Ability to Separate Situations with a Priori Coalition Structures by Means of Symmetric Solutions

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Abstract: We say that two situations described by cooperative games are inseparable by a family of solutions, when they obtain the same allocation by all solution concept of this family. The situation of separability by a family of linear solutions reduces to separability from the null game. This is the case of the family of solutions based on marginal contributions weighted by coefficients only dependent of the coalition size: the semivalues. It is known that for games with four or more players, the spaces of inseparable games from the null game contain games different to zero-game. We will prove that for five or more players, when a priori coalition blocks are introduced in the situation described by the game, the dimension of the vector spaces of inseparable games from the null game decreases in an important manner.

1 INTRODUCTION

Probabilistic values as a solution concept for cooperative games were introduced in (Weber, 1988). The payoff that a probabilistic value assigns to each player is a weighted sum of its marginal contributions to the coalitions, where the weighting coefficients form a probabilistic distribution over the coalitions to which it belongs. A particular type of probabilistic values is formed by the semivalues that were defined in (Dubey et al., 1981). In this case the weighting coefficients are independent of the players and they only depend on the coalition size. Semivalues represent a natural generalization of both the Shapley value (Shapley, 1953) and the Banzhaf value (Banzhaf, 1965; Owen, 1975). According to this approach, many works deal with the semivalues, with general properties as in (Carreras and Giménez, 2011), or applied to simple games as in (Carreras et al., 2003), and many others.

It is possible to find two cooperative games that obtain the same payoff vector for each semivalue. We say that these games are inseparable by semivalues. By the linearity property of semivalues, we can reduce the problem of separability between games to separability from the null game. The vector subspace of inseparable games from the null game by semivalues is called in (Amer et al., 2003) shared kernel and its dimension is $2^n - n^2 + n - 2$, where n denotes the number of players. For spaces of cooperative games with four or more players, the shared kernel contains games different to zero-game

The semivalues form an important family of solutions. We can evaluate their amplitude according to their faculty to separate games. Two games are separable if their difference does not belong to the shared kernel. The dimension of this subspace would mark the separation impossibility. In this paper we consider coalition structures in the player set. It is not difficult to find in the literature many papers devoted to the modified semivalues by coalition structures, for instance (Albizuri, 2009) or (Giménez and Puente, 2015), among others. Our purpose is to reduce the dimension of the vector subspace of inseparable games from the null game. For cooperative games with five or more players, modified semivalues for games with coalition structure (Amer and Giménez, 2003) are able to reduce in a significant way the dimension of the shared kernel.

In addition, once an a-priori ordering is chosen in the player set, we can see in (Amer et al., 2003) that the shared kernel is spanned by specific $\{-1,0,1\}$-valued games. These games are known as commutation games. Now, we will prove that the vector subspace of inseparable games from the null game by modified semivalues is spanned by games introduced here with the name of expanded commutation games.

The paper is organized as follows. In Section 2 we remember the solution concepts of semivalue and semivalue modified for games with a coalition structure whose allocations can be computed by means of

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the multilinear extension (Owen, 1972) of each game. Also, nomenclature and main results for inseparable games by semivalues are described. Section 3 shows that commutation games that are the solution for the problem of separability by semivalues does not have in general the same properties with respect to separability by modified semivalues. In section 4 two sufficient conditions for separability by modified semivalues are proposed. Finally, in Section 5 we determine the dimension and a basis of the vector subspace of inseparable games from the null game by modified semivalues.

2 PRELIMINARIES

2.1 Cooperative Games and Semivalues

A *cooperative game* with transferable utility is a pair (N,v), where N is a finite set of *players* and $v : 2^N \rightarrow \mathbb{R}$ is the so-called *characteristic function*, which assigns to every *coalition* $S \subseteq N$ a real number $v(S)$, the *worth* of coalition S, and satisfies the natural condition $v(\emptyset) = 0$. With G_N we denote the set of all cooperative games on N. For a given set of players N, we identify each game (N,v) with its characteristic function v.

The *multilinear extension* MLE (Owen, 1972) of cooperative game $v \in G_N$ is a function $f_v : [0,1]^N \rightarrow \mathbb{R}$ defined as

$$f_v(x_1,x_2,...,x_n) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \notin S} (1-x_j) v(S), \quad (1)$$

so that it provides all information of the game contained in its characteristic function v.

A function $\psi : G_N \rightarrow \mathbb{R}^N$ is called a *solution* and it represents a method to measure the negotiation strength of the players in the game. The payoff vector space $\mathbb{R}^N$ is also called the allocation space. The *semivalues* (Dubey et al., 1981) as solution concept were introduced and axiomatically characterized by Dubey, Neyman and Weber in 1981. The payoff to the players for a game $v \in G_N$ by a semivalue ψ is an average of marginal contributions of each player:

$$\psi_i[v] = \sum_{S \ni i} p_s [v(S) - v(S \backslash \{i\})] \quad \forall i \in N, \quad (2)$$

where the weighting coefficients p_s only depend on the coalition size and verify $\sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1$ and $p_s \geq 0$ for $1 \leq s \leq n$. With $Sem(G_N)$ we denote the set of all semivalues on G_N.

