Bijective Fuzzy Relations
A Graded Approach

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Abstract: Bijectivity is one of crucial mathematical notions. In this paper, we will present a fuzzy bijective mapping as a fuzzy relation that has several special properties. These properties come with degrees and so the bijectivity is also a graded property. We will focus on properties of this type of relations and show graded versions of theorems on fuzzy bijections that are known from traditional Fuzzy Set Theory.

1 INTRODUCTION

In this work, we will present a gradual version of results on fuzzy bijective functions from (Demirci, 2000; Demirci, 2001). The presented results lay foundations for developing a theory of partial fuzzy functions and partial bijective fuzzy relations over a simple system of fuzzy partial propositional logic (Běhounek and Novák, 2015). There we deal with membership functions that admit undefined truth degrees.

A basis for the notion of fuzzy bijective function is the notion of fuzzy function. This notion took many forms by their definitions; e.g., it is any mapping that assigns a fuzzy set to a fuzzy set or fuzzy set to a point (Novák et al., 1999); it is a fuzzy relation that meets two properties, namely, extensionality and functionality for a partial fuzzy function, and if it is in addition total then it is called a perfect fuzzy function (Demirci, 1999; Demirci, 2001; Demirci and Recasens, 2004; Perfiljeva I., 2014); etc. Overviews together with applications can be found in the following exemplary sources (Klawonn, 2000; Demirci, 2001; Bělohlávek, 2002).

As noted in (Demirci, 2000), not all notions of fuzzy function do coincide with the classical notion for crisp functions. We will avoid this problem and all the subsequent definitions will be consistent with the classical notion whenever applied on crisp inputs.

We follow up (Daňková, 2010a) where the graded property of extensionality has been studied and (Daňková, 2010b; Daňková, 2011) where the graded property of functionality has been explored and used to make an implicative model of fuzzy IF–THEN rules. In (Daňková, 2010b), the framework was that of Fuzzy Class Theory (Běhounek and Cintula, 2005). It enables formulation of notions in the standard mathematical notation whereas the background machinery provides the non-classical interpretation. Since this framework might be difficult to read for fuzzy mathematicians, we choose the common algebraic one for this paper and, in addition, review results presented in (Daňková, 2010b).

2 BASIC NOTIONS

In the following, we will work with algebraic structures used as basic structures for fuzzy logic of left-continuous t-norms, the so called monoidal t-norm based logic.

Definition 1. *An* MTL*-algebra $\mathcal{L}$ is a bounded residuated lattice*

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle \tag{1}$$

with four binary operations and two constants such that

- $\mathcal{L} = \langle L, \vee, \wedge, 0, 1\rangle$ *is a lattice with the largest element* 1 *and the least element* 0 *w.r.t. the lattice ordering* $\leq$*,*
- $\mathcal{L} = \langle L, *, 1\rangle$ *is a commutative semigroup with the unit element* 1*, i.e.,* $*$ *is commutative, associative, and* $1 * x = x$ *for all* $x \in L$*,*
- $*$ *and* $\rightarrow$ *form an adjoint pair, i.e.,*

$$z \leq (x \rightarrow y) \quad \text{iff} \quad x * z \leq y \quad \text{for all } x, y, z \in L,$$

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- *satisfying the pre-linearity equation:*

$$(x \to y) \vee (y \to x) = 1.$$

In the sequel, let us assume $\mathcal{L}$ be an MTL-algebra of the form (1). We will call the operation $*$ *product* and $\to$ *residuum*. Moreover, we define the operations called *bi-residuum* and *powers* w.r.t. the product $*$:

$$x \leftrightarrow y =_{\mathrm{df}} (x \to y) \wedge (y \to x)$$
$$\varphi^n =_{\mathrm{df}} \underbrace{\varphi * \ldots * \varphi}_{n\text{-times}}$$

To reduce the number of parenthesis used in mathematical expressions we set that $*$ has the highest priority and $\to$ the lowest priority out of all operations that are at the disposal.

Throughout the whole text, we will deal with fuzzy relations whose membership functions are defined on non-empty sets and take values from the support of $\mathcal{L}$, and will denote this fact by $\underset{\sim}{\subset}$. In the sequel, let $X,Y \neq \emptyset$, $R,S \underset{\sim}{\subset} X \times Y$, $\approx_1 \underset{\sim}{\subset} X \times X$ and $\approx_2 \underset{\sim}{\subset} Y \times Y$ are fuzzy relations such that all infima and suprema needed for the definition of the truth-value of any expression exist in $\mathcal{L}$.

Which variables belong to which sets is always clear from the context. Therefore it is not necessary to specify membership of variables in sets in expressions, including infima and suprema; e.g., we write $\bigvee_{x,y} R(x,y)$ instead of $\bigvee_{x\in X, y\in Y} R(x,y)$. And for the sake of brevity we use Rxy instead of $R(x,y)$.

