On Bipartite Fuzzy Stochastic Differential Equations

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Abstract: The paper contains a discussion on solutions to new type of fuzzy stochastic differential equations. The equations under study possess drift and diffusion terms at both sides of equations. We claim that such the equations have unique solutions in the case that equations' coefficients satisfy a certain generalized Lipschitz condition. We use approximation sequences to reach solutions.

1 INTRODUCTION

In modelling dynamical systems in presence of uncertainty the stochastic differential equations are used (Gihman and Skorohod, 1972; Mao, 2007). However, in the real-life phenomena there is often a sources of uncertainty that does not come from randomness and stochastic noises. This uncertainty is well treated by fuzzy set theory (Zadeh, 1965). The fuzzy sets theory has been successfully applied to deterministic fuzzy differential equations (Kaleva, 1987; Bede and Gal, 2005; Nieto and Rodríguez-López, 2006). There are also many attempts to use two kinds of uncertainties in modelling real-world systems, for instance (Li et al., 2003; Möller et al., 2003; Zmeškal, 2010). An apparatus to model dynamical systems with randomness and fuzziness in a form of random fuzzy differential equations with fuzzy derivative were studied too (Feng, 2000; Malinowski, 2009; Malinowski, 2012b; Park and Jeong, 2013; Malinowski, 2015c). However, these models are not enough when some stochastic noises in terms of Brownian motions appear. In such the situations fuzzy stochastic differential equations are more appropriate to be applied (Malinowski, 2012c; Malinowski, 2013a; Malinowski, 2013b; Malinowski, 2015d; Malinowski, 2015e; Malinowski, 2015a; Malinowski, 2015b; Malinowski and Agarwal, 2015).

The latter topic on fuzzy stochastic differential equations is new and needs further investigations. In this paper we proceed with a discussion on bipartite fuzzy stochastic differential equation in its integral form

$$\begin{aligned} x(t) \oplus (-1) \odot \int_0^t f(s,x(s))ds \\ \oplus \Big\langle \ (-1) \int_0^t g(s,x(s))d\mathfrak{B}(s) \Big\rangle \\ = x_0 \oplus \int_0^t \tilde{f}(s,x(s))ds \\ \oplus \Big\langle \int_0^t \tilde{g}(s,x(s))d\tilde{\mathfrak{B}}(s) \Big\rangle, \ t \in [0,T] \end{aligned} \tag{1.1}$$

which contains drift and diffusion parts at both sides and is driven by m-dimensional and n-dimensional Brownian motions $\mathfrak{B}$ and $\tilde{\mathfrak{B}}$, respectively. A detailed description of this form is included in Section 3. Such the equations were introduced by (Malinowski, 2016a). A fundamental problem on existence of solutions to such the equations was considered with the Lipschitz type assumptions imposed on equations' coefficients. In the current paper we intend to show that these equations have solutions when one relax the assumptions to a certain generalized condition involving a certain function instead of the Lipschitz constant. We will use a sequence of approximations to achieve it. We limit ourselves to prove the main results only, since there is a pages' limitation for this submission.

2 PRELIMINARIES

In this section, we give some definitions and useful facts and introduce necessary notation which will be used throughout the paper. Most of it can be found, for example, in (Malinowski, 2014; Malinowski, 2016b; Malinowski, 2016c).

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Let $\mathcal{K}(\mathbb{R}^d)$ be the family of all nonempty, compact and convex subsets of $\mathbb{R}^d$. In $\mathcal{K}(\mathbb{R}^d)$ we consider the Hausdorff metric d_H which is defined by $d_H(A,B) := \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\}$, where $\|\cdot\|$ denotes a norm in $\mathbb{R}^d$. The addition and scalar multiplication in $\mathcal{K}(\mathbb{R}^d)$ are defined as usual, i.e., for $A,B\in\mathcal{K}(\mathbb{R}^d)$, $\lambda\in\mathbb{R}$ we have $A+B := \{a+b : a\in A, b\in B\}$, $\lambda A := \{\lambda a : a\in A\}$.

Let $(\Omega,\mathcal{A},P)$ be a complete probability space and $\mathcal{M}(\Omega,\mathcal{A};\mathcal{K}(\mathbb{R}^d))$ denote the family of $\mathcal{A}$-measurable set-valued random variables. A set-valued random variable $F\in\mathcal{M}(\Omega,\mathcal{A};\mathcal{K}(\mathbb{R}^d))$ is said to be $\mathcal{L}^p$-integrally bounded, $p\geqslant 1$, if there exists $h\in L^p(\Omega,\mathcal{A},P;\mathbb{R})$ such that $\|a\|\leqslant h(\omega)$ for any a and ω with $a\in F(\omega)$. Let us denote $\mathcal{L}^p(\Omega,\mathcal{A},P;\mathcal{K}(\mathbb{R}^d)) := \big\{F\in\mathcal{M}(\Omega,\mathcal{A};\mathcal{K}(\mathbb{R}^d)) : \omega\mapsto d_H(F(\omega),\{0\}) \text{ is in } L^p(\Omega,\mathcal{A},P;\mathbb{R})\big\}$.