Given a number $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, we call *binomial semivalue* ψ_α to the semivalue whose coefficients are $p_{\alpha,s} = \alpha^{s-1}(1-\alpha)^{n-s}$. The extreme cases correspond to values $\alpha = 0$ and $\alpha = 1$. For $\alpha = 0$ we obtain the dictatorial index ψ_0, with coefficients $(1,0,...,0)$, whereas for $\alpha = 1$ we obtain the marginal index ψ_1, with coefficients $(0,...,0,1)$:

$$(\psi_0)_i[v] = v(\{i\}) \;\; \forall i \in N,$$
$$(\psi_1)_i[v] = v(N) - v(N \setminus \{i\}) \;\; \forall i \in N.$$

It is proven in (Amer and Giménez, 2003) that n different binomial semivalues form a reference system for the set of semivalues on G_N. Given n different numbers α_j in $[0,1]$, for every semivalue $\psi \in Sem(G_N)$ they exist unique coefficients λ_j, $1 \leq j \leq n$, such that $\psi = \sum_{j=1}^{n} \lambda_j \psi_{\alpha_j}$.

The Banzhaf value (Banzhaf, 1965; Owen, 1975) is the binomial semivalue for $\alpha = 1/2$. As it happens for the Banzhaf value, we see in (Amer and Giménez, 2003) that the allocation by every binomial semivalue can calculate replacing in the partial derivatives of MLE the variables by value α:

$$(\psi_\alpha)_i[v] = \frac{\partial f_v}{\partial x_i}(\overline{\alpha}) \quad \forall i \in N, \text{ where } \overline{\alpha} = (\alpha,\ldots,\alpha).$$

In addition, the allocation for every semivalue can be computed by means of a product of two matrices,

$$\psi[v] = B\,\Lambda, \quad (3)$$

where the matrix B depends on each reference system of semivalues $B = (b_{ij})_{1 \leq i,j \leq n}$ with $b_{ij} = (\psi_{\alpha_j})_i[v] = \frac{\partial f_v}{\partial x_i}(\overline{\alpha_j})$ and Λ is the column matrix of the coefficients of ψ in this reference system, $\Lambda^t = (\lambda_1 \lambda_2 \cdots \lambda_n)$ if $\psi = \sum_{j=1}^{n} \lambda_j \psi_{\alpha_j}$. Thus, a $(n \times n)$-matrix summarizes the payments by any semivalue to all players of a given game v.

2.2 Cooperative Games and Coalition Structures

The formation of coalition blocks in the player set N gives rise to the construction of modified solutions in attention to this circumstance. It is the case of the Owen coalition value (Owen, 1977) from the Shapley value (Shapley, 1953) or the modified Banzhaf value for games with coalition structure (Owen, 1981) from the Banzhaf value. If we denote by $B = \{B_1, B_2, ..., B_m\}$ the coalition structure in N, in both cases, the construction of the modified solutions follows a parallel way. It is considered a modified quotient game for each coalition $S \subseteq B_j$ and it is applied the Shapley or Banzhaf value. This action defines a game in B_j and there it is now applied the same solution obtaining for each $i \in B_j$ the modified allocations.

Given a semivalue $\psi \in Sem(G_N)$ with weighting coefficients p_s^n, the recursively obtained numbers

$$p_s^m = p_s^{m+1} + p_{s+1}^{m+1} \quad 1 \leq s \leq m < n,$$

define a *induced semivalue* ψ^m (Dragan, 1999) on the space of cooperative games with m players. Adding the own semivalue, the family of induced semivalues $\{\psi^m \in Sem(G_M) \,/\, 1 \leq m \leq n\}$ allows us to define the concept of *semivalue modified for games with coalition structure* (Amer and Giménez, 2003) following the same procedure as above. For a player i belongs to coalition block B_j the modified allocation has by expression

$$\psi_i[v;B] = \sum_{S\subseteq B_j\setminus\{i\}} \sum_{T\subseteq M\setminus\{j\}} p^{b_j}_{s+1}\, p^m_{t+1} \Big[v\big(\bigcup_{t\in T} B_t \cup S \cup \{i\}\big) - v\big(\bigcup_{t\in T} B_t \cup S\big)\Big]. \tag{4}$$

For the extreme coalition structures, individual blocks and grand coalition, the modified allocations agree with the allocation by the initial semivalue. Also, the allocations by modified semivalues can be computed by means of a product of matrices, once a reference system of binomial semivalues has been chosen:

$$\psi_i[v;B] = \Lambda^t\, A(i)\, \Lambda. \tag{5}$$

Matrix Λ is like in expression (3). The terms $a_{pq}(i)$, $1 \leq p,q \leq n$, of matrix $A(i)$ can be obtained by means of the following rules:

(i) Obtain the MLE $f_v = f_v(x_1,...,x_n)$ of game v.

(ii) For each $t \in M$, $t \neq j$, and each $m \in B_t$ replace the variable x_m by y_t. Thus, a new function of the variables x_k, y_t for $k \in B_j$ and $t \in M\setminus\{j\}$ is obtained.

(iii) In the above function, reduce all exponents that appear in y_t to 1, that is, replace y_t^r $(r > 1)$ by y_t, obtaining another multilinear function $g_j(x_k,y_t)$ $k \in B_j$ and $t \in M\setminus\{j\}$.

(iv) Calculate the derivative of the function g_j with respect to variable x_i.

(v) Replace each x_k with α_p and each y_t with α_q.

Then,

$$a_{pq}(i) = \frac{\partial g_j}{\partial x_i}(\overline{\alpha_p},\overline{\alpha_q}) \quad \text{for} \quad 1 \leq p,q \leq n. \tag{6}$$

2.3 Separability in Cooperative Games

We say that two cooperative games $v, v' \in G_N$ are *separable* by a solution ψ on G_N if $\psi[v] \neq \psi[v']$ for $v \neq v'$. When we study separability between games according to semivalues, we can only consider separability from the null game, since these solutions verify linearity property.

