and Shaked, 2013). The point of view that was taken in the latter work is similar to the one taken for the solution of both the continuous-time state-feedback control and filtering problems in (Gershon and Shaked, 2011). This solution is based on the input-output approach of (Kao and Lincoln, 2004), (Fridman and Shaked, 2006) that was developed for deterministic systems. The latter approach is based on the representation of the system's delay action by linear operators, with no delay, which in turn allows one to replace the underlying system with an equivalent one which possesses a norm-bounded uncertainty, and therefore may be treated by the theory of norm bounded uncertain, non-retarded systems with state-multiplicative noise (Gershon et al., 2005).

In the robust polytopic filtering case, the solution in (Gershon and Shaked, 2013) draws on the solution of the nominal uncertainty-free system. Basically, it

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applies the same Lyapunov function over the whole uncertainty polytope, leading to a considerably conservative solution [i.e the quadratic solution]. In the present paper we extend the results achieved in (Gershon and Shaked, 2013) for the robust case by applying a different approach which make use of the Finsler lemma. Unlike the quadratic solution, in this work we assign a different Lyapunov function to each vertex of the uncertainty polytope thus allowing for a possibly less conservative solution. In order to derive the latter solution we first develop a vertex-dependent stability condition followed by a vertex-dependent BRL solution.

In our system we allow for a time-varying delay where the uncertain stochastic parameters multiply both the delayed and the non delayed states in the state space model of the system. This paper is organized as follows: Based on the input-output approach, the solution of the robust stability issue is brought in Section 3 and newly developed for the vertex-dependent case, followed by the solution of the robust BRL in Section 4. The robust vertex-dependent filtering problem is treated in Section 5, for a general-type filter, resulting in a less conservative solution, compared to the already solved quadratic one.

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, $\mathcal{N}$ is the set of natural numbers and the notation $P>0$, (respectively, $P\geq 0$) for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (respectively, semi-definite). We denote by $L^2(\Omega,\mathcal{R}^n)$ the space of square-integrable $\mathcal{R}^n-$ valued functions on the probability space $(\Omega,\mathcal{F},\mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subset of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_k)_{k\in\mathcal{N}}$ we denote an increasing family of σ-algebras $\mathcal{F}_k\subset\mathcal{F}$. We also denote by $\tilde{l}^2(\mathcal{N};\mathcal{R}^n)$ the n-dimensional space of nonanticipative stochastic processes $\{f_k\}_{k\in\mathcal{N}}$ with respect to $(\mathcal{F}_k)_{k\in\mathcal{N}}$ where $f_k\in L^2(\Omega,\mathcal{R}^n)$. On the latter space the following l^2-norm is defined:

$$||\{f_k\}||^2_{\tilde{l}_2}=E\{\textstyle\sum_0^\infty||f_k||^2\}=\sum_0^\infty E\{||f_k||^2\}<\infty,\quad \{f_k\}\in\tilde{l}_2(\mathcal{N};\mathcal{R}^n), \tag{1}$$

where $||\cdot||$ is the standard Euclidean norm. We denote by $\mathrm{Tr}\{\cdot\}$ the trace of a matrix and by δ_{ij} the Kronecker delta function. Throughout the manuscript we refer to the notation of exponential l^2 stability, or internal stability, in the sense of (Bouhtouri et al., 1999) (see Definition 2.1, page 927, there).

2 PROBLEM FORMULATION

We consider the following linear retarded system:

$$\begin{array}{l} x_{k+1}=(A_0+D\nu_k)x_k+(A_1+F\mu_k)x_{k-\tau(k)}\\ +B_1w_k\quad x_l=0,\ l\leq 0,\\ y_k=C_2x_k+D_{21}n_k\end{array} \tag{2a,c}$$

with the objective vector

$$z_k=C_1x_k, \tag{3}$$

where $x_k\in\mathcal{R}^n$ is the system state vector, $w_k\in\mathcal{R}^q$ is the exogenous disturbance signal, $n_k\in\mathcal{R}^p$ is the the measurement noise signal, $y_k\in\mathcal{R}^m$ is the measured output and $z_k\in\mathcal{R}^r$ is the state combination (objective function signal) to be regulated and where the time delay bound is denoted by h. The variables $\{\mu_k\}$ and $\{\nu_k\}$ are zero-mean real scalar white-noise sequences that satisfy:

$$E\{\nu_k\nu_j\}=\delta_{kj},\ E\{\mu_k\mu_j\}=\delta_{kj}$$
$$E\{\mu_k\nu_j\}=0,\ \forall k,j\quad\geq 0.$$

The matrices in (2a,c), (3) are constant matrices of appropriate dimensions.

