A New Distance on a Specific Subset of Fuzzy Sets

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Abstract: In this paper, first we propose a definition for fuzzy LR sets and then we present a method to assigning distance between these form of fuzzy sets. We show that this distance is a metric on the set of all trapezoidal fuzzy sets with the same height and all trapezoidal fuzzy numbers and is a pseudo-metric on the set of all fuzzy sets.

1 INTRODUCTION

There are lots of works that the authors investigated fuzzy sets, in order to find the nearest approximation of an arbitrary fuzzy numbers. Approximation of a fuzzy number can be done in three ways. Some authors assigned a single crisp number to a fuzzy number as a ranking method. In this case many information of the fuzzy number will be lost. The other method is using an interval as an approximations of a fuzzy number (Chanas, 2001; Grzegorzewski, 2002). But, in this case, the modal value (the core with height 1) of the fuzzy number will be lost. In some works such as (Abbasbandy and Asady, 2004; Delgado et al., 1998; Grzegorzewski and Mrózka, 2005), the authors tried to solve an optimization problem in order to obtain a trapezoidal fuzzy number as a nearest approximation. Some works were done on approximation of a fuzzy number (Anzilli et al., 2014; Ban et al., 2011; Cano et al., 2016). Some distances and their properties were done in (Abbasbandy and Amirfakhrian, 2006a; Abbasbandy and Amirfakhrian, 2006b).

In this work we introduce a fuzzy LR set and we present a distance to find the nearest trapezoidal fuzzy set to an arbitrary LR fuzzy set. The motivation behind this distance is trying to compare fuzzy sets of special format and the same height.

The structure of the this paper is as follows. In Section 2 the basic concepts of our work are introduced, then we introduce LR fuzzy set. In Section 3 we introduce a distance over all fuzzy LR sets with the same height and we named it h-source distance, which is a metric on the set of all trapezoidal LR fuzzy set, with the same height. In Section 4 the nearest trapezoidal fuzzy number to an arbitrary trapezoidal LR fuzzy set was introduced and a simple method for computing it, was presented. Section 5 contains some numerical examples.

2 PRELIMINARIES

Let \( F(\mathbb{R}) \) be the set of all normal and convex fuzzy numbers on the real line.

**Definition 2.1.** A generalized LR fuzzy number \( \tilde{a} \) with the membership function \( \mu_{\tilde{a}}(x), x \in \mathbb{R} \) can be defined as

\[
\mu_{\tilde{a}}(x) = \begin{cases} 
\tilde{l}(x), & a \leq x \leq b, \\
1, & b \leq x \leq c, \\
\tilde{r}(x), & c \leq x \leq d, \\
0, & \text{otherwise}, 
\end{cases}
\]

(2.1)

where \( \tilde{l}(x) \) is the left membership function that is an increasing function on \([a, b]\) and \( \tilde{r}(x) \) is the right membership function that is a decreasing function on \([c, d]\). Furthermore we want to have \( \tilde{l}(a) = \tilde{r}(d) = 0 \) and \( \tilde{l}(b) = \tilde{r}(c) = 1 \). In addition, if \( \tilde{l}(x) \) and \( \tilde{r}(x) \) are linear, then \( \tilde{a} \) is a trapezoidal fuzzy number which is denoted by \((a, b, c, d)\). If \( b = c \), we denoted it by \((a, c, d)\), which is a triangular fuzzy number.

For \( 0 < \alpha \leq 1 \), \( \alpha \)-cut of a fuzzy number \( \tilde{a} \) is defined by,

\[
[\tilde{a}]^{[\alpha]} = \{ t \in \mathbb{R} | \mu_{\tilde{a}}(t) \geq \alpha \}.
\]

(2.2)

**Definition 2.2.** (Voxman, 1998). A continuous function \( s : [0, 1] \to [0, 1] \) with the following properties is a regular reducing function:

1. \( s(r) \) is increasing.
2. \( s(0) = 0 \).
3. \( s(1) = 1 \).
4. \( \int_0^1 s(r) dr = \frac{1}{2} \).
In (Chong-Xin and Ming, 1991), the authors represented a fuzzy number \( \tilde{u} \) by an ordered pair of functions \((\chi(r), \tau(r))\):

The parametric form of a fuzzy number is shown by \( \tilde{v} = (\chi(r), \tau(r))\), where functions \( \chi(r) \) and \( \tau(r) \); \( 0 \leq r \leq 1 \) satisfy the following requirements:

1. \( \chi(r) \) is monotonically increasing left continuous function.
2. \( \tau(r) \) is monotonically decreasing left continuous function.
3. \( \chi(r) \leq \tau(r) \), \( 0 \leq r \leq 1 \).
4. \( \tau(r) = \chi(r) = 0 \) for \( r < 0 \) or \( r > 1 \).