For each G_N, the linear subspace of all cooperative games inseparable by semivalues from the null game is called in (Amer et al., 2003) *shared kernel* C_N. It is proven that the dimension of C_N is $2^n - n^2 + n - 2$, since games in C_N have to satisfy conditions:

$$\sum_{S\ni i,\,|S|=s} v(S) = 0 \quad \text{for all } i \in N \text{ and } 1 \leq s \leq n. \tag{7}$$

Grouping these conditions according to coalition sizes, the freedom degrees for each s with $2 \leq s \leq n-2$ are $\binom{n}{s} - n$, whereas $v(S) = 0$ for $|S| = 1, n-1, n$. This way, the dimension of C_N is $2^n - n^2 + n - 2$ for $|N| = n \geq 2$ and $C_N = \{0\}$ if $|N| = 2, 3$.

In game spaces G_N with cardinality $|N| \geq 4$, for a given coalition $S \subseteq N$ and players $i, j \in S$ and $k,l \in N\setminus S$, we define the *commutation game* $v_{S,i,j,k,l}$ as

$$v_{S,i,j,k,l} = 1_S + 1_{S\cup\{k,l\}\setminus\{i,j\}} - 1_{S\cup\{k\}\setminus\{i\}} - 1_{S\cup\{l\}\setminus\{j\}}, \tag{8}$$

where 1_S is the unity game in G_N ($1_S(S) = 1$ and $1_S(T) = 1$ otherwise). If $v \in G_N$ is a commutation game, then $v \in C_N$. In (Amer et al., 2003), it is proven that the shared kernel is spanned by commutation games. Since each commutation game takes non null values uniquely on coalitions of a single size, the number of selected games in the proof of this property is $\binom{n}{s} - n$ for coalitions S with $2 \leq s \leq n-2$ ($|S| = s$).

3 COMMUTATION GAMES AND COALITION STRUCTURES

Let us remember that with C_N we denote the linear subspace of all cooperative games in G_N inseparable from the null game by semivalues.

Proposition 3.1. *Let $f_v = f_v(x_1,x_2,...,x_n)$ be the MLE of game $v \in G_N$.*

$$v \in C_N \;\Leftrightarrow\; \nabla f_v(\overline{\alpha}) = 0\; \forall\alpha \in [0,1],\; \overline{\alpha} = (\alpha,\ldots,\alpha).$$

Proof. If $v \in C_N$, then $\psi[v] = 0$ $\forall\psi \in Sem(G_N)$. In particular, for all binomial semivalue ψ_α with $\alpha \in [0,1]$, $\psi_\alpha[v] = \nabla f_v(\overline{\alpha}) = 0$ where $\overline{\alpha} = (\alpha,\alpha,\ldots,\alpha)$.

Conversely, since n binomial semivalues form a reference system in $Sem(G_N)$, every semivalue $\psi \in Sem(G_N)$ can uniquely be written like $\psi = \sum_{j=1}^n \lambda_j \psi_{\alpha_j}$ with $\alpha_j \in [0,1]$ for $1 \leq j \leq n$. Then,

$$\psi[v] = \sum_{j=1}^n \lambda_j \psi_{\alpha_j}[v] = \sum_{j=1}^n \lambda_j \nabla f_v(\overline{\alpha_j}) = 0$$

and game v belongs to the shared kernel C_N. $\square$

Example. Let $N = \{i, j, k, l\}$ be the set of players. For cooperative games with four players the coalition S in the commutation games is only composed by two players. For short, when $S = \{i, j\}$ we write the commutation game $v_{S,i,j,k,l}$ as $v_{i,j,k,l}$, i. e.,

$$v_{i,j,k,l} = 1_{\{i,j\}} + 1_{\{k,l\}} - 1_{\{j,k\}} - 1_{\{i,l\}}.$$

The MLE of this game is $f_{v_{i,j,k,l}} = x_i x_j + x_k x_l - x_j x_k - x_i x_l$. It is easy to see that $\nabla f_{v_{i,j,k,l}}(\overline{\alpha}) = 0\ \forall \alpha \in [0,1]$, $\overline{\alpha} = (\alpha,\alpha,\alpha,\alpha)$.

Definition 3.2. *We say that a cooperative game $v \in G_N$ is inseparable from the null game by semivalues modified for games with coalition structure if and only if $\psi[v;B] = 0$ for every semivalue ψ on G_N and every coalition structure B in N*

The above definition introduces our central concept of separability between games by modified semivalues; linearity of these solutions allows us to reduce the problem to separability from the null game. Now, the commutation games that give the solution to the problem of separability by semivalues, offer a different answer according to the cardinality of the player set.

Proposition 3.3. *Let G_N be the vector space of cooperative games with four players, $|N| = 4$. Condition of inseparable by semivalues is equivalent to condition of inseparable by semivalues modified for games with coalition structure.*

Proof. For case $|N| = 4$, the shared kernel C_N has dimension 2. According to development in (Amer et al., 2003), a basis for C_N is formed by commutation games $v_{1,4,3,2}$ and $v_{2,4,3,1}$. For the commutation games in a basis of C_N, we will prove that condition of inseparability from the null game by semivalues extends to condition of inseparability from the null game by modified semivalues. For the remaining games in C_N, the property is verified by linearity.