Let us define the following graded properties of fuzzy relations:

- Reflexivity:

$$\mathrm{Refl}(R) \equiv_{\mathrm{df}} \bigwedge_x Rxx$$

- Symmetry:

$$\mathrm{Sym}(R) \equiv_{\mathrm{df}} \bigwedge_{x,y} (Rxy \to Ryx)$$

- Transitivity:

$$\mathrm{Trans}(R) \equiv_{\mathrm{df}} \bigwedge_{x,y,z} [(Rxy * Ryz) \to Rxz]$$

- Similarity:

$$\mathrm{Sim}(R) \equiv_{\mathrm{df}} \mathrm{Refl}(R) * \mathrm{Sym}(R) * \mathrm{Trans}(R)$$

- Subsethood:

$$R \subseteq S \equiv_{\mathrm{df}} \bigwedge_{x,y} (R(x,y) \to S(x,y))$$

- Strong set-similarity:

$$R \cong S \equiv_{\mathrm{df}} \bigwedge_{x,y} (R(x,y) \leftrightarrow S(x,y))$$

- Set-similarity:

$$R \approx S \equiv_{\mathrm{df}} (R \subseteq S) * (S \subseteq R)$$

- Totality:

$$\mathrm{Tot}(R) \equiv_{\mathrm{df}} \bigwedge_x \bigvee_y Rxy$$

- Surjectivity:

$$\mathrm{Sur}(R) \equiv_{\mathrm{df}} \bigwedge_y \bigvee_x Rxy$$

- Injectivity:

$$\mathrm{Inj}_{\approx_{1,2}}(R) \equiv_{\mathrm{df}} \bigwedge_{x,x',y,y'} [((y \approx_2 y') * Rxy * Rx'y') \to (x \approx_1 x')]$$

In the sequel, we will freely use the class notation that has been formally developed in (Běhounek and Cintula, 2005). It means that for a fuzzy set $A \underset{\sim}{\subset} X$

$$\begin{array}{rll} \{x \mid Ax\} & \text{respresents} & A \\ y \in \{x \mid Ax\} & \text{stands for} & Ay \end{array}$$

and for a fuzzy relation $R \underset{\sim}{\subset} X \times Y$

$$\begin{array}{rll} \{xy \mid Rxy\} & \text{respresents} & R \\ x'y' \in \{xy \mid Rxy\} & \text{stands for} & Rx'y' \end{array}$$

and analogously we proceed for an arbitrary expression φ.

Let us introduce the following relational operations:

$$\begin{array}{rll} R^{\mathrm{T}} =_{\mathrm{df}} & \{yx \mid Rxy\} & \text{inverse} \\ R \cap S =_{\mathrm{df}} & \{xy \mid Rxy * Sxy\} & \text{strong intersection} \\ R \sqcup S =_{\mathrm{df}} & \{xy \mid Rxy \vee Sxy\} & \text{union} \\ R \sqcap S =_{\mathrm{df}} & \{xy \mid Rxy \wedge Sxy\} & \text{intersection} \end{array}$$

We will additionally deal with relational compositions, defined using a class notation. A systematic study can be find in (Bělohlávek, 2002). We will use three basic relational compositions:

- sup-T composition:

$$R \circ S =_{\mathrm{df}} \{xy \mid \bigvee_z (Rxz * Szy)\}$$

- BK-subproduct:

$$R \lhd S =_{\mathrm{df}} \{xy \mid \bigwedge_z (Rxz \to Szy)\}$$

- BK-superproduct:

$$R \rhd S =_{\mathrm{df}} \{xy \mid \bigwedge_z (Szy \to Rxz)\}$$

The crisp identity $=$ and crisp inclusion $\sqsubseteq$ of fuzzy relation are defined standardly:

$$R = S \quad \text{if} \quad Rxy = Sxy, \text{ for all } x,y;$$
$$R \sqsubseteq S \quad \text{if} \quad Rxy \leq Sxy, \text{ for all } x,y.$$

Let the following fuzzy relations be of the appropriate types; then we can summarize properties of sup-T-composition (Bělohlávek, 2002):

1. Transposition: $(R \circ S)^{\mathrm{T}} = S^{\mathrm{T}} \circ R^{\mathrm{T}}$
2. Monotony: $(R_1 \subseteq R_2) \leq (R_1 \circ S \subseteq R_2 \circ S)$
3. Union: $\left(\bigcup_{R\in\mathcal{A}} R\right) \circ S = \bigcup_{R\in\mathcal{A}} (R \circ S)$
4. Intersection: $\left(\bigcap_{R\in\mathcal{A}} R\right) \circ S \sqsubseteq \bigcap_{R\in\mathcal{A}} (R \circ S)$
5. Associativity: $(R \circ S) \circ T = R \circ (S \circ T)$

Properties of BK-products:

1. Transposition: $(R \lhd S)^{\mathrm{T}} = S^{\mathrm{T}} \rhd R^{\mathrm{T}}$
2. Monotony of $\lhd$:
$$(R_1 \subseteq R_2) \leq (R_2 \lhd S \subseteq R_1 \lhd S)$$
$$(S_1 \subseteq S_2) \leq (R \lhd S_1 \subseteq R \lhd S_2)$$
3. Monotony of $\rhd$:
$$(R_1 \subseteq R_2) \leq (R_1 \rhd S \subseteq R_2 \rhd S)$$
$$(S_1 \subseteq S_2) \leq (R \rhd S_2 \subseteq R \rhd S_1)$$
4. Intersection:
$$\bigcap_{R\in\mathcal{A}} (R \lhd S) = \left(\bigcup_{R\in\mathcal{A}} R\right) \lhd S$$
$$\bigcap_{S\in\mathcal{A}} (R \lhd S) = R \lhd \bigcap_{S\in\mathcal{A}} S$$
5. Union:
$$\bigcup_{R\in\mathcal{A}} (R \lhd S) \sqsubseteq \left(\bigcap_{R\in\mathcal{A}} R\right) \lhd S$$
$$\bigcup_{S\in\mathcal{A}} (R \lhd S) \sqsubseteq R \lhd \bigcup_{S\in\mathcal{A}} S$$
Converse inclusions have crisp counter-examples.
6. Residuation: $R \lhd (S \lhd T) = (R \circ S) \lhd T$
7. Exchange: $R \lhd (S \rhd T) = (R \lhd S) \rhd T$