We treat the following two problems:

i) H_∞ Filtering.
We consider the system of (2a,b) and (3) where and consider the estimator of the following general form:

$$\begin{array}{l}\hat{x}_{k+1}=A_c\hat{x}_k+B_cy_k,\\ \hat{z}_k=C_c\hat{x}_k.\end{array} \tag{4}$$

We denote

$$e_k=x_k-\hat{x}_k,\quad\text{and}\quad\bar{z}_k=z_k-\hat{z}_k, \tag{5}$$

and we consider the following cost function:

$$J_F\stackrel{\Delta}{=}||\bar{z}_k||^2_{\tilde{l}_2}-\gamma^2[||w_k||^2_{\tilde{l}_2}+||n_{k+1}||^2_{\tilde{l}_2}]. \tag{6}$$

Given $\gamma>0$, we seek an estimate $C_c\hat{x}_k$ of C_1x_k over the infinite time horizon $[0,\infty)$ such that J_F given by (6) is negative for all nonzero w_k,n_k where $w_k\in\tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$, $n_k\in\tilde{l}^2_{\mathcal{F}_k}([0,\infty];\mathcal{R}^p)$.

i) Robust Filtering. In the robust stochastic H_∞ estimation problem treated here, we assume that the system parameters lie within the following polytope:

$$\bar{\Omega}\stackrel{\Delta}{=}\left[\begin{array}{cccccccc}A_0 & A_1 & B_1 & C_1 & C_2 & D_{21} & D & F\end{array}\right], \tag{7}$$

which is described by the vertices:

$$\bar{\Omega}=\mathcal{C}o\{\bar{\Omega}_1,\bar{\Omega}_2,...,\bar{\Omega}_N\}, \tag{8}$$

where $\bar{\Omega}_i\stackrel{\Delta}{=}$

$$\left[\begin{array}{cccccccc}A_0^{(i)} & A_1^{(i)} & B_1^{(i)} & C_1^{(i)} & C_2^{(i)} & D_{21}^{(i)} & D^{(i)} & F^{(i)}\end{array}\right] \tag{9}$$

and where N is the number of vertices. In other words:

$$\bar{\Omega}=\sum_{i=1}^N\bar{\Omega}_if_i,\quad\sum_{i=1}^N f_i=1\quad,f_i\geq 0. \tag{10}$$

3 ROBUST MEAN-SQUARE EXPONENTIAL STABILITY

In order to solve the above two problems we bring first the stability result for the retarded, stochastic, discrete-time system that was already derived in (Gershon and Shaked, 2013). Considering the system of (2a) with $B_1 = 0$, we obtain the following theorem, which is brought here for the sake of completeness :

Theorem 1. (Gershon and Shaked, 2013) The exponential stability in the mean square sense of the system (2a) with $B_1 = 0$, is guaranteed if there exist $n \times n$ matrices $Q > 0$, $R_1 > 0$, $R_2 > 0$ and M that satisfy the following inequality:

$$\bar{\Gamma} \triangleq \begin{bmatrix} \bar{\Gamma}_{1,1} & \bar{\Gamma}_{1,2} & 0 & 0 & \bar{\Gamma}_{1,5} \\ * & -Q & Q(A_1-M) & QM & 0 \\ * & * & \bar{\Gamma}_{3,3} & 0 & \bar{\Gamma}_{3,5} \\ * & * & * & -R_2 & -hM^T R_2 \\ * & * & * & * & -R_2 \end{bmatrix} < 0.$$

where

$$\begin{aligned} &\bar{\Gamma}_{1,1} = -Q + D^T(Q+h^2R_2)D + R_1, \\ &\bar{\Gamma}_{1,2} = (A_0+M)^T Q, \\ &\bar{\Gamma}_{1,5} = h(A_0^T + M^T)R_2 - R_2 h, \\ &\bar{\Gamma}_{3,3} = -R_1 + F^T(Q+h^2R_2)F, \\ &\bar{\Gamma}_{3,5} = h(A_1^T - M^T)R_2. \end{aligned} \tag{11a-e}$$

We note that inequality (11a) is bilinear in the decision variables because of the terms QM and R_2M. In order to remain in the linear domain, we can define $Q_M = QM$ and choose $R_2 = \varepsilon Q$ where ε is a positive tuning scalar. The resulting LMI can be found in (Gershon and Shaked, 2013).

In the polytopic uncertain case we obtain two results, the first of which is the quadratic solution which appears in (Gershon and Shaked, 2013) and is based on assigning the same Lyapunov function over the all uncertainty polytope. A new result is obtained here by applying a vertex-dependent Lyapunov function. Using Schur's complement, (11a) can be written as:

$$\Psi + \Phi Q \Phi^T < 0, \tag{12}$$

with

$$\Psi \triangleq \begin{bmatrix} \Psi_{1,1} & 0 & 0 & h(A_0^T + M^T)R_2 - R_2 h \\ * & \Psi_{2,2} & 0 & h(A_1^T - M^T)R_2 \\ * & * & -R_2 & -hM^T R_2 \\ * & * & * & -R_2 \end{bmatrix},$$

$$\Phi \triangleq \begin{bmatrix} A_0^T + M^T \\ A_1^T - M^T \\ M^T \\ 0 \end{bmatrix}, \Psi_{1,1} = -Q + D^T(Q+h^2R_2)D + R_1$$

and

$$\Psi_{2,2} = -R_1 + F^T(Q+h^2R_2)F. \tag{13a-d}$$

The following result is thus obtained:

Lemma 1. Inequality (12) is satisfied iff there exist matrices: $0 < Q \in \mathcal{R}^{n\times n}$, $G \in \mathcal{R}^{n\times 4n}$, $M \in \mathcal{R}^{n\times n}$ and $H \in \mathcal{R}^{n\times n}$ that satisfy the following inequality

$$\Omega \triangleq \begin{bmatrix} \Psi + G^T\Phi^T + \Phi G & -G^T + \Phi H \\ -G + H^T\Phi^T & -H - H^T + Q \end{bmatrix} < 0. \tag{14}$$