We consider, for example, game $v_{2,4,3,1}$ and simultaneously all possible types of coalition structures in $N = \{1,2,3,4\}$. (a) Four individual blocks. (b) One bipersonal block where game $v_{2,4,3,1}$ takes non-null value and two individual blocks. (c) Like in (b) but taking null value. (d) Two bipersonal blocks where game $v_{2,4,3,1}$ takes non-null values. (e) Like in (d) but taking null values. (f) One coalition block with three players. (g) Only one coalition block with four players.

In cases (a) and (g), both allocations coincide: $\psi[v_{2,4,3,1};B] = \psi[v_{2,4,3,1}] = 0\ \forall\psi \in Sem(G_N)$, $B = \{\{1\},\{2\},\{3\},\{4\}\}$ or $B = \{\{1,2,3,4\}\}$.

From now, we will use the MLE $f_{v_{2,4,3,1}} = x_2x_4 + x_1x_3 - x_3x_4 - x_1x_2$.

Case (b). We consider, for instance, coalition structure $B = \{\{1,2\},\{3\},\{4\}\}$. According to rules that lead to coefficients in expression (6) for obtaining value $\psi_1[v_{2,4,3,1};B]$ by means of a product of matrices as in (5), we first determine modified MLE g_1:

$$g_1(x_1,x_2,y_2,y_3) = x_2y_3 + x_1y_2 - y_2y_3 - x_1x_2\,;$$

$$\frac{\partial g_1}{\partial x_1} = y_2 - x_2 \Rightarrow a_{pq}(1) = \frac{\partial g_1}{\partial x_1}(\overline{\alpha_p},\overline{\alpha_q}) = \alpha_q - \alpha_p$$

for $1 \le p,q \le 4$.

Written any semivalue ψ as linear combination of four different binomial semivalues, we can conclude that

$$\psi_1[v_{2,4,3,1};B] = \Lambda^t A(1)\,\Lambda = 0 \quad \forall\psi \in Sem(G_N),$$

since, in this case, matrix $A(1)$ satisfies $a_{pq}(1) = -a_{qp}(1)$ for $1 \le p,q \le 4$. In a similar way, $\psi_2[v_{2,4,3,1};B] = 0\ \forall\psi \in Sem(G_N)$.

Now, for obtaining value $\psi_3[v_{2,4,3,1};B]$, we determine modified MLE g_2:

$$g_2(y_1,x_3,y_3) = y_1y_3 + y_1x_3 - x_3y_3 - y_1\,;$$

$$\frac{\partial g_2}{\partial x_3} = y_1 - y_3 \Rightarrow a_{pq}(3) = \frac{\partial g_2}{\partial x_3}(\overline{\alpha_p},\overline{\alpha_q}) = 0$$

for $1 \le p,q \le 4$.

Then $\psi_3[v_{2,4,3,1};B] = 0$ and, also, $\psi_4[v_{2,4,3,1};B] = 0$.

Case (c). Possible coalition structure $B = \{\{1,4\},\{2\},\{3\}\}$.

$$g_1(x_1,x_4,y_2,y_3) = y_2x_4 + x_1y_3 - y_3x_4 - x_1y_2\,;$$

$$\frac{\partial g_1}{\partial x_1} = y_3 - y_2 \Rightarrow a_{pq}(1) = \frac{\partial g_1}{\partial x_1}(\overline{\alpha_p},\overline{\alpha_q}) = 0$$

for $1 \le p,q \le 4$.

Consequently, $\psi_1[v_{2,4,3,1};B] = 0$. In a similar way, $\psi_4[v_{2,4,3,1};B] = 0$ and $\psi_2[v_{2,4,3,1};B] = \psi_3[v_{2,4,3,1};B] = 0$.

Similar manipulations of MLE $f_{v_{2,4,3,1}}$ in cases (d), (e) and (g) give rise to the same conclusion $\psi[v_{2,4,3,1};B] = 0$.

Conversely, if a game is inseparable from the null game by modified semivalues, in particular, it is inseparable from the null game by semivalues. It suffices to consider the coalition structure formed by individual blocks. □

Proposition 3.4. *For vector spaces of cooperative games G_N with five or more players, every commutation game is separable from the null game by semivalues modified for games with coalition structure.*

Proof. In G_N with $|N| \ge 5$, the commutation game$v_{S,i,j,k,l} = 1_S + 1_{S\cup\{k,l\}\setminus\{i,j\}} - 1_{S\cup\{k\}\setminus\{i\}} - 1_{S\cup\{l\}\setminus\{j\}}$, with $i,j \in S$ and $k,l \in N\setminus S$, has by MLE

$$f_{v_{S,i,j,k,l}} = [x_ix_j + x_kx_l - x_jx_k - x_ix_l] \prod_{p\in S\setminus\{i,j\}} x_p \prod_{q\in N\setminus(S\cup\{k,l\})} (1-x_q).$$

For coalitions S with $2 \le |S| < n-2$, we consider coalition structure $B_S = \{S, N\setminus S\}$. The modified MLE g_1 for players in block S is

$$g_1 = x_ix_j(1-y_2) \prod_{p\in S\setminus\{i,j\}} x_p$$

and

$$\frac{\partial g_1}{\partial x_i} = x_j(1-y_2)\prod_{p\in S\setminus\{i,j\}} x_p\,,$$

where $N\setminus(S\cup\{k,l\})\neq\emptyset$ since $|S|<n-2$.