2.1 Graded Theorems and Their Reading

Let us observe the above definitions of relational properties. For example the totality of R may attain any degree from L and whenever it attains the degree 1, i.e. $\mathrm{Tot}(R) = 1$, it is total in the classical sense and we have the correspondence with the classical notion of totality: for all $x \in X$ there exists $y \in Y$ such that $Rxy = 1$. For an arbitrary property φ introduced in this paper, it is valid that if $\varphi = 1$ then φ is identical to the correspondent classical property.

Statements that will be presented in this paper are called gradual or graded theorems. It means that instead of a *classical theorem* of fuzzy mathematics

$$\text{If } (\varphi_1 = 1) \text{ and } \dots \text{ and } (\varphi_k = 1) \text{ then } (\psi = 1) \tag{2}$$

we search for a more informative and general (non-equivalent) form of this statement, the so called *graded theorem*:

$$\varphi_1^{n_1} * \dots * \varphi_k^{n_k} \leq \psi. \tag{3}$$

By the properties of $\to$, it is equivalent to

$$(\varphi_1^{n_1} * \dots * \varphi_k^{n_k} \to \psi) = 1 \tag{4}$$

where $*$ interprets strong conjunction and $\varphi_1, \dots, \varphi_k$, ψ represent the formalization of premises and the conclusion in a form that enjoys degrees of truths. Part of the analysis is finding out how many times the antecedents $\varphi_1, \dots, \varphi_k$ need be used to provide a lower bound for the degree of the consequent ψ; the result is encoded in the degrees $n_1, \dots, n_k$.

Graded theorems seem to be difficult for non-experienced readers; therefore, some translations will be added as a guideline. The proposed reading of graded theorems is analogous to the classical case (as when using classical mathematical logic) and it is distinguished by a special typeface; for the chosen operations of $\mathcal{L}$, we set:

Expression	Reading
$\varphi \to \psi$	IF φ THEN ψ
$\varphi \leftrightarrow \psi$	φ IFF ψ
$\varphi \wedge \psi$	φ AND ψ
$\varphi \vee \psi$	φ OR ψ
$\varphi * \psi$	φ and ψ
$\varphi = 1$	φ is TRUE

Hence, (4) can be read as

"It is TRUE that IF $\varphi_1^{n_1}$ and ... and $\varphi_k^{n_k}$ THEN ψ."

Explicitly using degrees, we have that

If $\varphi_1^{n_1} = d_1$ and ... and $\varphi_k^{n_k} = d_k$ then $\psi \geq d_1 * \dots * d_k$.[1]

As an example of a classical theorem (2), we can assume $R, S \mathrel{\underset{\sim}{\subset}} X \times X$ and consider only two antecedents: let φ_1 be "R is a subset of S", which means that $Rxy \leq Sxy$ for all $x, y \in X$; let φ_2 be interpreted as "R is reflexive", which means $Rxx = 1$ for all $x \in X$; and let the consequent ψ be "S is reflexive". We can prove the following classical theorem:

If R is a subset of S and R is reflexive then S is reflexive.

[1]Compare with (2).

It can be shown that there is also a graded theorem (Běhounek et al., 2008) for this statement:

$$(R \subseteq S) * \mathrm{Refl}(R) \leq \mathrm{Refl}(S)$$

that can be read analogously to the above classical theorem:

> "It is TRUE that IF $R \subseteq S$ and $\mathrm{Refl}(R)$ THEN $\mathrm{Refl}(S)$."

In this case, we have incorporated also additional hidden degrees for both properties: if R is reflexive to a degree r and the degree of subsethood is equal to d then S is reflexive as much as R is reflexive and R is a subset of S, which is expressed in degrees as $r * d \leq \mathrm{Refl}(S)$.

The main difference between classical and graded theorems is now obvious: having a classical theorem, we use the language of classical mathematical logic to read the proved theorems, while in the case of graded theorems, we may read proved inequalities of the form (3) directly using the generalized language proposed in the above table. Let us quote from (Daňková, 2007a):

> "We write classically, but we think in grades."

3 EXTENSIONALITY AND ITS PROPERTIES

The extensionality of a fuzzy relation $F \underset{\sim}{\subset} X \times Y$ w.r.t. $\approx_1 \underset{\sim}{\subset} X^2, \approx_2 \underset{\sim}{\subset} Y^2$ is defined as follows:

$$\mathrm{Ext}_{\approx_{1,2}}(F) \equiv_{\mathrm{df}} \bigwedge_{x,x',y,y'} [(x \approx_1 x') * (y \approx_2 y') * Fxy \to Fx'y'].$$

The extensionality is one of the most important properties of fuzzy mathematics. It is used in fuzzy control, fuzzy logic, fuzzy relational calculus etc., see (Hájek, 1998; Bělohlávek, 2002; Perfilieva, 2001; Daňková, 2007b).