**Definition 2.3.** (Voxman, 1998), The value and ambiguity of a fuzzy number \( \tilde{u} \) are defined by

\[
Val(\tilde{u}) := \int_{0}^{1} s(r)[\tau(r) - \chi(r)]dr,
\]  
(2.3)

and

\[
Amb(\tilde{u}) := \int_{0}^{1} s(r)[\tau(r) - \chi(r)]dr,
\]  
(2.4)

respectively.

**Definition 2.4.** A fuzzy set \( \tilde{u} \) is a generalized LR fuzzy set, if there exist a positive number \( h \in (0, 1] \) such that

\[
\mu_h(x) = \begin{cases} 
0, & \text{if } x \in [a, b], \\
\frac{l(x)}{r(c)}, & \text{if } b < x \leq c, \\
\frac{r(c)}{r(c)}, & \text{if } c < x < d, \\
\frac{l(x)}{r(c)}, & \text{if } x \in [c, d], \\
0, & \text{otherwise},
\end{cases}
\]  
(2.5)

where \( l(x) \) is nondecreasing on \([a, b]\) and \( r(x) \) is nonincreasing on \([c, d]\) such that \( l(a) = r(d) = 0 \) and \( l(b) = r(c) = h \). If \( h = 1 \) then \( \tilde{u} \) is an LR fuzzy number.

In addition, if \( l(x) \) and \( r(x) \) are linear, then \( \tilde{u} \) is a trapezoidal fuzzy set which is denoted by \((a, b, c, d, h)\). In this case if \( b = c \), we denote it by \((a, b, c, d, h)\), which is a trapezoidal LR fuzzy set. Also if \( TF(\mathbb{R}) \) and \( TF_h(\mathbb{R}) \) be the set of all trapezoidal fuzzy numbers and all trapezoidal LR fuzzy set with height \( h \) on \( \mathbb{R} \), respectively:

\[
\begin{cases} 
TF(\mathbb{R}) = \{(a, b, c, d) : a < b < c < d\}, \\
TF_h(\mathbb{R}) = \{(a, b, c, d, h) : a < b < c < d\}.
\end{cases}
\]  
(2.6)

**Definition 2.5.** A function \( s \in C[0, h] \) with the following properties is a source function

1. \( s(\alpha) \geq 0 \), \( \alpha \in [0, h] \)
2. \( s(0) = 0 \),
3. \( s(h) = h \),
4. \( \int_{0}^{h} s(\alpha)d\alpha = \frac{1}{2}h^2 \).

For an LR fuzzy set \( \tilde{u} \) and \( \alpha \in [0, h] \) we define \( u(\alpha) \) and \( \pi(\alpha) \) as follows

\[
u(\alpha) = \inf\{x | \mu_h(x) \geq \alpha\}, \quad 0 \leq \alpha < h, \]  
(2.7)

\[
\pi(\alpha) = \sup\{x | \mu_h(x) \geq \alpha\}, \quad 0 \leq \alpha < h. \]  
(2.8)

For a trapezoidal fuzzy set which is denoted by \( \tilde{u} = (a, b, c, d, h) \), we have

\[
u(\alpha) = a + \frac{b-a}{h}\alpha, \]  
(2.9)

\[
\pi(\alpha) = d - \frac{d-c}{h}\alpha. \]  
(2.10)

**Definition 2.6.** We define Value and Ambiguity of an LR fuzzy set \( \tilde{u} \) by the following relations:

1. \( V_h(\tilde{u}) = \int_{0}^{h} s(h)(\pi(\alpha) + u(\alpha))d\alpha, \)
2. \( A_h(\tilde{u}) = \int_{0}^{h} s(h)(\pi(\alpha) - u(\alpha))d\alpha. \)

**Definition 2.7.** Let \( s \) be a source function, then \( I_{s,h} \) defined bellow is source number with respect to \( s \).