A fuzzy set u in $\mathbb{R}^d$ is characterized by its membership function (denoted by u again) $u\colon\mathbb{R}^d\to[0,1]$ and $u(x)$ (for each $x\in\mathbb{R}^d$) is interpreted as the degree of membership of x in the fuzzy set u. For fuzzy set $u\colon\mathbb{R}^d\to[0,1]$ one defines so-called α-levels $[u]^\alpha := \{a\in\mathbb{R}^d : u(a)\geqslant\alpha\}$ for $\alpha\in(0,1]$ and $[u]^0 := \mathrm{cl}\{a\in\mathbb{R}^d : u(a)>0\}$. Let $\mathcal{F}(\mathbb{R}^d)$ denote a set of fuzzy sets $u\colon\mathbb{R}^d\to[0,1]$ such that $[u]^\alpha\in\mathcal{K}(\mathbb{R}^d)$ for every $\alpha\in[0,1]$ and the mapping $\alpha\mapsto[u]^\alpha$ is d_H-continuous on $[0,1]$. By $\langle r\rangle$ we mean the characteristic function of the singleton $\{r\}$, $r\in\mathbb{R}^d$. Obviously, $\langle r\rangle\in\mathcal{F}(\mathbb{R}^d)$. The addition $u\oplus v$ and scalar multiplication $\lambda\odot u$ in $\mathcal{F}(\mathbb{R}^d)$ can be defined levelwise, i.e. $[u\oplus v]^\alpha = [u]^\alpha+[v]^\alpha$, $[\lambda\odot u]^\alpha = \lambda[u]^\alpha$, where $u,v\in\mathcal{F}(\mathbb{R}^d)$, $\lambda\in\mathbb{R}$ and $\alpha\in[0,1]$. If for $u,v\in\mathcal{F}(\mathbb{R}^d)$ there exists $w\in\mathcal{F}(\mathbb{R}^d)$ such that $u = v\oplus w$ then w is said to be the fuzzy Hukuhara difference of u and v and we denote it by $u\ominus v$. In $\mathcal{F}(\mathbb{R}^d)$ we consider the metric $d_\infty(u,v) := \sup_{\alpha\in[0,1]} d_H([u]^\alpha,[v]^\alpha)$.

An $x\colon\Omega\to\mathcal{F}(\mathbb{R}^d)$ is called a fuzzy random variable, if $[x]^\alpha\colon\Omega\to\mathcal{K}(\mathbb{R}^d)$ is a random set for all $\alpha\in[0,1]$. It is known that in the framework considered here, this definition is equivalent to $\mathcal{A}|\mathcal{B}_{d_\infty}$-measurability of $x\colon\Omega\to\mathcal{F}(\mathbb{R}^d)$, see (Joo et al., 2006). A fuzzy random variable $x\colon\Omega\to\mathcal{F}(\mathbb{R}^d)$ is said to be L^p-integrably bounded, $p\geq 1$, if $[x]^0$ belongs to $\mathcal{L}^p(\Omega,\mathcal{A},P;\mathcal{K}(\mathbb{R}^d))$. By $\mathcal{L}^p(\Omega,\mathcal{A},P;\mathcal{F}(\mathbb{R}^d))$ we denote the set of the all L^p-integrably bounded fuzzy random variables. In the set $\mathcal{L}^2(\Omega,\mathcal{A},P;\mathcal{F}(\mathbb{R}^d))$ one can define a metric ρ by $\rho(x,y) := \big(\mathbb{E}d_\infty^2(x,y)\big)^{1/2}$. Then the metric space $\big(\mathcal{L}^2(\Omega,\mathcal{A},P;\mathcal{F}(\mathbb{R}^d)),\rho\big)$ is complete, see (Feng, 1999).