This notion generalizes the classical one that is of the following form: we say that F is extensional w.r.t. $\approx_1, \approx_2$ if

$$(x \approx_1 x') * (y \approx_2 y') * Fxy \leq Fx'y', \quad \text{for all } x, x' \in X, y, y' \in Y.$$

Let us recall that $\bigwedge, \bigvee$ represent generalized universal and existential quantifiers, respectively; therefore, the generalization is natural. Moreover, let us emphasize that we do not place any requirements on $\approx_1, \approx_2$. We would like to keep requirements as week as possible. The reason is that some applications may not be satisfied with use of similarities, but for example proximities (i.e., reflexive and symmetric fuzzy relations) would be enough, such as in case of Region Connected Calculus (Batyrshin et al., 2008)

3.1 Extensionality of Set Operations and Relational Compositions

Let us summarize some properties of extensionality (Daňková, 2010c).

Proposition 2. *For arbitrary $F, E \underset{\sim}{\subset} X \times Y$, the following inequalities are valid:*

$$\mathrm{Ext}_{\approx_{1,2}}(F) * \mathrm{Ext}_{\approx_{1,2}}(E) \leq \mathrm{Ext}_{\approx^2_{1,2}}(F \cap E), \tag{5}$$

$$\mathrm{Ext}_{\approx_{1,2}}(F) * \mathrm{Ext}_{\approx_{1,2}}(E) \leq \mathrm{Ext}_{\approx^2_{1,2}}(F \cup E), \tag{6}$$

$$\mathrm{Ext}_{\approx_{1,2}}(F) \wedge \mathrm{Ext}_{\approx_{1,2}}(E) \leq \mathrm{Ext}_{\approx_{1,2}}(F \sqcap E). \tag{7}$$

Readings of the above results:
It is TRUE that (5) – "IF F and E are extensional THEN their strong intersection is extensional."
(6) – "IF F and E are extensional THEN their strong union is extensional."
(7) – "IF F and E are extensional THEN their intersection is extensional."

In the following, the extensionality of supersets as well as similar sets will be studied.

Proposition 3. *For arbitrary $F, E \underset{\sim}{\subset} X \times Y$ we have:*

$$(F \subseteq E)^2 \leq [\mathrm{Ext}_{\approx_{1,2}}(F) \to \mathrm{Ext}_{\approx_{1,2}}(E)], \tag{8}$$

$$(F \approx E)^2 \leq [\mathrm{Ext}_{\approx_{1,2}}(F) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}(E)]. \tag{9}$$

Observe that (8) and (9) express the extensionality of a higher order for $\mathrm{Ext}_{\approx_{1,2}}$ w.r.t. $\subseteq^2$ and $\approx^2$, respectively. It means, extensionality when the variables are fuzzy relations instead of ordinary elements of X, Y and both (8) and (9) are valid for arbitrary $F, E \underset{\sim}{\subset} X \times Y$.

Readings of the results:
(8) – "IF F is a subset of E (we need this requirement twice) and F is extensional THEN E is extensional."
(9) – "IF F and E are similar sets (we need this requirement twice) THEN F is extensional IFF E is extensional."

The inequalities (8) and (9) together with properties of relational compositions produces a long list of consequences.

Corollary 4. *Let*

$$C_1 \equiv_{\mathrm{df}} (E_1 \subseteq E_2)^2,$$
$$C_2 \equiv_{\mathrm{df}} (F_1 \subseteq F_2)^2,$$
$$C_3 \equiv_{\mathrm{df}} (E_1 \approx E_2)^2,$$
$$C_4 \equiv_{\mathrm{df}} (F_1 \approx F_2)^2.$$

Then the following inequalities are valid for arbitrary $F, F_1, F_2 \subsetsim X \times Y$ and $E, E_1, E_2 \subsetsim Y \times Z$:

$$C_1 \le [\mathrm{Ext}_{\approx_{1,2}}(F \circ E_2) \to \mathrm{Ext}_{\approx_{1,2}}(F \circ E_1)],$$
$$C_1 \le [\mathrm{Ext}_{\approx_{1,2}}(F \triangleleft E_1) \to \mathrm{Ext}_{\approx_{1,2}}(F \triangleleft E_2)],$$
$$C_1 \le [\mathrm{Ext}_{\approx_{1,2}}(F \triangleright E_2) \to \mathrm{Ext}_{\approx_{1,2}}(F \triangleright E_1)],$$

$$C_2 \le [\mathrm{Ext}_{\approx_{1,2}}(F_2 \triangleleft E) \to \mathrm{Ext}_{\approx_{1,2}}(F_1 \triangleleft E)],$$
$$C_2 \le [\mathrm{Ext}_{\approx_{1,2}}(F_1 \triangleright E) \to \mathrm{Ext}_{\approx_{1,2}}(F_2 \triangleright E)],$$

$$C_3 \le [\mathrm{Ext}_{\approx_{1,2}}(F \circ E_2) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}(F \circ E_1)],$$
$$C_3 \le [\mathrm{Ext}_{\approx_{1,2}}(F \triangleleft E_2) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}(F \triangleleft E_1)],$$
$$C_3 \le [\mathrm{Ext}_{\approx_{1,2}}(F \triangleright E_1) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}(F \triangleright E_2)],$$

$$C_4 \le [\mathrm{Ext}_{\approx_{1,2}}(F_2 \triangleleft E) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}(F_1 \triangleleft E)],$$
$$C_4 \le [\mathrm{Ext}_{\approx_{1,2}}(F_1 \triangleright E) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}(F_2 \triangleright E)].$$