Proof. Substituting $G = 0$ and $H = Q$ in (14), inequality (12) is obtained. To show that (14) leads to (12) we consider

$$\begin{bmatrix} I & \Phi \\ 0 & I \end{bmatrix} \Omega \begin{bmatrix} I & 0 \\ \phi^T & I \end{bmatrix} = \begin{bmatrix} \Psi + \Phi Q \Phi^T & -G^T - \Phi H^T + \Phi Q \\ -G - H\Phi^T + Q\Phi^T & -H - H^T + Q \end{bmatrix}.$$

Inequality (12) thus follows from the fact that the (1,1) matrix block of the latter matrix is the left side of (12).

Taking $H = G[I_n\, 0\, 0\, 0]^T$, $R_2 = \varepsilon_r H$ where $\varepsilon_r > 0$ is a scalar tuning parameter and denoting $M_H = H^T M$, we note that in (14) the system matrices, excluding D and F, do not multiply Q. It is thus possible to choose vertex dependent $Q^{(i)}$ while keeping H and G constant. We thus arrive at the following result:

Corollary 1. The exponential stability in the mean square sense of the system (2a) where $B_1 = 0$ and where the system matrices lie within the polytope $\bar{\Omega}$ of (7) is guaranteed if there exist matrices $0 < Q_j \in \mathcal{R}^{n\times n}$, $\forall j = 1,...N$, $0 < R_1 \in \mathcal{R}^{n\times n}$, $M_H \in \mathcal{R}^{n\times n}$, $G \in \mathcal{R}^{n\times 4n}$ and a tuning scalar $\varepsilon_r > 0$ that satisfy the following set of inequalities:

$$\Omega_j = \begin{bmatrix} \Psi_j + G^T\Phi^{j,T} + \Phi_j G & -G^T + \Phi_j H \\ -G + H^T\Phi^{j,T} & -H - H^T + Q_j \end{bmatrix} < 0, \tag{15}$$

$\forall j,\ j = 1,2,....,N$, where $H \in \mathcal{R}^{n\times n} = G[I_n\, 0\, 0\, 0]^T$

$$\Psi_j \triangleq \begin{bmatrix} \Psi_{j,1,1} & 0 & 0 & \Psi_{j,1,4} \\ * & \Psi_{j,2,2} & 0 & h\varepsilon_r(A_1^{j,T}H - M_H^T) \\ * & * & -\varepsilon_r H & -h\varepsilon_r M_H^T \\ * & * & * & -\varepsilon_r H \end{bmatrix},$$

$$\Phi_j H = \begin{bmatrix} A_0^{j,T}H + M_H^T \\ A_1^{j,T}H - M_H^T \\ M_H^T \\ 0 \end{bmatrix}, \tag{16}$$

and where

$$\begin{aligned} &\Psi_{j,1,1} = -Q_j + D^T(Q_j + h^2\varepsilon_r H)D + R_1, \\ &\Psi_{j,1,4} = h\varepsilon_r(A_0^{j,T}H + M_H^T) - \varepsilon_r hH, \\ &\Psi_{j,2,2} = -R_1 + F^T(Q_j + h^2\varepsilon_r H)F. \end{aligned}$$

4 ROBUST BOUNDED REAL LEMMA

Based on the stability result of Corollary 1, the following result is readily obtained where we consider the system (2a) with $z_k = C_1 x_k$ and the following index of performance:

$$J_B = E\{x_{k+1}^T Q x_{k+1}\} - x_k^T Q x_k + z_k^T z_k - \gamma^2 w_k^T w_k.$$

Theorem 2 (Gershon and Shaked, 2013)**.** Consider the system (2a) and (3). The system is exponentially stable in the mean square sense and, for a prescribed scalar $\gamma > 0$ and a given scalar tuning parameter $\varepsilon_b > 0$, the requirement of $J_B < 0$ is achieves for all nonzero $w \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$, if there exist $n \times n$ matrices $Q > 0$, $R_1 > 0$ and a $n \times n$ matrix Q_m that satisfy $\tilde{\Gamma} < 0$ where $\tilde{\Gamma} =$

$$\begin{bmatrix} \tilde{\Gamma}_{11} & \tilde{\Gamma}_{12} & 0 & 0 & \tilde{\Gamma}_{15} & 0 & C_1^T \\ * & -Q & \tilde{\Gamma}_{23} & Q_m & 0 & QB_1 & 0 \\ * & * & \tilde{\Gamma}_{33} & 0 & \tilde{\Gamma}_{35} & 0 & 0 \\ * & * & * & -\varepsilon_b Q & -h\varepsilon_b Q_m^T & 0 & 0 \\ * & * & * & * & -\varepsilon_b Q & \varepsilon_b h Q B_1 & 0 \\ * & * & * & * & * & -\gamma^2 I_q & 0 \\ * & * & * & * & * & * & -I_r \end{bmatrix} \quad (17)$$

where

$$\begin{aligned} \tilde{\Gamma}_{11} &= -Q + D^T Q[1+\varepsilon_b h^2]D + R_1, \\ \tilde{\Gamma}_{12} &= A_0^T Q + Q_m^T, \\ \tilde{\Gamma}_{15} &= \varepsilon_b h[A_0^T Q + Q_m^T] - \varepsilon_b h Q, \\ \tilde{\Gamma}_{23} &= QA_1 - Q_m, \\ \tilde{\Gamma}_{33} &= -R_1 + (1+\varepsilon_b h^2)F^T Q F, \\ \tilde{\Gamma}_{35} &= \varepsilon_b h[A_1^T Q - Q_m^T]. \end{aligned}$$

