Nonlinear Second Cumulant/H-infinity Control with Multiple Decision Makers

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Abstract: This paper studies a second cumulant/h-infinity control problem with multiple players for a nonlinear stochastic system on a finite-horizon. The second cumulant/h-infinity control problem, which is a generalization of the higher-order multi-objective control problem, involves a control method with multiple performance indices. The necessary condition for the existence of Nash equilibrium strategies for the second cumulant/h-infinity control problem is given by the coupled Hamilton-Jacobi-Bellman (HJB) equations. In addition, a three-player Nash strategy is derived for the second cumulant/h-infinity control problem. A simulation example is given to illustrate the application of the proposed theoretical formulations.

1 INTRODUCTION

Higher-order control problems (Won et al., 2010) for stochastic systems have been investigated in recent years and related to multi-objective control theoretical game formulations (Lee et al., 2010). In multi-objective control problems, the control method must concern itself with multiple performance indices. A typical multi-objective control problem for both stochastic and deterministic systems can be formulated as mixed $H_2/H_{\infty}$ control, where the control wishes to minimize an $H_2$ norm while keeping the $H_{\infty}$ norm constrained. In fact, $H_2/H_{\infty}$ control problem is a robust control method which requires a controller to minimize the $H_2$ performance while attenuating the worst case external disturbance. This approach was investigated in (Bernstein and Hassas, 1989), while the Nash game approach to the problem was given in (Limebeer et al., 1994). In (Basar and Olsder, 1999), a two-player game involving control and disturbance was analyzed, where both players wished to optimize their respective performance indices when the other player plays their equilibrium strategy.

In this paper, mixed second cumulant/h-infinity (second cumulant/$H_{\infty}$) control problem with multiple players is investigated for a nonlinear stochastic system. Why second cumulant/$H_{\infty}$ as compared to first cumulant/$H_{\infty}$ or ($H_2/H_{\infty}$). Earlier studies in (Won et al., 2010) have shown that higher-order cumulants offer the control engineer additional degrees of freedom to improve system performance through the shaping of the cost function distribution. As a result of this opportunity, there is need to investigate higher-order cumulant to worst case disturbance effects on dynamic systems. The second cumulant/h-infinity control problem involves simultaneous optimization of the higher-order statistical properties of each individual player’s cost function distribution through cumulants while keeping the $H_{\infty}$ norm constrained. The optimization of cost function distribution through cost cumulant was initiated by Sain (Sain, 1966), (Sain and Liberty, 1971). Linear quadratic statistical game with related application such as satellite systems was investigated in (Lee et al., 2010) while an output feedback approach to higher-order statistical game was studied in (Aduba and Won, 2015).

As an extension of the foregoing studies in (Lee et al., 2010), (Aduba and Won, 2015) and the references there in, a nonlinear system of three players with quadratic cost function which is a non trivial extension is considered. Typical multi-objective control problem applications are in large-scale systems such as computer communications networks, electric power grid networks and manufacturing plant networks (Bauso et al., 2008), (Charilas and Panagopoulos, 2010) while the higher-order multi-objective control application has been reported for satellite network (Lee et al., 2010). The rest of this paper is organized as follows. In Section 2, the mathematical preliminaries and second cumulant/h-infinity control problem for a completely observed nonlinear system with multiple players; which is formulated as a nonzero-
sum differential game problem are given. Section 3
states and proves the necessary condition for the exis-
tence of Nash equilibrium strategies while Section 4
derives the optimal players strategy based on solving
coupled Hamilton-Jacobi-Bellman equations which is
the main result of this paper. Section 5 gives the
numerical approximate method for solving the coupled
Nash game Hamilton-Jacobi-Bellman equations while a numerical example is demonstrated in Section
6. Finally, the conclusions are drawn in Section 7.