Intersection:

$$\mathrm{Ext}_{\approx_{1,2}}\Big(\Big(\bigcap_{F \in \mathcal{A}} F\Big) \circ E\Big) \le \mathrm{Ext}_{\approx_{1,2}}\Big(\bigcap_{F \in \mathcal{A}} (F \circ E)\Big),$$
$$\mathrm{Ext}_{\approx_{1,2}}\Big(\bigcup_{F \in \mathcal{A}} (F \triangleleft E)\Big) \le \mathrm{Ext}_{\approx_{1,2}}\Big(\Big(\bigcap_{F \in \mathcal{A}} F\Big) \triangleleft E\Big),$$
$$\mathrm{Ext}_{\approx_{1,2}}\Big(\bigcup_{E \in \mathcal{A}} (F \triangleleft E)\Big) \le \mathrm{Ext}_{\approx_{1,2}}\Big(F \triangleleft \bigcup_{E \in \mathcal{A}} E\Big).$$

Union:

$$\mathrm{Ext}_{\approx_{1,2}}\Big(\Big(\bigcup_{F \in \mathcal{A}} F\Big) \circ E\Big) = \mathrm{Ext}_{\approx_{1,2}}\Big(\bigcup_{F \in \mathcal{A}} (F \circ E)\Big),$$
$$\mathrm{Ext}_{\approx_{1,2}}\Big(\bigcap_{F \in \mathcal{A}} (F \triangleleft E)\Big) = \mathrm{Ext}_{\approx_{1,2}}\Big(\Big(\bigcup_{F \in \mathcal{A}} F\Big) \triangleleft E\Big),$$
$$\mathrm{Ext}_{\approx_{1,2}}\Big(\bigcap_{E \in \mathcal{A}} (F \triangleleft E)\Big) = \mathrm{Ext}_{\approx_{1,2}}\Big(F \triangleleft \bigcap_{E \in \mathcal{A}} E\Big).$$

4 FUNCTIONALITY AND ITS PROPERTIES

In this section, we will introduce a property of fuzzy relation called functionality that is a direct generalization of the related crisp notion.

Let the functionality property be given as follows:

$$\mathrm{Func}_{\approx_{1,2}}(F) \equiv_{\mathrm{df}} \bigwedge_{x,x',y,y'} [(x \approx_1 x') * Fxy * Fx'y' \to (y \approx_2 y')].$$

It generalizes the following classical notion of functionality of F (Demirci, 1999): we say that F is functional w.r.t. $\approx_2$ if

$$F(x,y) * F(x,y') \le y \approx_2 y', \text{ for all } x \in X,\ y,y' \in Y.$$

Taking a closer look at the latter expression, we uncover the hidden crisp equality related to the variable x. Our setting determines distinguishability on the input space by $\approx_1$. Therefore we have incorporated this fact by modifying the left side of the above inequality. Finally, we replace the universal quantifier with its graded generalization (as infimum) and obtain our definition.

Example 5. *Let $\mathcal{L}$ be an arbitrary* MTL*-algebra, $f : X \to Y$ be an arbitrary function and $d \in L$. The following relation is functional to the degree* 1 *w.r.t. the crisp equality* $=, =$:

$$F_1(x,y) = \begin{cases} d & \text{for } y = f(x), \\ 0 & \text{otherwise}. \end{cases}$$

Example 6. *Let $X = Y = [0,1]$ and $\mathcal{L}$ be the standard MV-algebra, i.e., $L = [0,1]$,*

$$x * y =_{\mathrm{df}} 0 \vee (x + y - 1)$$
$$x \to y =_{\mathrm{df}} 1 \wedge (1 - x + y).$$

Let $\sim$ be a similarity on L defined as:

$$x \sim y =_{\mathrm{df}} 0 \vee (1 - 3 \cdot |x - y|).$$

Then we find out that:

1. $F_2(x,y) = (y \sim \sin(x))$ *is functional to the degree* 1 *w.r.t.* $\sim, \sim$.
2. $F_3(x,y) = (y \sim f(x))$, *where*
$$f(x) = \begin{cases} 1.7x^2, & x \in [0, 0.5], \\ \cos(0.9x) - 0.5, & \text{otherwise}, \end{cases}$$
is functional to the degree 1 *w.r.t.* $=, \sim$.
3. $F_4(x,y) = F_2 \vee F_3$ *is functional to the degree* 0 *w.r.t.* $\sim, \sim$. *Take $x = x' = 1$ and $y = 0.1$, $y' = 0.8$; then $F(x,y) \otimes F(x',y') = 1$, but $y \sim y' = 0$.*

4.1 Functionality of Relational Operations and Compositions

Let us summarize some properties of functionality (Daňková, 2010c).

Proposition 7. *The following inequalities are valid for arbitrary $F, E \subsetsim X \times Y$:*

$$\mathrm{Func}_{\approx_{1,2}}(F) * \mathrm{Func}_{\approx_{2,3}}(E) \le \mathrm{Func}_{\approx_{1,3}}(F \circ E),$$
$$\mathrm{Func}_{\approx_{1,2}}(F) * \mathrm{Func}_{\approx_{1,2}}(E) \le \mathrm{Func}_{\approx^2_{1,2}}(F \cap E),$$
$$\mathrm{Func}_{\approx_{1,2}}(F) \wedge \mathrm{Func}_{\approx_{1,2}}(E) \le \mathrm{Func}_{\approx_{1,2}}(F \sqcap E).$$

We can also prove the following properties of relations analogous to those of the classical notions.