Similarly to the stability condition for the uncertain case of Section 3 we obtain two results for the robust BRL solution. The first one, which is referred to as the quadratic solution, is simply obtained by assigning the same Lyapunov function over the all uncertainty polytope and thus is solved similarly to the robust quadratic condition . A new, possibly less conservative condition is obtained by applying the following vertex-dependent Lyapunov function:

$$\begin{bmatrix} \Psi & \begin{bmatrix} 0 & C_1^T \\ 0 & 0 \\ 0 & 0 \\ hR_2 B_1 & 0 \end{bmatrix} \\ \begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix} & \begin{bmatrix} -\gamma^2 I_q & 0 \\ * & I_r \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \Phi \\ \\ B_1^T \\ 0 \end{bmatrix} Q \begin{bmatrix} \Phi^T & B_1 & 0 \end{bmatrix} \triangleq \hat{\Psi} + \hat{\Phi} Q \hat{\Phi}^T < 0,$$

where Ψ and Φ are given in (13a,b).

Following the derivation of the LMI of Corollary 1, the following result is readily derived:

$$\begin{bmatrix} \hat{\Psi} + \hat{G}^T \hat{\Phi}^T + \hat{\Phi}\hat{G} & -\hat{G}^T + \hat{\Phi} H \\ -\hat{G} + H^T \hat{\Phi}^T & -H - H^T + Q \end{bmatrix} < 0, \quad (18)$$

where now $\hat{G} \in \mathcal{R}^{n \times 4n+q+r}$ and $H \in \mathcal{R}^{n \times n}$. We thus arrive at the following result for the uncertain case, taking $H = \hat{G}[I_n\ 0\ 0\ 0]^T$, $R_2 = \varepsilon_r H$, $M_H = H^T M$:

Corollary 2. Consider the system (2a) and (3) where the system matrices lie within the polytope $\bar{\Omega}$ of (7). The system is exponentially stable in the mean square sense and, for a prescribed $\gamma > 0$ and given tuning parameter ε_r, the requirement of $J_B < 0$ is achieved for all nonzero $w \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$, if there exist $0 < Q \in \mathcal{R}^{n \times n}$, $0 < R_1 \in \mathcal{R}^{n \times n}$, $M_H \in \mathcal{R}^{n \times n}$, and $\hat{G} \in \mathcal{R}^{n \times 4n+q+r}$, that satisfy the following set of LMIs:

$$\begin{bmatrix} \hat{\Psi}_j + \hat{G}^T \hat{\Phi}^{j,T} + \hat{\Phi}_j \hat{G} & -\hat{G}^T + \hat{\Phi}_j H \\ -\hat{G} + H^T \hat{\Phi}^{j,T} & -H - H^T + Q_j \end{bmatrix} < 0, \quad (19)$$

where $H \in \mathcal{R}^{n \times n} = \hat{G}[I_n\ 0\ 0\ 0]^T$

$$\hat{\Psi}_j = \begin{bmatrix} \Psi_j & \begin{bmatrix} 0 & C_1^{j,T} \\ 0 & 0 \\ 0 & 0 \\ h\varepsilon_r B_1^j & 0 \end{bmatrix} \\ \begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix} & \begin{bmatrix} -\gamma^2 I_q & 0 \\ * & I \end{bmatrix} \end{bmatrix},$$

$$\hat{\Phi}_j H = \begin{bmatrix} \Phi_j H \\ \\ B_1^{j,T} H \\ 0 \end{bmatrix}, \forall j,\ j = 1,2,....,N,$$

where Ψ_j and $\Phi_j H$ are given in (16).

5 DELAYED FILTERING

In this section we address the filtering problem of the delayed state-multiplicative noisy system. We start with the nominal case and then we bring the quadratic solution of the uncertain polytopic case. We consider the system of (2a-c) and (3) and the general type filter of (4). Denoting $\xi_k^T \triangleq [x_k^T\ \ \hat{x}_k^T]$, $\bar{w}_k^T \triangleq [w_k^T\ \ n_k^T]$ we obtain the following augmented system:

$$\begin{aligned} \xi_{k+1} &= \tilde{A}_0 \xi_k + \tilde{B}\bar{w}_k + \tilde{A}_1 \xi_{k-\tau(k)} + \tilde{D}\xi_k \nu_k + \tilde{F}\xi_{k-\tau(k)}\mu_k, \\ \tilde{z}_k &= \tilde{C}\xi_k, \quad \xi_l = 0,\ l \le 0 \end{aligned} \quad (20)$$