Denote $I := [0,T]$. We equip the probability space with a filtration $\{\mathcal{A}_t\}_{t\in I}$ satisfying the usual hypotheses. An $x\colon I\times\Omega\to\mathcal{F}(\mathbb{R}^d)$ is called the fuzzy stochastic process, if for every $t\in I$ the mapping $x(t,\cdot)\colon\Omega\to\mathcal{F}(\mathbb{R}^d)$ is a fuzzy random variable. It is d_∞-continuous, if almost all (with respect to the probability measure P) its trajectories, i.e. the mappings $x(\cdot,\omega)\colon I\to\mathcal{F}(\mathbb{R}^d)$ are d_∞-continuous functions. A fuzzy stochastic process x is said to be nonanticipating, if for every $\alpha\in[0,1]$ the mapping $[x(\cdot,\cdot)]^\alpha$ is measurable with respect to the σ-algebra $\mathcal{N}$, which is defined as follows $\mathcal{N} := \{A\in\mathcal{B}(I)\otimes\mathcal{A} : A^t\in\mathcal{A}_t \text{ for every } t\in I\}$, where $A^t = \{\omega : (t,\omega)\in A\}$. Let $p\geq 1$ and $L^p(I\times\Omega,\mathcal{N};\mathbb{R}^d)$ denote the set of all nonanticipating stochastic processes $h\colon I\times\Omega\to\mathbb{R}^d$ such that $\mathbb{E}\int_I\|h(s)\|^p ds<\infty$. A fuzzy stochastic process x is called L^p-integrably bounded ($p\geq 1$), if there exists a real-valued stochastic process $h\in L^p(I\times\Omega,\mathcal{N};\mathbb{R})$ such that $d_\infty(x(t,\omega),\langle 0\rangle)\leq h(t,\omega)$ for a.a. $(t,\omega)\in I\times\Omega$. By $\mathcal{L}^p(I\times\Omega,\mathcal{N};\mathcal{F}(\mathbb{R}^d))$ we denote the set of nonanticipating and L^p-integrably bounded fuzzy stochastic processes. For convenience, from now on, the phrase "with $P.1$" stands for "with probability one". Also we will write $x \overset{P.1}{=} y$ instead of $P\big(x=y\big) = 1$, where x,y are random elements. Also we will write $x(t) \overset{I\ P.1}{=} y(t)$ instead of $P\big(x(t) = y(t)\ \ \forall\, t\in I\big) = 1$, where x,y are the stochastic processes.

For $\tau,t\in I$, $\tau<t$, and $x\in\mathcal{L}^1(I\times\Omega,\mathcal{N};\mathcal{F}(\mathbb{R}^d))$ we can define, see (Malinowski, 2012a; Malinowski, 2012c), the fuzzy stochastic Lebesgue–Aumann integral $\Omega\ni\omega\mapsto\int_\tau^t x(s,\omega)ds\in\mathcal{F}(\mathbb{R}^d)$ which is a fuzzy random variable.

Lemma 2.1. *Let* $p\geqslant 1$. *If* $x,y\in\mathcal{L}^p(I\times\Omega,\mathcal{N};\mathcal{F}(\mathbb{R}^d))$ *then*

(i) $I\times\Omega\ni(t,\omega)\mapsto\int_0^t x(s,\omega)ds\in\mathcal{F}(\mathbb{R}^d)$ *belongs to* $\mathcal{L}^p(I\times\Omega,\mathcal{N};\mathcal{F}(\mathbb{R}^d))$,

(ii) the fuzzy process $(t,\omega)\mapsto\int_0^t x(s,\omega)ds$ *is* d_∞-*continuous,*

(iii)

$$sup_{u\in[0,t]}\quad d_\infty^p\Big(\int_0^u x(s)ds, \int_0^u y(s)ds\Big) \overset{I\ P.1}{\leqslant} t^{p-1}\int_0^t d_\infty^p\big(x(s),y(s)\big)ds,$$

(iv) for every $t\in I$ *it holds*

$$\mathbb{E}sup_{u\in[0,t]}\quad d_\infty^p\Big(\int_0^u x(s)ds, \int_0^u y(s)ds\Big) \leqslant t^{p-1}\mathbb{E}\int_0^t d_\infty^p\big(x(s),y(s)\big)ds.$$

As we mentioned e.g. in (Malinowski, 2012c; Malinowski, 2013c) it is not possible to define fuzzy stochastic integral of Itô type such the integral in such a fashion that it is not a crisp random variable. Hence, we consider the diffusion part of the fuzzy stochastic differential equation as the crisp stochastic Itô integral whose values are embedded into $\mathcal{F}(\mathbb{R}^d)$.

For convenience of the reader we give also formulation of the Bihari inequality that we will useful in the paper.