Proposition 8. *It can be proved that*

$$(E \subseteq F)^2 \leq [\mathrm{Func}_{\approx_{1,2}}(F) \to \mathrm{Func}_{\approx_{1,2}}(E)],$$
$$(F \approx E)^2 \leq [\mathrm{Func}_{\approx_{1,2}}(F) \leftrightarrow \mathrm{Func}_{\approx_{1,2}}(E)],$$

are valid for arbitrary $F, E \subsetsim X \times Y$.

We can generate a long list of corollaries the relational compositions analogous to Corollary 4. Since the list of corollaries is identical to that of Corollary 4 (only replacing Ext with Func), we do not feel the need to repeat it.

5 FUZZY FUNCTIONS

Let us start with the graded notion of partial fuzzy function:

$$\mathrm{Function}_{\approx_{1,2}}(F) \equiv_{\mathrm{df}} \mathrm{Ext}_{\approx_{1,2}}(F) \wedge \mathrm{Func}_{\approx_{1,2}}(F),$$

which joins extensionality and functionality using $\wedge$. Hence, the degree of being a partial fuzzy function is computed as the minimum of the degrees of these two properties. Extensionality means we can substitute the original inputs by indistinguishable ones. The functionality tells us that the images of indistinguishable elements are indistinguishable. Therefore, we must still track what is indistinguishable, i.e., how we choose $\approx_{1,2}$. These relations represent the granularity of X and Y, respectively, which means that they should be the coarsest relations in our system enabling the distinguishability of elements.

Let us fix for this section that $F, F_f, F_{f_F} \subsetsim X \times Y$, $\approx_1 \subsetsim X^2$, $\approx_2 \subsetsim Y^2$ and $f, f_F, f_{F_f} : X \mapsto Y$.

5.1 Fuzzy Functions and Their Crisp Counterparts

In classical mathematics, crisp functions can be expressed as functional relations and vice-versa. This relation is given by $y = f(x)$ for $f : X \mapsto X$. In fuzzy mathematics we deal with $\approx_1, \approx_2$ that do not need to be similarity or equality relations and moreover, we have to consider the graded compatibility property w.r.t. $\approx_{1,2}$ (generalization of (Bělohlávek, 2002)) defined as

$$\mathrm{Comp}_{\approx_1,\approx_2}(f) \equiv_{\mathrm{df}} \bigwedge_{x,y} (x \approx_1 y) \to (f(x) \approx_2 f(y)).$$

For any function f, the condition $\mathrm{Comp}_{=,=}(f)$ is trivially valid (to the degree 1). Similarly, by definition, $\mathrm{Comp}_{=,\approx_2}(f) = \bigwedge_x (f(x) \approx_2 f(x))$; thus, provided that $\approx_2$ is reflexive, $\mathrm{Comp}_{=,\approx_2}(f) = 1$ for each f.

Let us first explore how the relation between compatibility and functionality/extensionality looks like for a specially chosen relation $F_f(x,y) =_{\mathrm{df}} y \approx_2 f(x)$.

Lemma 9. *Let*

$$F_f(x,y) =_{\mathrm{df}} y \approx_2 f(x),$$
$$C \equiv_{\mathrm{df}} \mathrm{Sym}(\approx_2) * (\mathrm{Trans}(\approx_2))^2.$$

Then:

$$\mathrm{Tot}(f) \leq \mathrm{Tot}(F_f),$$
$$C \leq [\mathrm{Comp}_{\approx_1,\approx_2}(f) \to \mathrm{Func}_{\approx_1,\approx_2}(F_f)],$$
$$C \leq [\mathrm{Comp}_{\approx_1,\approx_2}(f) \to \mathrm{Ext}_{\approx_1,\approx_2}(F_f)].$$

Theorem 10. *Let F_f and C be as in Lemma 9, and moreover let*

$$\mathrm{Definition}(F,f) \equiv_{\mathrm{df}} \bigwedge_x [F(x,f(x)) \leftrightarrow \bigvee_y F(x,y)],$$
$$C' \equiv_{\mathrm{df}} \mathrm{Refl}(\approx_2) * \mathrm{Definition}(F_f, f_{F_f}) * \mathrm{Tot}(f).$$

Then the following estimations are valid:

$$C \leq \mathrm{Comp}_{\approx_1,\approx_2}(f) \to \mathrm{Function}_{\approx_1,\approx_2}(F_f), \quad (10)$$
$$C' \leq f_{F_f} \approx f. \quad (11)$$

Reading of the results: "It is TRUE that
(10) – IF $\approx_2$ is symmetric and transitive (we need the transitivity twice) and f is compatible, THEN $y \approx_2 f(x)$ is fuzzy function."
(11) – IF $\approx_2$ is reflexive and f is total and f_{F_f} is such that for an arbitrary x: $[f_{F_f}(x) \approx_2 f(x)$ IFF there exists y: $y \approx_2 f(x)]$, THEN f_{F_f} is similar to f."

The reverse problem can be formulated as follows: consider a fuzzy relation F and let us find a crisp function f_F such that it is compatible with $(\approx_1, \approx_2)$ and its extension to fuzzy relation F_{f_F} is similar to F.