where

$$\tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ B_c C_2 & A_c \end{bmatrix},\ \tilde{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_c D_{21} \end{bmatrix},$$
$$\tilde{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{D} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{C}^T = \begin{bmatrix} C_1^T \\ -C_c^T \end{bmatrix} \tag{21}$$

and $\tilde{F} = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$. Using the BRL result of Section 4 we obtain the inequality condition $\tilde{\Upsilon} < 0$ where $\tilde{\Upsilon} =$

$$\begin{bmatrix} \tilde{\Upsilon}_{11} & \tilde{\Upsilon}_{12} & 0 & 0 & \tilde{\Upsilon}_{15} & 0 & \tilde{C}^T & \tilde{\Upsilon}_{18} \\ * & -\tilde{Q} & \tilde{\Upsilon}_{23} & \tilde{Q}_M & 0 & \tilde{Q}\tilde{B} & 0 & 0 \\ * & * & \tilde{\Upsilon}_{33} & 0 & \tilde{\Upsilon}_{35} & 0 & 0 & 0 \\ * & * & * & -\varepsilon_f\tilde{Q} & \tilde{\Upsilon}_{45} & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_f\tilde{Q} & \tilde{\Upsilon}_{56} & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & * & * & -I_r & 0 \\ * & * & * & * & * & * & * & -\tilde{Q} \end{bmatrix} \tag{22}$$

where

$$\begin{aligned} &\tilde{\Upsilon}_{11} = -\tilde{Q} + \tilde{R}_1, \\ &\tilde{\Upsilon}_{12} = \tilde{A}_0^T\tilde{Q} + \tilde{Q}_M^T, \\ &\tilde{\Upsilon}_{15} = \varepsilon_f h[\tilde{A}_0^T\tilde{Q} + \tilde{Q}_M^T] - \varepsilon_f h\tilde{Q}, \\ &\tilde{\Upsilon}_{18} = \tilde{D}^T\tilde{Q}\sqrt{1+\varepsilon_f h^2}, \\ &\tilde{\Upsilon}_{23} = \tilde{Q}\tilde{A}_1 - \tilde{Q}_M, \\ &\tilde{\Upsilon}_{33} = -\tilde{R}_1 + (1+\varepsilon_f h^2)\tilde{F}^T\tilde{Q}\tilde{F}, \\ &\tilde{\Upsilon}_{35} = \varepsilon_f h[\tilde{A}_1^T\tilde{Q} - \tilde{Q}_M^T], \\ &\tilde{\Upsilon}_{45} = -h\varepsilon_f\tilde{Q}_M^T, \\ &\tilde{\Upsilon}_{56} = h\varepsilon_f\tilde{Q}\tilde{B}. \end{aligned}$$

Defining $\tilde{P} = \tilde{Q}^{-1}$, denoting the following partitions

$$\tilde{P} = \begin{bmatrix} X & M^T \\ M & T \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Y & N^T \\ N & W \end{bmatrix},$$

and $J = \begin{bmatrix} X^{-1} & Y \\ 0 & N \end{bmatrix}$, we multiply (22) by

$$\hat{J} = diag\{\tilde{P}J, \tilde{P}J, \tilde{P}J, \tilde{P}J, \tilde{P}J, I, I, \tilde{P}J\}$$

from the right and by $\hat{J}^T$, from the left. We obtain, denoting $\bar{R}_p = J^T\tilde{P}\tilde{R}_1\tilde{P}J$,

$$\left[\begin{array}{ccccccccc} \hat{\Upsilon}_{11} & \hat{\Upsilon}_{12} & 0 & 0 & \hat{\Upsilon}_{15} & 0 & J^T\tilde{P}\tilde{C}^T & \bar{\varepsilon}J^T\tilde{P}\tilde{D}^T J & 0 \\ * & -J^T\tilde{P}J & \hat{\Upsilon}_{23} & J^T\tilde{P}\tilde{Q}_M\tilde{P}J & 0 & J^T\tilde{B} & 0 & 0 & 0 \\ * & * & -\bar{R}_p & 0 & \hat{\Upsilon}_{35} & 0 & 0 & 0 & \bar{\varepsilon}J^T\tilde{P}\tilde{F}^T J \\ * & * & * & -\varepsilon_f J^T\tilde{P}J & \hat{\Upsilon}_{45} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_f J^T\tilde{P}J & h\varepsilon_b J^T\tilde{B} & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I_{q+p} & 0 & 0 & 0 \\ * & * & * & * & * & * & -I_r & 0 & 0 \\ * & * & * & * & * & * & * & -J^T\tilde{P}J & 0 \\ * & * & * & * & * & * & * & * & -J^T\tilde{P}J \end{array}\right] < 0, \tag{23}$$

where

$$\begin{aligned} &\hat{\Upsilon}_{11} = -J^T\tilde{P}J + \bar{R}_p, \\ &\hat{\Upsilon}_{12} = J^T\tilde{P}\tilde{A}_0^T J + J^T\tilde{P}\tilde{Q}_M^T\tilde{P}J, \\ &\hat{\Upsilon}_{15} = \varepsilon_f h[J^T\tilde{P}\tilde{A}_0^T J + J^T\tilde{P}\tilde{Q}_M^T\tilde{P}J] - \varepsilon_f hJ^T\tilde{P}J, \\ &\hat{\Upsilon}_{23} = J^T\tilde{A}_1\tilde{P}J - J^T\tilde{P}\tilde{Q}_M\tilde{P}J, \\ &\hat{\Upsilon}_{35} = \varepsilon_f h[J^T\tilde{P}\tilde{A}_1^T J - J^T\tilde{P}\tilde{Q}_M^T\tilde{P}J], \\ &\hat{\Upsilon}_{45} = -h\varepsilon_f J^T\tilde{P}\tilde{Q}_M^T\tilde{P}J, \\ &\bar{\varepsilon}^2 = 1 + \varepsilon_f h^2. \end{aligned}$$