Lemma 2.2. *(Bihari's inequality, see e.g. Theorem 1.8.2 in (Mao, 2007)). Let $T > 0$ and $c > 0$. Let $\kappa\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function such that $\kappa(t) > 0$ for every $t > 0$. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0,T]$, and let $v(\cdot)$ be a nonnegative integrable function on $[0,T]$. If $u(t) \leqslant c + \int_0^t v(s)\kappa(u(s))ds$ for every $t \in [0,T]$, then $u(t) \leqslant J^{-1}\left(J(c) + \int_0^t v(s)ds\right)$ holds for all such $t \in [0,T]$ that $J(c) + \int_0^t v(s)ds \in \mathrm{Dom}(J^{-1})$, where $J(r) = \int_1^r \frac{ds}{\kappa(s)}$, $r > 0$, and J^{-1} is the inverse function of J. Moreover, if $c = 0$ and $\int_{0+} \frac{ds}{\kappa(s)} = \infty$ then $u(t) = 0$ for every $t \in [0,T]$.*

3 MAIN RESULTS

In (Malinowski, 2016a) we introduced a new type of fuzzy stochastic differential equations (1.1) which is by far the most general one. More precisely, we considered an expanded integral form of these equations

$$x(t) \oplus (-1) \odot \int_0^t f(s,x(s))ds \oplus \left\langle \sum_{i=1}^m \int_0^t (-1) g^i(s,x(s))dB^i(s) \right\rangle \overset{I\ P.1}{=} x_0 \oplus \int_0^t \tilde{f}(s,x(s))ds \oplus \left\langle \sum_{j=1}^n \int_0^t \tilde{g}^j(s,x(s))d\tilde{B}^j(s) \right\rangle, \tag{3.1}$$

where $f,\tilde{f}\colon I\times\Omega\times\mathcal{F}(\mathbb{R}^d) \to \mathcal{F}(\mathbb{R}^d)$, $g\colon I\times\Omega\times\mathcal{F}(\mathbb{R}^d) \to \mathbb{R}^d\times\mathbb{R}^m$, $\tilde{g}\colon I\times\Omega\times\mathcal{F}(\mathbb{R}^d) \to \mathbb{R}^d\times\mathbb{R}^n$, $x_0\colon \Omega \to \mathcal{F}(\mathbb{R}^d)$ is a fuzzy random variable, and $B^1, B^2, \ldots, B^m, \tilde{B}^1, \tilde{B}^2, \ldots, \tilde{B}^n$ are the independent, one-dimensional $\{\mathcal{A}_t\}_{t\in I}$-Brownian motions.

It has been noticed that without loss of generality one can study an equivalent form of (3.1), i.e.

$$x(t) \overset{I\ P.1}{=} \left[\left(x_0 \oplus \int_0^t \tilde{f}(s,x(s))ds\right) \ominus \left((-1)\odot \int_0^t f(s,x(s))ds\right)\right] \oplus \left\langle \sum_{k=1}^{\ell} \int_0^t h^k(s,x(s))dW^k(s) \right\rangle, \tag{3.2}$$

where $\ell = m+n$, $h^1, h^2, \ldots, h^\ell\colon I\times\Omega\times\mathcal{F}(\mathbb{R}^d) \to \mathbb{R}^d$ and $W^1, W^2, \ldots, W^\ell$ are the independent one-dimensional $\{\mathcal{A}_t\}_{t\in I}$-Brownian motions, x_0 is a fuzzy random variable. The existence of solutions to such the equations is a fundamental issue. Below we explain what we mean by a solution to (3.2). Let $\tilde{T} \in (0,T]$, $\tilde{I} = [0,\tilde{T}]$.

Definition 3.1. *A fuzzy stochastic process $x\colon \tilde{I}\times\Omega \to \mathcal{F}(\mathbb{R}^d)$ is said to be the solution to* (3.2) *if it satisfies: (i) $x \in \mathcal{L}^2(\tilde{I}\times\Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$, (ii) x is d_∞-continuous, (iii) it holds* (3.2)*. If $\tilde{T} < T$ then x is called a local solution, and if $\tilde{T} = T$, then x is called the global solution. A solution $x\colon \tilde{I}\times\Omega \to \mathcal{F}(\mathbb{R}^d)$ to equation* (3.2) *is said to be unique, if $x(t) \overset{\tilde{I}\ P.1}{=} y(t)$, where $y\colon \tilde{I}\times\Omega \to \mathcal{F}(\mathbb{R}^d)$ is any other local solution to* (3.2).

In what follows we begin our study with a first and most important issue of existence and uniqueness of solutions to (3.2). In the paper we require that $x_0\colon \Omega \to \mathcal{F}(\mathbb{R}^d)$, $f,\tilde{f}\colon I\times\Omega\times\mathcal{F}(\mathbb{R}^d) \to \mathcal{F}(\mathbb{R}^d)$, $h^k\colon I\times\Omega\times\mathcal{F}(\mathbb{R}^d) \to \mathbb{R}^d$ ($k = 1,2,\ldots,\ell$) satisfy:

(A0) $x_0 \in \mathcal{L}^2(\Omega, \mathcal{A}_0, P; \mathcal{F}(\mathbb{R}^d))$,