Theorem 11. *Let*

$$D \equiv_{\mathrm{df}} \mathrm{Tot}(F) * \mathrm{Definition}(F, f_F),$$
$$D' \equiv_{\mathrm{df}} D * \mathrm{Refl}(\approx_1) * \mathrm{Sym}(\approx_2).$$

Then it can be proved that

$$D^2 \leq \mathrm{Func}_{\approx_{1,2}}(F) \to \mathrm{Comp}_{\approx_{1,2}}(f_F), \quad (12)$$
$$D' \leq \mathrm{Function}_{\approx_{1,2}}(F) \to (F_{f_F} \approx F). \quad (13)$$

Reading of the results: "It is TRUE that
(12) – IF F is functional and total (we need totality twice) and f_F is such that for an arbitrary x: $[F(x, f_F(x)$ IFF there exists y: $F(x,y)]$ (also needed twice) THEN f_F is the compatible function."
(13) – IF F is a total fuzzy function and f_F is as above and $\approx_1$ is reflexive and $\approx_2$ is symmetric THEN F_{f_F} is indistinguishable from F."

6 FUZZY BIJECTIONS

Traditionally, bijectivity refers to a mapping while here we are dealing with fuzzy relations generally. Let us look how the classical notion can be generalize to fuzzy relations and whether we get analogical results.

Let $a,b \in \mathbb{N}_+$, $R \underset{\sim}{\subset} X \times Y$ and $\approx_1 \underset{\sim}{\subset} X^2$, $\approx_2 \underset{\sim}{\subset} Y^2$ for this section.

We define (a,b)-bijection as a property of a fuzzy relation R w.r.t. $\approx_{1,2}$ as follows:

$$\mathrm{Bij}^{a,b}_{\approx_{1,2}}(R) \equiv_{\mathrm{df}} (\mathrm{Inj}_{\approx_{1,2}}(R))^a * (\mathrm{Sur}(R))^b.$$

The coefficients a,b refer to the numbers of times the respective properties are used in the above definition. E.g., $(2,1)$-bijection is defined as:

$$\mathrm{Inj}_{\approx_{1,2}}(R) * \mathrm{Inj}_{\approx_{1,2}}(R) * \mathrm{Sur}(R)\,.$$

It means that a relation R that is injective (to a degree m) and surjective (to a degree n) is $(2,1)$-bijective (to the degree $m*m*n$).

Let us summarize the well known results on the particular properties important for fuzzy bijections.

Proposition 12. *Let* Prop *be one of the properties* $\{\mathrm{Tot}, \mathrm{Sur}, \mathrm{Func}, \mathrm{Inj}\}$; *then the following expressions are valid:*

$$\mathrm{Tot}(R) = \mathrm{Sur}(R^{\mathrm{T}})$$
$$\mathrm{Func}_{\approx_{1,2}}(R) = \mathrm{Inj}_{\approx_{1,2}}(R^{\mathrm{T}})$$
$$(R_1 \approx R_2) * \mathrm{Tot}(R_1) \le \mathrm{Tot}(R_2)$$
$$(R_1 \approx R_2) * \mathrm{Sur}(R_1) \le \mathrm{Sur}(R_2)$$
$$(R_1 \approx R_2)^2 * \mathrm{Func}_{\approx_{1,2}}(R_1) \le \mathrm{Func}_{\approx_{1,2}}(R_2)$$
$$(R_1 \approx R_2)^2 * \mathrm{Inj}_{\approx_{1,2}}(R_1) \le \mathrm{Inj}_{\approx_{1,2}}(R_2)$$
$$\mathrm{Prop}(R) * \mathrm{Prop}(S) \le \mathrm{Prop}(R \circ S)$$

for an arbitrary $R, R_1, R_2 \underset{\sim}{\subset} X \times Y$, $S \underset{\sim}{\subset} Y \times Z$.

Guidelines for the proof of this proposition can be found in (Bělohlávek, 2002).

From the above proposition we easily deduce the following identity.

Theorem 13.

$$(\mathrm{Tot}(R))^b * (\mathrm{Func}_{\approx_{1,2}}(R))^a = \mathrm{Bij}^{a,b}_{\approx_{1,2}}(R^{\mathrm{T}})$$

The next theorem shows relationships between compositions of a fuzzy bijection R and R^{T} and the indistinguishability relations $\approx_1$ and $\approx_2$.

Theorem 14.

$$\mathrm{Refl}(\approx_1) * (\mathrm{Sur}(R))^2 * \mathrm{Function}_{\approx_{1,2}}(R) \le [(R^{\mathrm{T}} \circ R) \approx \approx_2] \quad (14)$$

$$\mathrm{Refl}(\approx_2) * (\mathrm{Sur}(R^{\mathrm{T}}))^2 * \mathrm{Function}_{\approx_{1,2}}(R^{\mathrm{T}}) \le [(R \circ R^{\mathrm{T}}) \approx \approx_1] \quad (15)$$

$$\mathrm{Refl}(\approx_1) * \mathrm{Refl}(\approx_2) * (\mathrm{Tot}(R))^2 * \mathrm{Function}_{\approx_{1,2}}(R) \le (\mathrm{Bij}^{1,2}_{\approx_{1,2}}(R) \to [[(R \circ R^{\mathrm{T}}) \approx \approx_1] \wedge [(R^{\mathrm{T}} \circ R) \approx \approx_2]]) \quad (16)$$

The proof of this theorem is too long to fit the page number limitation, therefore it is omitted and left for a full journal paper that is already in preparation.