Then, modified Banzhaf value β separates game $v_{S,i,j,k,l}$, $2\le|S|<n-2$, from the null game:

$$\beta_i[v_{S,i,j,k,l};B_S] = \frac{\partial g_1}{\partial x_i}(\overline{1/2},\overline{1/2}) = \frac{1}{2^s}\neq 0.$$

For case $|S|=n-2$, $S=N\setminus\{k,l\}$ and the MLE is

$$f_{v_{N\setminus\{k,l\},i,j,k,l}} = [x_ix_j+x_kx_l-x_jx_k-x_ix_l]\prod_{p\in N\setminus\{i,j,k,l\}} x_p.$$

Now, we consider coalition structure $B_{N\setminus\{k,l\}}=\{N\setminus\{k,l\},\{k,l\}\}$ and we obtain the modified MLE g_1 for players in block $N\setminus\{k,l\}$:

$$g_1 = [x_ix_j+y_2-x_jy_2-x_iy_2]\prod_{p\in N\setminus\{i,j,k,l\}} x_p,$$

where $N\setminus\{i,j,k,l\}\neq\emptyset$ since $|N|\ge 5$. Let h be a player in $N\setminus\{i,j,k,l\}$. Again, modified Banzhaf value β separates game $v_{N\setminus\{k,l\},i,j,k,l}$ from the null game:

$$\frac{\partial g_1}{\partial x_h} = [x_ix_j+y_2-x_jy_2-x_iy_2]\prod_{p\in N\setminus\{h,i,j,k,l\}} x_p$$

and

$$\beta_h[v_{N\setminus\{k,l\},i,j,k,l};B_{N\setminus\{k,l\}}] = \frac{\partial g_1}{\partial x_h}(\overline{1/2},\overline{1/2}) = \frac{1}{2^{n-3}}\neq 0.$$

□

4 SUFFICIENT CONDITIONS OF SEPARABILITY

For games with five or more players, the commutation games are not a solution for the problem of inseparability by semivalues modified for games with coalition structure. In this section we provide two sufficient conditions of separability, that is, two necessary conditions of inseparability from the null game by modified semivalues.

Proposition 4.1. *Let us consider vector spaces of cooperative games G_N with $|N|\ge 4$. If there exists a coalition S with $v(S)\neq v(N\setminus S)$, then game v is separable from the null game by semivalues modified for games with coalition structure.*

Proof. Let us suppose S' a coalition with smallest size that verifies $v(S')\neq v(N\setminus S')$. If $|S'|=1$, game v is separable from the null game by semivalues and also by modified semivalues. We can consider that $|S'|=s'\ge 2$ and $s'\le n/2$. Then, the MLE of game v can be written as

$$\begin{aligned} f_v = \sum_{S:2\le|S|\le s'}\Big[&\prod_{i\in S}x_i\prod_{j\in N\setminus S}(1-x_j)v(S)+\\ &\prod_{i\in N\setminus S}x_i\prod_{j\in S}(1-x_j)v(N\setminus S)\Big]+\\ &\sum_{S:s'<|S|<n-s'}\prod_{i\in S}x_i\prod_{j\in N\setminus S}(1-x_j)v(S). \end{aligned}$$

Now, we choose the coalition structure $B_{S'}=\{S',N\setminus S'\}$. In such a case, the modified MLE g_1 for players in coalition block S' has by expression

$$\begin{aligned} g_1 = \sum_{S\subset S',s\ge 2}\Big[&(1-y_2)\prod_{i\in S}x_i\prod_{j\in S'\setminus S}(1-x_j)+\\ &y_2\prod_{i\in S'\setminus S}x_i\prod_{j\in S}(1-x_j)\Big]v(S)+\\ &(1-y_2)\prod_{i\in S'}x_i\,v(S')+y_2\prod_{j\in S'}(1-x_j)\,v(N\setminus S'), \end{aligned}$$

because terms for coalitions S containing elements as much in S' as in $N\setminus S'$ vanish in MLE g_1. If k is a player in S',

$$\begin{aligned} \frac{\partial g_1}{\partial x_k} = \sum_{S\subset S',s\ge 2,S\ni k}\Big[&(1-y_2)\prod_{i\in S\setminus\{k\}}x_i\prod_{j\in S'\setminus S}(1-x_j)-\\ &y_2\prod_{i\in S'\setminus S}x_i\prod_{j\in S\setminus\{k\}}(1-x_j)\Big]v(S)+\\ \sum_{S\subset S',s\ge 2,S\not\ni k}\Big[&-(1-y_2)\prod_{i\in S}x_i\prod_{j\in S'\setminus(S\cup\{k\})}(1-x_j)+\\ &y_2\prod_{i\in S'\setminus(S\cup\{k\})}x_i\prod_{j\in S}(1-x_j)\Big]v(S)+\\ &+(1-y_2)\prod_{i\in S'\setminus\{k\}}x_i\,v(S')-y_2\prod_{j\in S'\setminus\{k\}}(1-x_j)\,v(N\setminus S'). \end{aligned}$$

Then

$$\frac{\partial g_1}{\partial x_k}(\overline{1/2},\overline{1/2}) = \frac{1}{2^{s'}}\left[v(S')-v(N\setminus S')\right]$$

and the modified Banzhaf value β separates game v from the null game:

$$\beta_k[v;B_{S'}] = \frac{\partial g_1}{\partial x_k}(\overline{1/2},\overline{1/2})\neq 0 \quad \text{for } k\in S'. \quad \square$$

Proposition 4.2. *For spaces of cooperative games G_N with $|N|\ge 6$, let us consider a game v that satisfies $v(S)=v(N\setminus S)$ $\forall S\subseteq N$ and $v(\{i\})=0$ $\forall i\in N$. If there exists a coalition S with*

$$v(S)\neq\sum_{T\subset S,|T|=2}v(T) \quad \textit{and} \quad 3\le|S|\le n/2, \qquad (9)$$

then game v is separable from the null game by semivalues modified for games with coalition structure.