2 PROBLEM FORMULATION

Consider a 3-player nonlinear stochastic state dynam-
ics given by the following Itô-type differential equa-
tion:

\[
dx(t) = f(t, x(t), u_1(t), u_2(t), v(t))dt + \sigma(x(t))dw(t)
\]

where \( t \in [t_0, t_F] = T \), \( x(t) \in \mathbb{R}^n \) is the state and
\( x(t_0) = x_0 \), \( u_k(t) \in U_k \subset \mathbb{R}^m \) is the k-th player strategy,
\( k = 1, 2 \), \( v(t) \in V_k \subset \mathbb{R}^m \) is the external disturbance player
and \( dw(t) \) is a Gaussian random process of dimen-
sion \( d \) with zero mean, covariance of \( W(t) dt \). Let
\( Q_0 = [t_0, t_F] \times \mathbb{R}^n \) and \( \tilde{Q}_0 \) is the closure of \( Q_0 \).

\( f \) and \( \sigma \) are Borel measurable functions given as:
\( f: C^1(\tilde{Q}_0 \times U_k \times U_j \times V_j) \) and \( \sigma: C^1(\tilde{Q}_0) \). In addition, \( f \) and \( \sigma \) satisfy Lipschitz and linear growth
conditions (Arnold, 1974) while \( x(t) \) is the regul-
ated output of the stochastic system. Let \( u_k(t) = u_k(t, x(t), v(t)) = v(t, x(t), t) \in T \) be memoryless state feedback strategies with \( \mu_k(t, v(t), x(t)) \) satisfying Lip-
schitz and linear growth condition and thus are admis-
sible strategies. It is shown in (Fleming and Rishel, 1975) that a process \( x(t) \) from (1) having admissible strategies together with polynomial growth condition
ensures that \( E \|x(t)\|^2 \) is finite.

The backward evolution operator, \( O(\mu_1, \mu_2, \nu) \)
(Sain et al., 2000): \( O = O_1 + O_2 \) is introduced

\[
O_1(\mu_1, \mu_2, \nu) = \frac{\partial}{\partial t} + f'(t, x, \mu_1, \mu_2, \nu) \frac{\partial}{\partial x}
\]

\[
O_2(\mu_1, \mu_2, \nu) = \frac{1}{2} tr \left( \sigma W \sigma' \frac{\partial^2}{\partial x^2} \right),
\]

where \( tr \) is the trace operator The cost function \( J_k \)
for the k-th player is given as:

\[
J_k(t, x, \mu_1, \mu_2, \nu) = \int_{t}^{t_F} L_k(s, x(s), \mu_1, \mu_2, \nu) ds
+ \psi_k(x(t_F))
\]

\[
J_k(t, x, \mu_1, \mu_2, \nu) = \int_{t}^{t_F} \frac{\partial}{\partial t} z_k(t) z_k(t) ds + \psi_k(x(t_F)),
\]

where \( k = 1, 2 \), \( L_k \) is the running cost and \( \psi_k \) is the
terminal cost with both \( (L_k, \psi_k) \) satisfying polyno-
mial growth condition. The \( z_k \) in (3) is defined as
\( z_k(t) = x(t)Q(t)x(t) + u_k(t, \mu_1, \mu_2, \nu) \), \( Q(t) = \dot{Q}(t) \geq 0 \),
\( R_k = R_k > 0 \).

The cost function \( J \) for \( \nu \) is given as:

\[
J(t, x, \mu_1, \mu_2, \nu) = \int_{t}^{t_F} L(s, x(s), \mu_1, \mu_2, \nu, \nu) ds
+ \psi(x(t_F)) \quad \text{or}
\]

\[
J(t, x, \mu_1, \mu_2, \nu) = \int_{t}^{t_F} \left( p^2 \nu'(t) \nu(t) - z'(t) z(t) \right) ds
+ \psi(x(t_F)),
\]

where \( L \) is the running cost and \( \psi \) is the terminal cost
with both \( (L, \psi) \) satisfying polynomial growth condition.
Also, \( \rho > 0 \) is the constraint on the \( H_m \) of the
system.