Reading of the results: "It is TRUE that
(14) – IF $\approx_1$ is reflexive and R is a surjective (needed twice) fuzzy function THEN $(R^{\mathrm{T}} \circ R) \approx \approx_2$."
(15) – IF $\approx_2$ is reflexive and R^{T} is a surjective (needed twice) fuzzy function THEN $(R \circ R^{\mathrm{T}}) \approx \approx_1$."
(16) – IF $\approx_1$ is reflexive and $\approx_2$ is reflexive and R is a total (needed twice) fuzzy function THEN IF R is $(1,2)$-bijective (because we need surjectivity twice and injectivity only once) THEN $(R^{\mathrm{T}} \circ R) \approx \approx_2$ AND $(R \circ R^{\mathrm{T}}) \approx \approx_1$."

Interesting is the fact that we have to require from $\approx_1$ and $\approx_2$ only reflexivity to prove the above inequalities. And of course, that there is a need for having powers of surjectivity and totality. It is obviously a non-trivial generalization of the following classical theorem (Demirci, 2000).

Theorem 15. *Let* $\approx_1, \approx_2$ *be similarities, i.e., reflexive, symmetric and transitive fuzzy relations.*

If R *is a bijective total fuzzy function w.r.t.* $\approx_{1,2}$ *then* $(R^{\mathrm{T}} \circ R) \approx \approx_2$ *and* $(R \circ R^{\mathrm{T}}) \approx \approx_1$.

Example 16. *Let* X, Y *be non-empty sets of elements with* $\approx_1 \underset{\sim}{\subset} X$, $\approx_2 \underset{\sim}{\subset} Y$, $\mathfrak{F}$ *be the set of all functions* $f : X \mapsto Y$ *and* $\mathfrak{R}$ *be the set of all fuzzy relations* $F, F' \underset{\sim}{\subset} X \times Y$. *Moreover, let* $\sim_1 \underset{\sim}{\subset} \mathfrak{F}$ *and* $\sim_2 \underset{\sim}{\subset} \mathfrak{R}$ *defined as:*

$$f \sim_1 f' \equiv_{\mathrm{df}} \bigwedge_{x \in X} (f(x) \approx_2 f'(x)),$$
$$F \sim_2 F' \equiv_{\mathrm{df}} \bigwedge_{x \in X, y \in Y} (F(x,y) \leftrightarrow F'(x,y)).$$

Define a mapping $M : \mathfrak{F} \mapsto \mathfrak{R}$ *by the following assignment:*

$$M[f,F](x,y) =_{\mathrm{df}} F(x,y) = (y \approx_2 f(x)).$$

In other words, M *is a crisp function from* $\mathfrak{F}$ *to* $\mathfrak{R}$ *assigning a fuzzy relation* F *to a crisp function* f.

It can be proved that M *is injective provided that* $\approx_2$ *is symmetric and transitive. Generally, we can show that*

$$\mathrm{Sym}(\approx_2) * \mathrm{Trans}(\approx_2) \le \mathrm{Inj}_{\sim_{1,2}}(M),$$

which means that as much $\approx_2$ is symmetric and *transitive, at least that much R is injective.*

To show surjectivity we incorporate (13):

$$D * \mathrm{Tot}(F) * \mathrm{Function}_{\approx_{1,2}}(F) \leq (F_{f_F} \sim_2 F),$$

where

$$D \equiv_{\mathrm{df}} \mathrm{Refl}(\approx_1) * \mathrm{Sym}(\approx_2) * \mathrm{Definition}(F, f_F).$$

We would like to have $=$ instead of $\sim_2$ and that will work only with crisp properties on the left of the inequality. Thus M is surjective (to degree 1) provided that F is a total fuzzy function, $\approx_1$ is reflexive, $\approx_2$ is symmetric and f_F is such that $y = f_F(x)$ if and only if $F(x, f_F(x)) = \bigvee_{y \in Y} F(x,y)$.

All these properties led us to the following conclusion: Let $\approx_1$ be reflexive and $\approx_2$ be symmetric. Then M is $(1,1)$-bijection to the degree that is bounded from below by degrees of symmetry (a) and *transitivity (b) of $\approx_2$. Hence M is a $(1,1)$-bijection at least to degree $a * b$ between $\mathfrak{F}$ and a subset of $\mathfrak{R}$ of total fuzzy functions.*

7 CONCLUSIONS

In this contribution, we have reviewed the results on extensionality and functionality properties in the algebraic framework. Also the well known representation theorem for fuzzy functions has been presented in the graded form. Moreover, we have introduced the graded notion of a bijective relation and showed graded theorems that generalize the known results of fuzzy mathematics. It appeared that we can relax a lot of requirements on relations describing indistinguishability of elements of the respective universes. The main advantage of the presented approach is that it incorporates the whole scale for degrees of truth. An evaluation of the degrees of the antecedents of a graded theorem provides the additional information about the estimation of the degree of the consequent.

We have shown that graded theorems bring a new light into the already well established classical theory of fuzzy functions and fuzzy bijective relations and generally, to fuzzy mathematics. An intended class of applications of the introduced theory of fuzzy functions and fuzzy bijective relations is connected with fuzzy rules, especially, the implicative model of fuzzy rules (Daňková, 2011; Štěpnička M., 2013). It is due to the fact that this model is closely connected with fuzzy functions; and by Theorem 14, fuzzy functions are closely related to bijections. Moreover, this research is a basis for developing a theory of partial fuzzy functions and partial bijective fuzzy relations over a simple system of fuzzy partial propositional logic, i.e., a fuzzy propositional logic which admits undefined truth degrees (Běhounek and Novák, 2015). It is a topic for future research to find some interesting examples that actively work with degrees and use the presented graded theorems.

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