Proof. The MLE of game v that satisfies the two first conditions of the statement can be written as

$$f_v = \sum_{S:\,2\leq|S|<n/2} \Big[\prod_{i\in S}x_i \prod_{j\in N\setminus S}(1-x_j) + \prod_{i\in N\setminus S}x_i\prod_{j\in S}(1-x_j)\Big]v(S) + \sum_{S:\,|S|=n/2}\prod_{i\in S}x_i\prod_{j\in N\setminus S}(1-x_j)\,v(S), \tag{10}$$

where the second sum only appears in case n even number. Let us suppose S' a coalition with smallest size that verifies (9) for $|S'|<n/2$. In such a case, we choose coalition structure $B_{S'}=\{S',N\setminus S'\}$ and write modified MLE g_1 for players in coalition block S':

$$g_1 = \sum_{S\subset S',\,2\leq s<s'} \Big[(1-y_2)\prod_{i\in S}x_i\prod_{j\in S'\setminus S}(1-x_j) + y_2\prod_{i\in S'\setminus S}x_i\prod_{j\in S}(1-x_j)\Big]v(S) + \Big[(1-y_2)\prod_{i\in S'}x_i + y_2\prod_{j\in S'}(1-x_j)\Big]v(S').$$

Next, we consider a player j_1 in block S', compute the partial derivative of MLE g_1 with respect to variable x_{j_1} and replace all variables by generic value α grouping the sums as follows:

$$\frac{\partial g_1}{\partial x_{j_1}}(\overline{\alpha},\overline{\alpha}) = \sum_{S\subset S',\,S\ni j_1,\,|S|=2}\big[\alpha(1-\alpha)^{s'-1}-\alpha^{s'-1}(1-\alpha)\big]v(S) + \sum_{S\subset S',\,S\ni j_1,\,2<s<s'}\big[\alpha^{s-1}(1-\alpha)^{s'-s+1}-\alpha^{s'-s+1}(1-\alpha)^{s-1}\big]v(S) + \sum_{S\subset S',\,S\not\ni j_1,\,2\leq s<s'-1}\big[\alpha^{s'-s}(1-\alpha)^{s}-\alpha^{s}(1-\alpha)^{s'-s}\big]v(S) + \big[\alpha(1-\alpha)^{s'-1}-\alpha^{s'-1}(1-\alpha)\big]\big[v(S'\setminus\{j_1\})-v(S')\big].$$

All terms for coalitions S with $S\not\ni j_1$ and $2\leq s<s'-1$ can be written by means of coalitions T with $T\ni j_1$ and $3\leq t<s'$. Then,

$$\frac{\partial g_1}{\partial x_{j_1}}(\overline{\alpha},\overline{\alpha}) = \alpha(1-\alpha)\big[(1-\alpha)^{s'-2}-\alpha^{s'-2}\big]\Big\{\sum_{S\subset S',\,S\ni j_1,\,|S|=2}v(S)+v(S'\setminus\{j_1\})-v(S')\Big\} + \sum_{S\subset S',\,S\ni j_1,\,2<s<s'}\big[\alpha^{s-1}(1-\alpha)^{s'-s+1}-\alpha^{s'-s+1}(1-\alpha)^{s-1}\big]v(S) + \sum_{T\subset S',\,T\ni j_1,\,2<t<s'}\big[\alpha^{s'-t+1}(1-\alpha)^{t-1}-\alpha^{t-1}(1-\alpha)^{s'-t+1}\big]v(T\setminus\{j_1\}).$$

We shorten polynomial $(1-\alpha)^{s'-2}-\alpha^{s'-2}$ by means of $p_{s'}(\alpha)$ and write $v(S'\setminus\{j_1\})$ as a sum of all values on contained bipersonal coalitions:

$$\frac{\partial g_1}{\partial x_{j_1}}(\overline{\alpha},\overline{\alpha}) = \alpha(1-\alpha)p_{s'}(\alpha)\Big[\sum_{S\subset S',\,S\ni j_1,\,|S|=2}v(S) + \sum_{T\subseteq S'\setminus\{j_1\},\,|T|=2}v(T)-v(S')\Big] + \sum_{S\subset S',\,S\ni j_1,\,2<s<s'}\big[\alpha^{s-1}(1-\alpha)^{s'-s+1}-\alpha^{s'-s+1}(1-\alpha)^{s-1}\big]\big[v(S)-v(S\setminus\{j_1\})\big]. \tag{11}$$