\[
I_{s,h} = \int_{0}^{h} s(\alpha)d\alpha. \]  
(2.11)

**Lemma 2.1.** For an arbitrary source function \( s \) over \((0, h]\), we have \( I_{s,h} < \frac{1}{2}h^2 \).

**Proof.** By using Mid-point Theorem, the proof is straightforward. \( \square \)

### 3 h-SOURCE DISTANCE BETWEEN FUZZY LR SETS

**Definition 3.1.** For two LR fuzzy sets \( \tilde{u} \) and \( \tilde{v} \), with same heights \( h \), we define \( h \)-source distance \( \mathcal{D} \) as follows,

\[
\mathcal{D}(\tilde{u}, \tilde{v}) = \frac{1}{2}\left\{ |V_h(\tilde{u}) - V_h(\tilde{v})| + |A_h(\tilde{u}) - A_h(\tilde{v})| \right\} + h^2d_H([\tilde{u}]^h, [\tilde{v}]^h).
\]

where \( d_H \) is Hausdorff metric and \([\tilde{w}]^h = \{x | \mu_h(x) \geq h\} \) is the \( h \)-cut of fuzzy number \( \tilde{w} \).

**Theorem 3.1.** For \( \tilde{u}, \tilde{v}, \tilde{w} \) in LR fuzzy sets, the \( h \)-source distance, \( \mathcal{D} \), satisfies the following properties:

1. \( \mathcal{D}(\tilde{u}, \tilde{u}) = 0 \),
2. \( \mathcal{D}(\tilde{u}, \tilde{v}) = \mathcal{D}(\tilde{v}, \tilde{u}) \),
3. \( \mathcal{D}(\tilde{u}, \tilde{w}) \leq \mathcal{D}(\tilde{u}, \tilde{v}) + \mathcal{D}(\tilde{v}, \tilde{w}) \).

Source distance between fuzzy numbers defined in (Abbasbandy and Amirfakhrian, 2006b) is a special case of \( h \)-source distance.
Example 3.1. Let \( \mu_\tilde{u}(x) = \begin{cases} h, & x = a, \\ 0, & \text{otherwise}, \end{cases} \)
\( \mu_\tilde{v}(x) = \begin{cases} h, & x = b, \\ 0, & \text{otherwise}. \end{cases} \)
\( D(\tilde{u}, \tilde{v}) \) is the source distance to find the fuzzy number of \( \tilde{u} \) is \( \tilde{v} = (6 - 2e, e, 2e(3e - 7)) \). In Figure 2, \( \tilde{u} \) and \( \tilde{v} \) are shown by solid and dashed lines, respectively.

4 NEAREST APPROXIMATION OF LR FUZZY SETS

In this section we use \( h \)-source distance to find the nearest approximation of an arbitrary fuzzy set. We start with a theorem on set of all fuzzy sets with the same height.

Theorem 4.1. Let \( \tilde{u}, \tilde{v} \in T_F(\mathbb{R}) \), then \( D(\tilde{u}, \tilde{v}) = 0 \), if and only if \( \tilde{u} = \tilde{v} \).

Proof. If \( \tilde{u} = \tilde{v} \), from Theorem 3.1 we have \( D(\tilde{u}, \tilde{v}) = 0 \). Let \( \tilde{u} = (a_u, b_u, c_u, d_u, h) \) and \( \tilde{v} = (a_v, b_v, c_v, d_v, h) \) are two trapezoidal fuzzy sets. If \( D(\tilde{u}, \tilde{v}) = 0 \) then

\[
\begin{align*}
\text{a) } & \max \{h^3|c_u - c_v|, h^3|b_u - b_v|\} = 0, \\
\text{b) } & V_h(\tilde{u}) = V_h(\tilde{v}), \\
\text{c) } & A_h(\tilde{u}) = A_h(\tilde{v}).
\end{align*}
\]