Denoting $\bar{X} = X^{-1}$ and

$$\bar{X}_y = \begin{bmatrix} \bar{X} & \bar{X} \\ \bar{X} & Y \end{bmatrix},\ \tilde{P}_M = J^T\tilde{P}\tilde{Q}_M^T\tilde{P}J,$$

we obtain:

$$\left[\begin{array}{ccccc} -\bar{X}_y + \bar{R}_p & \Psi_{12} & 0 & 0 & \Psi_{15} \\ * & -\bar{X}_y & \Psi_{23} & \tilde{P}_M & 0 \\ * & * & -\bar{R}_p & 0 & \Psi_{35} \\ * & * & * & -\varepsilon_f\bar{X}_y & \Psi_{45} \\ * & * & * & * & -\varepsilon_f\bar{X}_y \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right.$$

$$
\left.\begin{array}{cccc}
0 & J^T\tilde{P}\tilde{C}^T & \bar{\varepsilon}J^T\tilde{P}\tilde{D}^T J & 0 \\
J^T\tilde{B} & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{\varepsilon}J^T\tilde{P}\tilde{F}^T J \\
0 & 0 & 0 & 0 \\
h\varepsilon_b J^T\tilde{B} & 0 & 0 & 0 \\
-\gamma^2 I_{q+p} & 0 & 0 & 0 \\
* & -I & 0 & 0 \\
* & * & -\bar{X}_y & 0 \\
* & * & * & -\bar{X}_y
\end{array}\right] < 0, \quad (24)
$$

where

$$
\begin{aligned}
\Psi_{12} &= J^T\tilde{P}\tilde{A}_0^T J + \tilde{P}_M, \\
\Psi_{15} &= \varepsilon_f h[J^T\tilde{P}\tilde{A}_0^T J + \tilde{P}_M] - \varepsilon_f h \begin{bmatrix} \bar{X} & \bar{X} \\ \bar{X} & Y \end{bmatrix}, \\
\Psi_{23} &= J^T\tilde{A}_1\tilde{P}J - \tilde{P}_M, \\
\Psi_{35} &= \varepsilon_f h[J^T\tilde{P}\tilde{A}_1^T J - \tilde{P}_M], \\
\Psi_{45} &= -h\varepsilon_f\tilde{P}_M, \\
\bar{\varepsilon}^2 &= 1 + \varepsilon_f h^2.
\end{aligned}
$$

Carrying out the various multiplications and denoting $K_0 = N^T A_c M\bar{X}$, $U = N^T B_c$ and $Z = C_c M\bar{X}$, we obtain the following result:

$$
\left[\begin{array}{ccccc}
-\bar{X}_y + \bar{R}_p & \tilde{\Psi}_{12} & 0 & 0 & \tilde{\Psi}_{15} \\
* & -\bar{X}_y & \tilde{\Psi}_{23} & \tilde{P}_M & 0 \\
* & * & -\bar{R}_p & 0 & \tilde{\Psi}_{35} \\
* & * & * & -\varepsilon_f\bar{X}_y & -h\varepsilon_f\tilde{P}_M \\
* & * & * & * & -\varepsilon_f\bar{X}_y \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{array}\right.
$$

$$
\left.\begin{array}{cccc}
0 & \tilde{\Psi}_{17} & \tilde{\Psi}_{18} & 0 \\
\tilde{\Psi}_{26} & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{\Psi}_{39} \\
0 & 0 & 0 & 0 \\
\tilde{\Psi}_{56} & 0 & 0 & 0 \\
-\gamma^2 I_{q+p} & 0 & 0 & 0 \\
* & -I_r & 0 & 0 \\
* & * & -\bar{X}_y & 0 \\
* & * & * & -\bar{X}_y
\end{array}\right] < 0, \quad (25)
$$