It is possible to find coalitions S with $S\subset S'$, $S\ni j_1$ and $2<s<s'$ only in case $s'\geq 4$. Then, the last sum in the above expression can be written as

$$\sum_{S\subset S',\,S\ni j_1,\,3\leq s<1+s'/2}\big[\alpha^{s-1}(1-\alpha)^{s'-s+1}-\alpha^{s'-s+1}(1-\alpha)^{s-1}\big]\big[v(S)-v(S\setminus\{j_1\})\big] + \sum_{T\subset S',\,T\ni j_1,\,1+s'/2<t\leq s'-1}\big[\alpha^{t-1}(1-\alpha)^{s'-t+1}-\alpha^{s'-t+1}(1-\alpha)^{t-1}\big]\big[v(T)-v(T\setminus\{j_1\})\big],$$

where case $s=1+s'/2$ is not considered, since only for s' even number, cardinality of S can take value $s=1+s'/2$ but, in this case, coefficient $\alpha^{s-1}(1-\alpha)^{s'-s+1}-\alpha^{s'-s+1}(1-\alpha)^{s-1}$ vanish. In the above sums, we can identify coalitions S for $3\leq s<1+s'/2$ with coalitions T for $1+s'/2<t\leq s'-1$ by means relation $t=s'-s+2$. Then, both sums reduce to

$$\sum_{3\leq s<1+s'/2}\big[\alpha^{s-1}(1-\alpha)^{s'-s+1}-\alpha^{s'-s+1}(1-\alpha)^{s-1}\big]\Big\{\sum_{S\subset S',\,S\ni j_1,\,|S|=s}\big[v(S)-v(S\setminus\{j_1\})\big] - \sum_{T\subset S',\,T\ni j_1,\,|T|=s'-s+2}\big[v(T)-v(T\setminus\{j_1\})\big]\Big\}.$$

Let us suppose that $S'=\{j_1,j_2,\ldots,j_{s'}\}$. For a given cardinality s with $3\leq s<1+s'/2$, the last difference of sums vanish, because it can be written as

$$\sum_{S\subset S',\,S\ni j_1,\,|S|=s}\Big[\sum_{P\subset S,\,|P|=2}v(P)-\sum_{Q\subseteq S\setminus\{j_1\},\,|Q|=2}v(Q)\Big] - \sum_{T\subset S',\,T\ni j_1,\,|T|=s'-s+2}\Big[\sum_{P\subset T,\,|P|=2}v(P)-\sum_{Q\subset T\setminus\{j_1\},\,|Q|=2}v(Q)\Big] =$$

$$\sum_{S\subset S',\,S\ni j_1,\,|S|=s}\Big[\sum_{P\subset S,\,P\ni j_1,\,|P|=2}v(P)\Big] - \sum_{T\subset S',\,T\ni j_1,\,|T|=s'-s+2}\Big[\sum_{P\subset T,\,P\ni j_1,\,|P|=2}v(P)\Big] =$$

$$\sum_{i=2}^{s'}\left[\binom{s'-2}{s-2}-\binom{s'-2}{s'-s}\right]v(\{j_1,j_i\})=0.$$

Thus, from expression (11), we can write the modified binomial semivalue ψ_α for player $j_1 \in S'$ as

$$(\psi_\alpha)_{j_1}[v;B_{S'}]=\frac{\partial g_1}{\partial x_{j_1}}(\overline{\alpha},\overline{\alpha})=$$

$$\alpha(1-\alpha)p_{s'}(\alpha)\Big[\sum_{T\subset S',|T|=2}v(T)-v(S')\Big].$$

Since $\alpha = 1/2$ is the unique real zero of polynomial $p_{s'}$ for values $s' \geq 3$ and game v satisfies inequality (9) for coalition S', we conclude that $(\psi_\alpha)_{j_1}[v;B_{S'}] \neq 0$ for values $\alpha \in (0,1/2)\cup(1/2,1)$ and these modified semivalues separate game v from the null game.

It only lack to see case in which $|S| = n/2$ is the smallest size of coalitions that verify (9). Here, n is a even number and all coalitions in the second sum of expression (10) can be grouped by pairs: S and $N\setminus S$. The selected coalition S' will belong to one or another half of coalitions with size $n/2$; we choose half that contains coalition S' and describe the second sum with S and $N\setminus S$, as the same way that the first sum in (10). Then, by repeating the same procedure as in case $|S| < n/2$, we arrived at the same conclusion. □

5 EXPANDED COMMUTATION GAMES

We denote with D_N the vector subspace of all cooperative games in G_N inseparable from the null game by semivalues modified for games with coalition structure.

Definition 5.1. *In G_N with $|N| \geq 5$, we consider a commutation game with coalition size 2, $v_{i,j,k,l}$, $k,l \in N\setminus\{i,j\}$. The expanded game of commutation game $v_{i,j,k,l}$ is the sum of all commutation games in G_N, $v_{P,i,j,k,l}$, with the same commuted players, i.e.,*

$$v^e_{i,j,k,l}=\sum_{P\ni i,j,\,P\subseteq N\setminus\{k,l\}}v_{P,i,j,k,l}.$$

Lemma 5.2. *In G_N with $|N| \geq 5$ an expanded commutation game $v^e_{i,j,k,l}$, $k,l \in N\setminus\{i,j\}$, satisfies the following properties:*

(a) $v^e_{i,j,k,l}(S) = v^e_{i,j,k,l}(N\setminus S) \quad \forall S\subseteq N$;

(b) $v^e_{i,j,k,l}(S) = \sum_{T\subset S,|T|=2} v^e_{i,j,k,l}(T)$ $\forall S\subseteq N$ *and* $3 \leq |S| \leq |N|$;

(c) its MLE is $f_{v^e_{i,j,k,l}} = x_ix_j + x_kx_l - x_jx_k - x_ix_l$.