To study the cumulant game of cost function, the
\( m \)-th moments of cost functions \( M_m^k \) of the k-th player
is defined as:

\[
M_m^k(t, x, \mu_1, \mu_2, \nu) = E \left\{ (\nu)^m(t, x, \mu_1, \mu_2) | x(t) = x \right\},
\]

where \( m = 1, 2 \). The \( m \)-th cost cumulant function
\( V_m^k(t, x) \) of the k-th player is defined by (Smith, 1995),

\[
V_m^k(t, x) = M_m^k \sum_{i=0}^{m-2} \frac{(m-1)!}{(m-1-i)!} \nu_i \nu_{m-1-i+1}
\]

where \( t \in T = [t_0, t_F], x(t_0) = x_0, x(t) \in \mathbb{R}^n \). Next, the following definitions are given:

**Definition 2.1**: A function \( M^k, V^k: \tilde{Q}_0 \to \mathbb{R}^+ \) is an admissible \( l \)-th moment cost function if there exists a strategy \( \mu_k \) such that

\[
M^k_l(t, x, \mu_1, \mu_2, \nu),
\]

\[
V^k_l(t, x) = V^k_l(t, x, \mu_1, \mu_2, \nu),
\]

for \( t \in T, x \in \mathbb{R}^n, i = 1, 2 \).

**Definition 2.2**: The players equilibrium strategy \( \mu^*_1, \mu^*_2 \) is such that

\[
M^k_l(t, x, \mu^*_1, \mu^*_2, \nu) \leq M^k_l(t, x, \mu^*_1, \mu^*_2, \nu),
\]

\[
V^k_l(t, x) = V^k_l(t, x, \mu^*_1, \mu^*_2, \nu) \leq V^k_l(t, x, \mu^*_1, \mu^*_2, \nu),
\]

\[
M^k_l(t, x, \mu^*_1, \mu^*_2, \nu) \leq M^k_l(t, x, \mu^*_1, \mu^*_2, \nu),
\]

\[
V^k_l(t, x) = V^k_l(t, x, \mu^*_1, \mu^*_2, \nu) \leq V^k_l(t, x, \mu^*_1, \mu^*_2, \nu).
\]

The moment (5), moment-cumulant relationship (6),
definition 2.1 (7) and definition 2.2 (8) all hold for the
external disturbance player (\( \nu \)) as well.

**Problem Definition**: Consider an open set \( Q \subset Q_0 \) and let the k-th player and disturbance cost cumulant functions \( V^k_l(t, x), \tilde{V}_l(t, x) \in C^p_{\infty}(\tilde{Q}) \cap C(\tilde{Q}) \) be
an admissible cumulant function. Assume the existence of optimal players strategies $\mu_k^1, \mu_k^2, \nu^*$ and optimal players value functions $V_k^{1^*}(t,x), V_k^{2^*}(t,x)$, thus, the multi-player second cumulant/H$_\infty$ control problem is to find the Nash strategies $\mu_k^1, \mu_k^2, \nu^*$ which result in the minimal second value functions $V_k^{1^*}(t,x), V_k^{2^*}(t,x)$ while satisfying the system H$_\infty$ constraint. Thus, $\mu_k^2$ is the second cumulant/H$_\infty$ optimal strategy and $\nu^*$ is the external disturbance strategy.

Remark: To find the Nash strategies $\mu_k^1, \mu_k^2, \nu^*$, we constrain the candidates of the optimal players strategy to $U_{M^1}, U_{M^2}, U_{M^3}$ and the optimal value functions $V_k^{1^*}(t,x), V_k^{2^*}(t,x)$ are found with the assumption that lower order cumulants, $V_1^1, V_1^2$ are admissible.

3 SECOND CUMULANT HJB EQUATION

Theorem 3.1: Let $M_k^j(t,x) \in C_{r,p}^{1,2}(Q) \cap C(\overline{Q})$ be the admissible moment cost function, if there exists an optimal $k$-th player strategy $\mu_k^j$ such that $M_k^j(t,x) = M_k^j(t,x,\mu_k^1, \mu_k^2, \nu^*)$, $t \in T = [0,T_F]$, then,

$$O[M_k^j(t,x)] + jM_{k-1}^j(t,x)\mu_k^j(t,x,\mu_k^1, \mu_k^2, \nu^*) = 0,$$

where $M_k^j(t_F,x) = \psi_k^j(x(T_F))$, $j = 1,2$ and $k = 3,4$.

Remark: This theorem is an extension of Theorem 3.1 in (Won et al., 2010) which considered only a single player in a statistical optimal control problem. This theorem is applied in this multi-player game.