From (a), we have \( \max \{h^3|b_u - a_u|, h^3|b_v - a_v|\} = 0 \) and hence \( a_u = a_v \) and \( b_u = b_v \). From (b) and (c)

\[
\begin{align*}
V_h(\tilde{u}) + A_h(\tilde{u}) &= 2 \int_0^h \hat{s}(\alpha) \hat{p}(\alpha) d\alpha \\
&= 2 \int_0^h \hat{s}(\alpha) \hat{p}(\alpha) d\alpha \\
&= V_h(\tilde{v}) + A_h(\tilde{v}),
\end{align*}
\]

\[
\begin{align*}
V_h(\tilde{u}) - A_h(\tilde{u}) &= 2 \int_0^h \hat{s}(\alpha) \hat{u}(\alpha) d\alpha \\
&= 2 \int_0^h \hat{s}(\alpha) \hat{u}(\alpha) d\alpha \\
&= V_h(\tilde{v}) - A_h(\tilde{v}),
\end{align*}
\]

and hence

By considering \( \theta = h^3 - 2I_{s,h} \), the system (4.4) is equivalent to the following relations:

\[
\begin{align*}
hd_r\theta + 2c_rI_{s,h} &= hd_r\theta + 2c_rI_{s,h}, \\
hd_v\theta + 2b_vI_{s,h} &= hd_v\theta + 2b_vI_{s,h}.
\end{align*}
\]

Corollary 4.2. \( h \)-source distance, \( D \), is a metric on \( T_F(\mathbb{R}) \), for a fixed height \( h \).

Proof. By Theorems 3.1 and 4.1 the proof is clear.

Corollary 4.3. Let \( \tilde{u} \) be an arbitrary LR fuzzy set with height \( h \). Let

\[
\begin{align*}
I &= \int_0^h \hat{u}(\alpha)s(\alpha) d\alpha, \\
\bar{I} &= \int_0^h \bar{u}(\alpha)s(\alpha) d\alpha.
\end{align*}
\]

The nearest trapezoidal fuzzy set of \( \tilde{u} \) is \( \tilde{v} = (a_v, b_v, c_v, d_v, h) \), where

\[
\begin{align*}
a_v &= \frac{1}{(h^3 - 2I_{s,h})(-2b_v + 2h)}, \\
b_v &= \hat{u}(h), \\
c_v &= \bar{u}(h), \\
d_v &= \frac{1}{(h^3 - 2I_{s,h})(-2c_v + 2h)}.
\end{align*}
\]

Lemma 4.4. For an arbitrary LR fuzzy set \( \tilde{u} \) with height \( h \), the nearest trapezoidal fuzzy set exists and it is unique.

Proof. By Corollary 4.3 the proof is clear.

5 NUMERICAL EXAMPLES

In this section we present some numerical examples using the proposed method.

Example 5.1. Let \( \tilde{u} = (e', e^{2r}) \), \( \frac{1}{2} \) and \( s(r) = r \). The nearest trapezoidal fuzzy set of \( \tilde{u} \) is \( \tilde{v} = (24 - 14\sqrt{e}, \sqrt{e}, e^{3/2}, 24e^{2} - 38e^{3/2}) \). In Figure 1, \( \tilde{u} \) and \( \tilde{v} \) are shown by solid and dashed lines, respectively.

Example 5.2. Let \( \tilde{u} = (e', e^{2r}) \), and \( s(r) = r \). The nearest triangular fuzzy number of \( \tilde{u} \) is \( \tilde{v} = (6 - 2e, e, 2e(3e - 7)) \). In Figure 2, \( \tilde{u} \) and \( \tilde{v} \) are shown by solid and dashed lines, respectively.
Example 5.3.
Let $\tilde{u}_h = (\ln(2 + r^2, 4 - \ln(1 + r), h)$ and $\tilde{v}_h = (r^2 + 1, 5 - 2r^2, h)$. Using $s(r) = r$, the values of $h$-distance $D(\tilde{u}_h, \tilde{v}_h)$ between the fuzzy sets are shown in Table 1 for various values of $h$. See Figure 3.

Table 1: $\tilde{u}_h$ and $\tilde{v}_h$ for various values of $h$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$D(\tilde{u}_h, \tilde{v}_h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>3.95069</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>1.49782</td>
</tr>
<tr>
<td>$\frac{2}{3}$</td>
<td>0.76366</td>
</tr>
<tr>
<td>$\frac{1}{5}$</td>
<td>0.09769</td>
</tr>
</tbody>
</table>

6 CONCLUSIONS

In this work we presented a method to find the nearest trapezoidal fuzzy set of an arbitrary LR fuzzy set with the same height. The nearest fuzzy set was found by using a new distance between fuzzy sets. Numerical examples shows that this method is acceptable.

REFERENCES