where

$$
\begin{aligned}
\tilde{\Psi}_{12} &= \begin{bmatrix} A_0^T\bar{X} & A_0^T Y + C_2^T U^T + K_0^T \\ A_0^T\bar{X} & A_0^T Y + C_2^T U^T \end{bmatrix} + \tilde{P}_M, \\
\tilde{\Psi}_{15} &= \varepsilon_f h \begin{bmatrix} A_0^T\bar{X} & A_0^T Y + C_2^T U^T + K_0^T \\ A_0^T\bar{X} & A_0^T Y + C_2^T U^T \end{bmatrix} \\
&\quad + \varepsilon_f h\tilde{P}_M - \varepsilon_f h \begin{bmatrix} \bar{X} & \bar{X} \\ \bar{X} & Y \end{bmatrix}, \\
\tilde{\Psi}_{17} &= \begin{bmatrix} C_1^T - Z^T \\ C_1^T \end{bmatrix}, \\
\tilde{\Psi}_{18} &= \begin{bmatrix} D^T\bar{X} & D^T Y \\ D^T\bar{X} & D^T Y \end{bmatrix}, \\
\tilde{\Psi}_{23} &= \begin{bmatrix} \bar{X}A_1 & \bar{X}A_1 \\ YA_1 & YA_1 \end{bmatrix} - \tilde{P}_M, \\
\tilde{\Psi}_{26} &= \begin{bmatrix} \bar{X}B_1 & 0 \\ YB_1 & UD_{21} \end{bmatrix}, \\
\tilde{\Psi}_{35} &= \varepsilon_f h \begin{bmatrix} A_1^T\bar{X} & A_1^T Y \\ A_1^T\bar{X} & A_1^T Y \end{bmatrix} - \varepsilon_f h\tilde{P}_M, \\
\tilde{\Psi}_{39} &= \begin{bmatrix} F^T\bar{X} & F^T Y \\ F^T\bar{X} & F^T Y \end{bmatrix}, \\
\tilde{\Psi}_{56} &= h\varepsilon_f \begin{bmatrix} \bar{X}B_1 & 0 \\ YB_1 & UD_{21} \end{bmatrix}, \\
\bar{\varepsilon}^2 &= 1 + \varepsilon_f h^2.
\end{aligned}
$$

We thus arrive at the following theorem:

Theorem 3. Consider the system of (2a-c) and (3). For a prescribed scalar $\gamma > 0$ and a positive tuning scalar ε_f, there exists a filter of the structure (4) that achieves $J_F < 0$, where J_F is given in (6), for all nonzero $w \in \tilde{l}^2([0,\infty);\mathcal{R}^q)$, $n \in \tilde{l}^2([0,\infty);\mathcal{R}^p)$, if there exist $n \times n$ matrices $\bar{X} > 0$, $Y > 0$, $2n \times 2n$ matrix $\bar{R}_p > 0$, $n \times n$ matrices K_0 and U, $2n \times 2n$ matrix $\tilde{P}_M$ and a $n \times l$ matrix Z, that satisfy (25). In the latter case the filter parameters can be extracted using the following equations:

$$
A_c = N^{-T}K_0\bar{X}M^{-1},\ B_c = N^{-T}U,\ C_c = Z\bar{X}M^{-1}. \quad (26)
$$

Noting that $XY - M^T N = I$, the filter matrix parameters A_c, B_c, and C_c can be readily found, without any loss of generality, by a singular value decomposition of $I - XY$.

In the uncertain case a robust filter is obtained by either applying the quadratic solution (based on the quadratic BRL) or applying the following vertex dependent approach: Starting with (25), we define

$$\Gamma = diag\{E,E,E,E,E,I,I,E,E\},\ E = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}. \tag{27}$$

We then multiply (25) by Γ and Γ^t, from the left and the right respectively. We obtain then:

$$\begin{bmatrix}
-V^i+\hat{R}^i_p & \bar{\Psi}^i_{12} & 0 & 0 & \bar{\Psi}^i_{15} & 0 & \begin{bmatrix} -Z^T \\ C_1^{i,T} \end{bmatrix} & \bar{\Psi}_{18} & 0 \\
* & -V^i & \bar{\Psi}^i_{23} & \hat{P}^i_M & 0 & \begin{bmatrix} -\Delta B^i_1 & -UD^i_{21} \\ Y^iB^i_1 & UD^i_{21} \end{bmatrix} & 0 & 0 & 0 \\
* & * & -\hat{R}^i_p & 0 & h\varepsilon_f\bar{\Psi}^{i,T}_{23} & 0 & 0 & 0 & \bar{\Psi}_{39} \\
* & * & * & -\varepsilon_f V^i & -h\varepsilon_f\hat{P}^i_M & 0 & 0 & 0 & 0 \\
* & * & * & * & -\varepsilon_f V^i & h\varepsilon_f\begin{bmatrix} -\Delta B^i_1 & -UD^i_{21} \\ Y^iB^i_1 & UD^i_{21} \end{bmatrix} & 0 & 0 & 0 \\
* & * & * & * & * & -\gamma^2 I_{q+p} & 0 & 0 & 0 \\
* & * & * & * & * & * & -I & 0 & 0 \\
* & * & * & * & * & * & * & -V^i & 0 \\
* & * & * & * & * & * & * & * & -V^i
\end{bmatrix} < 0 \tag{28}$$

where

$$V^i = \begin{bmatrix} \Delta & -\Delta \\ -\Delta & Y^i \end{bmatrix},\ \Delta = Y^i - \bar{X}^i,\ \hat{R}^i_p = E\bar{R}^i_pE^T,$$

$$\bar{\Psi}^i_{12} = \begin{bmatrix} -K_0^T & K_0^T \\ -A_0^{i,T}\Delta - C_2^{i,T}U^T & A_0^{i,T}Y^i + C_2^{i,T}U^T \end{bmatrix} + \hat{P}^i_M,$$

$$\bar{\Psi}^i_{15} = h\varepsilon_f(\bar{\Psi}^i_{12} - V^i),\ \bar{\Psi}_{18} = \begin{bmatrix} 0 & 0 \\ -D_1^{i,T}\Delta & D_1^{i,T}Y^i \end{bmatrix},$$