Theorem 3.2: The necessary condition for Nash equilibrium using the $k$-th player; $k = 1, 2$ as reference is stated and proven. However, similar proof holds with disturbance $\nu$ as reference. Consider a 3-player nonlinear system (1) with cost functional (3), (4) of fixed duration $[0,T_F]$. Let $V_k^1(t,x), V_k^2(t,x) \in C_{r,p}^{1,2}(Q) \cap C(\overline{Q})$ be admissible value functions for the $k$-th player.

Similarly, let $\tilde{V}_1(t,x), \tilde{V}_2(t,x) \in C_{r,p}^{1,2}(Q) \cap C(\overline{Q})$ be admissible value functions for the disturbance player. Assume the existence of optimal player strategy $\mu_k^2$ and an optimal value function $V_k^{2^*}(t,x)$. Then, the minimal 2nd value function $V_k^{2^*}(t,x)$ satisfies in compact form the following HJB equation for the $k$-th player.

$$0 = \min_{\mu_k^2 \in U_{M^2}} \left\{ O(\mu_k^1, \mu_k^2, \nu^*) \left[ \frac{dV_k^L(t,x)}{dx} \right] + \frac{dV_k^L(t,x)}{dx} \right\},$$

where $\sigma(t,x) = \sigma(t,x)W(t)\sigma(t,x)^T$.

Theorem 4: Let $V_k^1(t,x), V_k^2(t,x) \in C_{r,p}^{1,2}(Q) \cap C(\overline{Q})$ be admissible value functions for the $k$-th player; $k =$
1.2. Also, \( \bar{V}_1(t,x) \in C^1_p(Q) \cap C(\bar{Q}) \) is the admissible value function for the external disturbance (\( \nu \)). The players full state-feedback Nash strategies are given as

\[
\mu^*_k(t,x) = -\frac{1}{2} R^{-1}_k B_k' \left( \frac{\partial \bar{V}^k}{\partial x} + \bar{V}'^k(t) \frac{\partial \bar{V}^k}{\partial x} \right),
\]

\[
v^*(t,x) = -\frac{1}{2p^2} B_1' \left( \frac{\partial \bar{V}^1}{\partial x} + \gamma(t) \frac{\partial \bar{V}^1}{\partial x} \right),
\]

(18)

with \( V^j_k(t_F,x_F) = V^j(t_F,x_F) = 0 \) where \( j = 1,2, p > 0 \) and \( V^2(t), \gamma(t) \) are the Lagrange multipliers. From (1), \( f(.) = g(x(t)) + B_1(x)u_1(t) + B_2(x)u_2(t) + B_3(x)\nu(t) \) and from (3), \( \mu_k = x(t)^\top Q(t)x(t) + \mu^*_k(x)R_k(t)\mu_k(x) \), with \( g : \bar{Q}_0 \to \mathbb{R}^n \) is \( C^1(\bar{Q}_0) \). \( B_i(x(t)) \), \( i = 1,2,3 \) are continuous real matrices and \( R_k(t) > 0 \) are symmetric matrices.

In addition, \( L = p^2 \gamma(t)(\nu(t) - \zeta(t)) = 0 \) in (1) and the matrices \( C, D_1, D_2 \) are continuous real matrices of appropriate dimensions with \( C = D_1^\top D_1 + D_2^\top D_2 = I \) and \( D_1^\top D_2 = D_2^\top D_1 = 0 \).

**Proof:** The minimal 3-player 2nd order value functions \( V^1(t,x), V^2(t,x), \bar{V}_1(t,x) \) satisfy the constrained coupled HJB equations as follows:

\[
G_1(\mu_1,\mu_2,\nu) = \mathcal{O}(V^1) + \left( \frac{\partial V^1}{\partial x} \right)' \sigma W \sigma' \left( \frac{\partial V^1}{\partial x} \right),
\]

\[
+ \lambda_1(t) \left( \mathcal{O}(V^1) + L_1(t,x,\mu_1,\mu_2,\nu) \right),
\]

(22)

where \( \lambda_1(t) \) is time-varying Lagrange multiplier.