Proof. It is easy to prove sections (a) and (b); it suffices to check if players i,j,k,l belong or not to coalitions S, since the only bipersonal coalitions that take non-null values in game $v^e_{i,j,k,l}$ are $\{i,j\}$, $\{k,l\}$, $\{j,k\}$ and $\{i,l\}$. In order to verify section (c) we can write MLE of game $v^e_{i,j,k,l}$ as

$$f_{v^e_{i,j,k,l}}=\left[x_ix_j+x_kx_l-x_jx_k-x_ix_l\right]\left[\prod_{q\in N\setminus\{i,j,k,l\}}(1-x_q)+f_{\sum_{Q\subseteq N\setminus\{i,j,k,l\}}1_Q}\right],$$

where games 1_Q are considered in $G_{N\setminus\{i,j,k,l\}}$. Since $\sum_{Q\subseteq N\setminus\{i,j,k,l\}} 1_Q(T) = 1$ $\forall T \subseteq N\setminus\{i,j,k,l\}$, $T \neq \emptyset$, $(Q \neq \emptyset)$, its MLE equals the unity in $N\setminus\{i,j,k,l\}$ and section (c) follows. □

Proposition 5.3. *In spaces of cooperative games G_N with $|N| \geq 5$, every expanded commutation game $v^e_{i,j,k,l}$, $k,l \in N\setminus\{i,j\}$ belongs to vector subspace D_N.*

Proof. Section (c) in above Lemma proves that MLE of expanded commutation game $v^e_{i,j,k,l}$, $k,l \in N\setminus\{i,j\}$ in G_N with $|N| \geq 5$ agrees with MLE of commutation game $v_{i,j,k,l}$ in a space of cooperative games with only four players, $\{i,j,k,l\}$.

In order to demonstrate that game $v^e_{i,j,k,l}$, $k,l \in N\setminus\{i,j\}$, is inseparable by modified semivalues, we can consider that players i,j,k,l are distributed in different coalition blocks in the same way that in the proof of Proposition 3.3. The remaining players $N\setminus\{i,j,k,l\}$ will be distributed in the different blocks next to players i,j,k,l or they will form new coalition blocks.

Since variables that correspond to players in $N\setminus\{i,j,k,l\}$ does not appear in the MLE of game $v^e_{i,j,k,l}$, when we compute allocations for players i,j,k,l by means of a product of matrices as in (5), we obtain the same result as in Proposition 3.3, that is, $\psi_p[v^e_{i,j,k,l},B] = 0$ for $p = i,j,k,l$, $\forall\psi \in Sem(G_N)$, $\forall B$ coalition structure in N.

For the remaining players, $\psi_q[v^e_{i,j,k,l},B] = 0$ $\forall q \in N\setminus\{i,j,k,l\}$, since variable x_q does not appear in the MLE. □

Theorem 5.4. *Let us consider vector spaces of cooperative games G_N with five or more players, $|N| \geq 5$. Then,*

(a) $\dim D_N = \binom{n}{2} - n$;

(b) the vector subspace D_N is spanned by expanded of commutation games with coalition size 2.

Proof. We can see in (Amer et al., 2003) that the shared kernel C_N for $|N| \geq 4$ is spanned by $2^n - n^2 + n - 2$ commutation games whose coalitions with non-null value vary from cardinality $s = 2$ to $n-2$. We choose the $\binom{n}{2} - n$ commutation games with coalition size 2. As they are linearly independent in G_N, its expanded games are also linearly independent and, by above Proposition, inseparable from the null game by modified semivalues. The linear subspace spanned by these expanded commutation games is contained in subspace D_N for $|N| \geq 5$.

In addition, as $D_N \subseteq C_N$, the freedom degrees in C_N by a consequence of conditions (7) for coalitions with sizes $s > n/2$ disappear according to necessary condition of inseparability from the null game in D_N: $v(S) = v(N \setminus S)$ (Proposition 4.1). Also, the freedom degrees for coalitions with size from $s = 3$ to $s = n/2$ disappear according to necessary condition $v(S) = \sum_{T\subset S, |T|=2} v(T)$ $\forall S \subseteq N$ with $3 \leq |S| \leq n/2$ (Proposition 4.2).

Only the $\binom{n}{2} - n$ freedom degrees for coalition size $s = 2$ in C_N remain in vector subspace D_N. Then, the vector subspace spanned by the $\binom{n}{2} - n$ expanded commutation games agrees with D_N. $\square$

6 CONCLUSION

It is known that every cooperative game with two or three players is separable from the null game by semivalues, so that dimension for the shared kernel C_N is zero in cases $n = 2, 3$. Consequently, vector subspace D_N is only formed by the null game in cases $n = 2, 3$. For games with four players, Proposition 3.3 proves that both separability concepts coincide: $D_N = C_N$ for $n = 4$.

Table 1 compares dimensions of C_N and D_N for cooperative games with few players.

Table 1: Dimensions of kernels according to N.

$\lvert N\rvert = n$	2	3	4	5	6	7	8
$\dim G_N$	3	7	15	31	63	127	255
$\dim C_N$	0	0	2	10	32	84	198
$\dim D_N$	0	0	2	5	9	14	20

For games with five or more players, the introduction of modified semivalues for games with coalition structure allows us to reduce in a significant way the dimension of the vector subspace of inseparable games from the null game. According to the linearity property, separability between two games is reduced by both concepts of solution to separability of their difference from the null game. The ability of separation by semivalues has considerably increased by introduction of a priori coalition structures.

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