(A1) the mappings $f,\tilde{f}\colon (I\times\Omega)\times\mathcal{F}(\mathbb{R}^d) \to \mathcal{F}(\mathbb{R}^d)$ are $\mathcal{N}\otimes\mathcal{B}_{d_\infty}|\mathcal{B}_{d_\infty}$-measurable and $h^1, h^2, \ldots, h^\ell\colon (I\times\Omega)\times\mathcal{F}(\mathbb{R}^d) \to \mathbb{R}^d$ are $\mathcal{N}\otimes\mathcal{B}_{d_\infty}|\mathcal{B}(\mathbb{R}^d)$-measurable,

(A2) P-a.a. it holds
$$d_\infty^2\big(f(t,\omega,u), f(t,\omega,v)\big) \leqslant \xi(d_\infty^2(u,v)),$$
$$d_\infty^2\big(\tilde{f}(t,\omega,u), \tilde{f}(t,\omega,v)\big) \leqslant \xi(d_\infty^2(u,v)),$$
for every $t\in I$ and for any $u,v \in \mathcal{F}(\mathbb{R}^d)$, and
$$\|h^k(t,\omega,u) - h^k(t,\omega,v)\|^2 \leqslant \xi(d_\infty^2(u,v)),$$
for every $t \in I$, for any $u,v \in \mathcal{F}(\mathbb{R}^d)$ and $k = 1,2,\ldots,\ell$, where $\xi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, concave, nondecreasing function such that $\xi(0) = 0$, $\xi(u) > 0$ for $u > 0$, and $\int_{0+} \frac{du}{\xi(u)} = \infty$,

(A3) there exists a constant $C > 0$ such that P-a.a. it holds: for every $t \in I$
$$d_\infty^2\big(f(t,\omega,\langle 0\rangle), \langle 0\rangle\big) \wedge d_\infty^2\big(\tilde{f}(t,\omega,\langle 0\rangle), \langle 0\rangle\big) \leqslant C,$$
and for every $t\in I$ and $k = 1,2,\ldots,\ell$
$$\|h^k(t,\omega,\langle 0\rangle)\|^2 \leqslant C,$$

(A4) there exists a constant $\tilde{T} \in (0,T]$ such that P-a.e. the fuzzy Hukuhara differences
$$\int_\tau^t \tilde{f}(s,\omega,x(s,\omega))ds \ominus \int_\tau^t f(s,\omega,x(s,\omega))ds$$
do exist, for every $\tau, t \in \tilde{I} = [0,\tilde{T}]$ and for every d_∞-continuous $x \in \mathcal{L}^2(\tilde{I}\times\Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$.

The condition (A2) is much more general than the Lipschitz condition used in (Malinowski, 2016a). The conditions (A0)-(A4) assure existence of a unique solution to (3.2). This fact constitutes a main result of the paper.

Theorem 3.2. *Let $x_0 \colon \Omega \to \mathcal{F}(\mathbb{R}^d)$, $f, \tilde{f} \colon (I \times \Omega) \times \mathcal{F}(\mathbb{R}^d) \to \mathcal{F}(\mathbb{R}^d)$ and $h^k \colon (I \times \Omega) \times \mathcal{F}(\mathbb{R}^d) \to \mathbb{R}^d$ ($k = 1, 2, \ldots, \ell$) satisfy (A0)-(A4). Then (3.2) possesses a unique solution $x \colon \hat{I} \times \Omega \to \mathcal{F}(\mathbb{R}^d)$.*

To prove this result we will use a sequence of successive approximations $\{y_n\}_{n\in\mathbb{N}}$ defined as follows:

$$
\begin{aligned}
&y_n(t) \overset{[-1,0]\ P.1}{=} x_0,\\
&y_n(t) \overset{\tilde{I}\ P.1}{=} \Big[\Big(x_0 \oplus \int_0^t \tilde{f}(s, y_n(s-\frac{1}{n}))ds\Big)\\
&\quad \ominus \Big((-1) \odot \int_0^t f(s, y_n(s-\frac{1}{n}))ds\Big)\Big]\\
&\quad \oplus \Big\langle \sum_{k=1}^{\ell} \int_0^t h^k(s, y_n(s-\frac{1}{n}))dW^k(s)\Big\rangle.
\end{aligned}
$$

Remark 3.3. *Let $x_0, f, \tilde{f}, h^k$ satisfy (A0)-(A4). Then $y_n \colon \tilde{I} \times \Omega \to \mathcal{F}(\mathbb{R}^d)$ are d_∞-continuous, nonanticipating fuzzy stochastic processes that belong to $\mathcal{L}^2(\tilde{I} \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$.*

It is intended to apply sequence $\{y_n\}$ to approach a solution to (3.2). However firstly we need to observe some useful properties of the approximations y_n. They will be used later on. Below we state, as a first observation, that $\{y_n\}$ is a bounded sequence.