$$\bar{\Psi}^i_{23} = \begin{bmatrix} 0 & -\Delta A^i_1 \\ 0 & Y^iA^i_1 \end{bmatrix} - \hat{P}^i_M,\ \bar{\Psi}_{39} = \begin{bmatrix} 0 & 0 \\ -F^{i,T}\Delta & F^{i,T}Y^i \end{bmatrix},$$

$$\hat{P}^i_M = E\bar{P}^i_ME^T, \qquad .$$

The latter can be written as

$$\Psi^i(\Delta, Y^i, K_o, U, \hat{P}^i_M Z) + \Delta^i_Y + \Delta^{i,T}_Y < 0$$

where $\Delta_Y + \Delta_Y^T$ is the part that includes products of Y^i with A^i_0, A^i_1, B^i_1, D^i_1 and F^i.

We readily find that $\Delta_Y =$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\begin{bmatrix} 0 \\ Y^i \end{bmatrix}\begin{bmatrix} 0 & A^i_0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ Y^i \end{bmatrix}\begin{bmatrix} 0 & A^i_1 \end{bmatrix} & 0 & 0 & \begin{bmatrix} 0 \\ Y^i \end{bmatrix}\begin{bmatrix} B^i_1 & 0 \end{bmatrix} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\varepsilon_f h\begin{bmatrix} 0 \\ Y^i \end{bmatrix}\begin{bmatrix} 0 & A^i_0 \end{bmatrix} & 0 & \varepsilon_f h\begin{bmatrix} 0 \\ Y^i \end{bmatrix}\begin{bmatrix} 0 & A^i_1 \end{bmatrix} & 0 & 0 & \varepsilon_f h\begin{bmatrix} 0 \\ Y^i \end{bmatrix}\begin{bmatrix} B^i_1 & 0 \end{bmatrix} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\begin{bmatrix} 0 \\ Y^i \end{bmatrix}\begin{bmatrix} 0 & D^i \end{bmatrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \begin{bmatrix} 0 \\ Y^i \end{bmatrix}\begin{bmatrix} 0 & F^i \end{bmatrix} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{29}$$

In the case where there is no uncertainty in F and D we find that $\Delta^i_Y \overset{\Delta}{=} \Theta^i\alpha^{i,T} =$

$$\begin{bmatrix} 0 \\ \begin{bmatrix} 0 \\ Y^i \end{bmatrix} \\ 0 \\ 0 \\ h\varepsilon_f\begin{bmatrix} 0 \\ Y^i \end{bmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times \alpha^{i,T}, \quad \alpha^{i,T} = \begin{bmatrix} \begin{bmatrix} 0 & A^i_0 \end{bmatrix}^T \\ 0 \\ \begin{bmatrix} 0 & A^i_1 \end{bmatrix}^T \\ 0 \\ 0 \\ \begin{bmatrix} B^i_1 & 0 \end{bmatrix}^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

In this case we have

$$\Psi^i + \Theta^i\alpha^{i,T} + \alpha^i\Theta^{i,T} < 0. \tag{31}$$

By Finsler's method the latter is equivalent to the following inequality:

$$\begin{bmatrix} \Psi^i + G\alpha^{i,T} + \alpha^iG^T & \Theta^i - G + \alpha^iH \\ * & -H - H^T \end{bmatrix} < 0. \tag{32}$$

The dimension of the new constant matrix G are identical to those of Θ^i. The constant decision matrix H is a square matrix. In the case of (30) it is a $n \times n$ matrix. In We thus arrive at the following possibly less conservative result:

Theorem 4. Consider the system of (2a-c) and (3) where the system matrices lie within the polytope $\bar{\Omega}$ of (7). For a prescribed scalar $\gamma > 0$ and a positive tuning scalar ε_f, there exists a filter of the structure (4) that achieves $J_F < 0$, where J_F is given in (6), for all nonzero $w \in \tilde{l}^2([0,\infty);\mathcal{R}^q)$, $n \in \tilde{l}^2([0,\infty);\mathcal{R}^p)$, if there exist $n \times n$ matrices $\bar{X} > 0$, $Y > 0$, $2n \times 2n$ matrix $\bar{R}_p > 0$, $n \times n$ matrices K_0 and U, $2n \times 2n$ matrix $\tilde{P}_M$ and a $n \times l$ matrix Z, and matrices H and G that satisfy (31). In the latter case the filter parameters can be extracted using (26) as explained in Theorem 3.

6 CONCLUSIONS

In this paper the theory of linear H_∞ filtering of state-multiplicative noisy discrete-time delayed systems, is extended to the robust polytopic vertex-dependant case. Delay dependent analysis and synthesis methods are developed for the robust case which are based on the input-output approach. Sufficient conditions are thus derived for the robust stability of the system and the existence of a solution to the corresponding robust BRL. Based on the robust vertex-dependant BRL derivation, the robust filtering problem is formulated and solved.

An inherent overdesign is admitted to our solution due to the use of the bounded operators which enable us to transform the retarded system to a norm-bounded one. Some additional overdesign is also admitted in our solution due to the special structure imposed on R_2.

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