Lemma 3.4. *Let $x_0, f, \tilde{f}, h^k$ satisfy (A0)-(A4). Then there exists a positive constant C_1 such that for every $n \in \mathbb{N}$*

$$\mathbb{E}sup_{t\in\tilde{I}} d_\infty^2(y_n(t), \langle 0\rangle) \leqslant C_1.$$

Lemma 3.5. *Let the assumptions of Lemma 3.4 be satisfied. Then there exists a positive constant C_2 such that for every $n \in \mathbb{N}$ and every $\tau, t \in \tilde{I}$, $\tau \leqslant t$*

$$\mathbb{E}d_\infty^2(y_n(t), y_n(\tau)) \leqslant C_2(t - \tau).$$

Lemma 3.6. *Let the assumptions of Lemma 3.4 be satisfied. Then*

$$\mathbb{E}sup_{t\in\tilde{I}} d_\infty^2(y_n(t), y_i(\tau)) \longrightarrow 0, \text{ as } n, i \to \infty.$$

Proof. Let us fix $n, i \in \mathbb{N}$. Without loss of generality we may assume that $n > i$. Observe that for $t \in \tilde{I}$ we have, using the property $d_\infty(u \ominus v, w \ominus z) \leqslant d_\infty(u, w) + d_\infty(v, z)$ together with Lemma 2.1 and Doob's inequality,

$$
\begin{aligned}
&\mathbb{E}\sup_{u\in[0,t]} d_\infty^2(y_n(u), y_i(u))\\
&\leqslant 8t\mathbb{E}\int_0^t d_\infty^2\Big(\tilde{f}(s, y_n(s-\frac{1}{n})), \tilde{f}(s, y_i(s-\frac{1}{n}))\Big)\\
&+ 8t\mathbb{E}\int_0^t d_\infty^2\Big(\tilde{f}(s, y_i(s-\frac{1}{n})), \tilde{f}(s, y_i(s-\frac{1}{i}))\Big)\\
&+ 8t\mathbb{E}\int_0^t d_\infty^2\Big(f(s, y_n(s-\frac{1}{n})), f(s, y_i(s-\frac{1}{n}))\Big)\\
&+ 8t\mathbb{E}\int_0^t d_\infty^2\Big(f(s, y_i(s-\frac{1}{n})), f(s, y_i(s-\frac{1}{i}))\Big)\\
&+ 16\ell \sum_{k=1}^{\ell} \mathbb{E}\int_0^t \Big\| h^k(s, y_n(s-\frac{1}{n})) - h^k(s, y_i(s-\frac{1}{n}))\Big\|^2\\
&+ 16\ell \sum_{k=1}^{\ell} \mathbb{E}\int_0^t \Big\| h^k(s, y_i(s-\frac{1}{n})) - h^k(s, y_i(s-\frac{1}{i}))\Big\|^2,
\end{aligned}
$$

Assumption (A2) lead us to

$$
\begin{aligned}
&\mathbb{E}\sup_{u\in[0,t]} d_\infty^2(y_n(u), y_i(u))\\
&\leqslant 16(t+\ell^2)\int_0^t \xi\big(\mathbb{E}\sup_{u\in[0,s]} d_\infty^2(y_n(u), y_i(u))\big)ds\\
&+ 16(t+\ell^2)\int_0^t \xi\big(\mathbb{E}d_\infty^2(y_i(s-\frac{1}{n}), y_i(s-\frac{1}{i}))\big)ds.
\end{aligned}
$$

By Lemma 3.5 we get

$$
\begin{aligned}
&\mathbb{E}\sup_{u\in[0,t]} d_\infty^2(y_n(u), y_i(u))\\
&\leqslant C_3\int_0^t \xi\big(\mathbb{E}\sup_{u\in[0,s]} d_\infty^2(y_n(u), y_i(u))\big)ds\\
&+ C_3\tilde{T}\xi\Big(C_2(\frac{1}{i} - \frac{1}{n})\Big),
\end{aligned}
$$

where $C_3 = 16(\tilde{T} + \ell^2)$. Application of Lemma 2.2 yields

$$
\begin{aligned}
&\mathbb{E}\sup_{u\in[0,t]} d_\infty^2(y_n(u), y_i(u))\\
&\leqslant J^{-1}(J(C_3\tilde{T}\xi\Big(C_2(\frac{1}{i} - \frac{1}{n})\Big) + C_3 t)
\end{aligned}
$$

for every $t \in \tilde{I}$. Owing to Lemma 2.2 and properties of function J from this lemma, we obtain

$$\lim_{n,i\to\infty} J^{-1}(J(C_3\tilde{T}\xi\Big(C_2(\frac{1}{i} - \frac{1}{n})\Big) + C_3\tilde{T}) = 0.$$

This allows us to infer that

$$\lim_{n,i\to\infty} \mathbb{E}\sup_{t\in\tilde{I}} d_\infty^2(y_n(t), y_i(t)) = 0.$$

□

Now, we present a scheme of the proof of our main result.

Proof of Theorem 3.2. Due to Lemma 3.6 we have

$$\lim_{n,i\to\infty} \rho^2(y_n(t), y_i(t)) = 0 \text{ for every } t \in \tilde{I},$$

where $\rho(x,y) = [\mathbb{E}d_\infty^2(x,y)]^{1/2}$ is a metric in $\mathcal{L}^2(\Omega,\mathcal{A}_t,P;\mathcal{F}(\mathbb{R}^d))$. Since $(\mathcal{L}^2(\Omega,\mathcal{A}_t,P;\mathcal{F}(\mathbb{R}^d)),\rho)$ is a complete metric space we infer that for every $t\in\tilde{I}$ there exists a unique fuzzy random variable $x_t\in\mathcal{L}^2(\Omega,\mathcal{A}_t,P;\mathcal{F}(\mathbb{R}^d))$ such that $\lim_{n\to\infty}\rho(y_n(t),x_t)=0$. Let us define $x\colon \tilde{I}\times\Omega\to\mathcal{F}(\mathbb{R}^d)$ as $x(t,\omega)=x_t(\omega)$. Then the fuzzy stochastic process x is $\{\mathcal{A}_t\}$-adapted. Due to the Markov inequality we obtain that for every $\varepsilon>0$

$$\lim_{n,i\to\infty} P\left(\sup_{t\in\tilde{I}} d_\infty(y_n(t),y_i(t))>\varepsilon\right)=0.$$

Hence we can infer that there exists a subsequence $\{y_{n_\ell}(\cdot,\cdot)\}$ of the sequence $\{y_n(\cdot,\cdot)\}$ such that

$$\lim_{\ell\to\infty}\sup_{t\in\tilde{I}} d_\infty(y_{n_\ell}(t),x(t)) \overset{P.1}{=} 0.$$

Thus the process x is d_∞-continuous and consequently it is measurable. Since x is also $\{\mathcal{A}_t\}$-adapted, it is nonanticipating. Also, since $x(t)\in\mathcal{L}^2(\Omega,\mathcal{A}_t,P;\mathcal{F}(\mathbb{R}^d))$ for every $t\in\tilde{I}$, we have $\mathbb{E}\int_{\tilde{I}} d_\infty^2(x(t),\langle 0\rangle)dt \leqslant \tilde{T}\sup_{t\in\tilde{I}}\mathbb{E}d_\infty^2(x(t),\langle 0\rangle)<\infty$. This implies that $x\in\mathcal{L}^2(\tilde{I}\times\Omega,\mathcal{N};\mathcal{F}(\mathbb{R}^d))$. Moreover, applying Lemma 3.4, we infer that $\mathbb{E}\sup_{t\in\tilde{I}} d_\infty^2(x(t),\langle 0\rangle)\leqslant C_1$. We can also infer that

$$\lim_{\ell\to\infty}\mathbb{E}\sup_{t\in\tilde{I}} d_\infty^2(y_{n_\ell}(t),x(t))=0. \tag{3.3}$$

In what follows we shall show that x is a solution to (3.2). To this aim, let us observe that

$$\begin{aligned}
&\mathbb{E}\sup_{u\in\tilde{I}} d_\infty^2\Big(x(u),\Big[\Big(x_0\oplus\int_0^t \tilde{f}(s,x(s))ds\Big)\\
&\ominus\Big((-1)\odot\int_0^t f(s,x(s))ds\Big)\Big]\\
&\oplus\Big\langle\sum_{k=1}^{\ell}\int_0^t h^k(s,x(s))dW^k(s)\Big\rangle\\
\leqslant\ & 2Q_\ell+2P_\ell,
\end{aligned}$$

where

$$Q_\ell=\mathbb{E}\sup_{u\in\tilde{I}} d_\infty^2\big(y_{n_\ell}(u),x(u)\big)$$

and

$$\begin{aligned}
P_\ell =\ & \mathbb{E}\sup_{u\in\tilde{I}} d_\infty^2\Big(\Big[\Big(x_0\oplus\int_0^t \tilde{f}(s,y_{n_\ell}(s-\frac{1}{n_\ell}))ds\Big)\\
&\ominus\Big((-1)\odot\int_0^t f(s,y_{n_\ell}(s-\frac{1}{n_\ell}))ds\Big)\Big]\\
&\oplus\Big\langle\sum_{k=1}^{\ell}\int_0^t h^k(s,y_{n_\ell}(s-\frac{1}{n_\ell}))dW^k(s)\Big\rangle,\\
&\Big[\Big(x_0\oplus\int_0^t \tilde{f}(s,x(s))ds\Big)\\
&\ominus\Big((-1)\odot\int_0^t f(s,x(s))ds\Big)\Big]\\
&\oplus\Big\langle\sum_{k=1}^{\ell}\int_0^t h^k(s,x(s))dW^k(s)\Big\rangle\Big).
\end{aligned}$$

By (3.3) the expression Q_ℓ converges to zero as ℓ goes to infinity, and it can be verified that

$$\begin{aligned}
P_\ell\ \leqslant\ & C_4\mathbb{E}\int_0^{\tilde{T}}\xi\Big(d_\infty^2(y_{n_\ell}(s-\frac{1}{n_\ell}),y_{n_\ell}(s))\Big)ds\\
&+C_4\mathbb{E}\int_0^{\tilde{T}}\xi\Big(d_\infty^2(y_{n_\ell}(s),x(s))\Big)ds,
\end{aligned}$$

where $C_4=16(\tilde{T}+1)$. Thus

$$\begin{aligned}
P_\ell\ \leqslant\ & C_4\int_0^{\tilde{T}}\xi\Big(\mathbb{E}d_\infty^2(y_{n_\ell}(s-\frac{1}{n_\ell}),y_{n_\ell}(s))\Big)ds\\
&+C_4\int_0^{\tilde{T}}\xi\Big(\mathbb{E}\sup_{u\in\tilde{I}} d_\infty^2(y_{n_\ell}(u),x(u))\Big)ds.
\end{aligned}$$

Applying Lemma 3.5 we obtain

$$\begin{aligned}
P_\ell\ \leqslant\ & C_4\tilde{T}\xi\Big(\frac{C_2}{n_\ell}\Big)\\
&+C_4\tilde{T}\xi\Big(\mathbb{E}\sup_{u\in\tilde{I}} d_\infty^2(y_{n_\ell}(u),x(u))\Big).
\end{aligned}$$

By properties of ξ and in view of (3.3) the right-hand side of the latter inequality converges to zero. Thus

$$\begin{aligned}
&\mathbb{E}\sup_{u\in\tilde{I}} d_\infty^2\Big(x(u),\Big[\Big(x_0\oplus\int_0^t \tilde{f}(s,x(s))ds\Big)\\
&\ominus\Big((-1)\odot\int_0^t f(s,x(s))ds\Big)\Big]\\
&\oplus\Big\langle\sum_{k=1}^{\ell}\int_0^t h^k(s,x(s))dW^k(s)\Big\rangle=0
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sup_{u\in\tilde{I}} d_\infty\Big(x(u),\Big[\Big(x_0\oplus\int_0^t \tilde{f}(s,x(s))ds\Big)\\
&\ominus\Big((-1)\odot\int_0^t f(s,x(s))ds\Big)\Big]\\
&\oplus\Big\langle\sum_{k=1}^{\ell}\int_0^t h^k(s,x(s))dW^k(s)\Big\rangle \overset{P.1}{=} 0.
\end{aligned}$$

This shows that x is a solution to (3.2). Now we notice that x is a unique solution. Indeed, assume that $y\colon \tilde{I}\times\Omega\to\mathcal{F}(\mathbb{R}^d)$ is another solution to (3.2). Then for $t\in\tilde{I}$ we have

$$\begin{aligned}
&\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\big(x(u),y(u)\big)\\
&\leqslant 4t\mathbb{E}\int_0^t d_\infty^2(\tilde{f}(s,x(s)),\tilde{f}(s,y(s)))ds\\
&+4t\mathbb{E}\int_0^t d_\infty^2(f(s,x(s)),f(s,y(s)))ds\\
&+8\ell\sum_{k=1}^{\ell}\mathbb{E}\int_0^t\Big\|h^k(s,x(s))-h^k(s,y(s))\Big\|^2 ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\big(x(u),y(u)\big)\\
&\leqslant 8(\tilde{T}+\ell^2)\int_0^t\xi\Big(\mathbb{E}\sup_{u\in[0,s]} d_\infty^2(x(s),y(s))\Big)ds.
\end{aligned}$$

Invoking Lemma 2.2, we get

$$\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\big(x(u),y(u)\big) \leqslant 0 \text{ for every } t \in \tilde{I}.$$

Therefore $\mathbb{E}\sup_{u\in\tilde{I}} d_\infty^2\big(x(u),y(u)\big) = 0$, which implies that $\sup_{u\in\tilde{I}} d_\infty\big(x(u),y(u)\big) \overset{P.1}{=} 0$. This proves uniqueness of the solution x. The proof is completed. □

4 CONCLUSION

In the paper we consider bipartite fuzzy stochastic differential equations. A main result treat on existence of a unique solution to such the equations in the case when coefficients satisfy a generalized Lipschitz condition. A continuous dependence of the solution with respect to initial value and drift and diffusion coefficients can be investigated in a future research.

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