Similarly, let \( G_2(\mu_1,\mu_2,\nu) \) be formulated by converting the constrained coupled HJB equation (20) to unconstrained coupled HJB equations as follows:

\[
G_2(\mu_1,\mu_2,\nu) = \mathcal{O}(V^2) + \left( \frac{\partial V^2}{\partial x} \right)' \sigma W \sigma' \left( \frac{\partial V^2}{\partial x} \right),
\]

\[
+ \lambda_2(t) \left( \mathcal{O}(V^2) + L_2(t,x,\mu_1,\mu_2,\nu) \right),
\]

(23)

where \( \lambda_2(t) \) is time-varying Lagrange multiplier.

Similarly, let \( G(\mu_1,\mu_2,\nu) \) be formulated by converting the constrained coupled HJB equation (21) to unconstrained coupled HJB equations as follows:

\[
G(\mu_1,\mu_2,\nu) = \mathcal{O}(V) + \left( \frac{\partial V}{\partial x} \right)' \sigma W \sigma' \left( \frac{\partial V}{\partial x} \right),
\]

\[
+ \lambda(t) \left( \mathcal{O}(V) + L(t,x,\mu_1,\mu_2,\nu) \right),
\]

(24)

where \( \lambda(t) \) is time-varying Lagrange multiplier.

At equilibrium state, the stationary conditions are given by the partial derivative of \( G_1(\mu_1,\mu_2,\nu), G_2(\mu_1,\mu_2,\nu), G(\mu_1,\mu_2,\nu) \) in (22), (23), (24), with respect to \( \mu_1, \lambda^*_1(t), \mu_2, \lambda^*_2(t), \nu, \lambda(t) \), which is zero. Thus, the full-state feedback Nash strategies \( \mu^*_1, \mu^*_2, \nu^* \) become

\[
\mu^*_1(t,x) = -\frac{1}{2} R_1^{-1} B_1' \left( \frac{\partial \bar{V}^1}{\partial x} + \frac{1}{\lambda^*_1(t)} \frac{\partial \bar{V}^1}{\partial x} \right),
\]

(25)

\[
\mu^*_2(t,x) = -\frac{1}{2} R_2^{-1} B_2' \left( \frac{\partial \bar{V}^2}{\partial x} + \frac{1}{\lambda^*_2(t)} \frac{\partial \bar{V}^2}{\partial x} \right),
\]

(25)

\[
v^*(t,x) = -\frac{1}{2p^2} B_1' \left( \frac{\partial \bar{V}^1}{\partial x} + \gamma(t) \frac{\partial \bar{V}^1}{\partial x} \right),
\]

(21)

Now, let the Lagrange multipliers in (25) be defined as

\[
\gamma^*_1(t) = \frac{1}{\lambda^*_1(t)}, \gamma^*_2(t) = \frac{1}{\lambda^*_2(t)}, \gamma(t) = \frac{1}{\lambda(t)}.
\]

(26)

Then, substituting (26) in (25) gives

\[
\mu^*_1(t,x) = -\frac{1}{2} R_1^{-1} B_1' \left( \frac{\partial \bar{V}^1}{\partial x} + \gamma^*_1(t) \frac{\partial \bar{V}^1}{\partial x} \right),
\]

(27)

\[
\mu^*_2(t,x) = -\frac{1}{2} R_2^{-1} B_2' \left( \frac{\partial \bar{V}^2}{\partial x} + \gamma^*_2(t) \frac{\partial \bar{V}^2}{\partial x} \right),
\]

(27)

\[
v^*(t,x) = -\frac{1}{2p^2} B_1' \left( \frac{\partial \bar{V}^1}{\partial x} + \gamma(t) \frac{\partial \bar{V}^1}{\partial x} \right).
\]
Thus, substituting for \( \mu^*_1, \mu^*_2, \nu^* \) to the 3-player 2\(^{nd}\) cost cumulant HJB equations (10) gives the closed loop system form of the second cumulant/H\(_{\infty}\) control. The theorem is proved.

**Remark:** The coupled cost cumulant HJB equation (10) provides the necessary condition for the Nash equilibrium solution of the 3-player second cumulant/H\(_{\infty}\) control.

However, substituting for \( \mu^*_1 \) in (19) or (20) for the first cumulant HJB equation (first line of (19) or (20)) gives

\[
\frac{\partial V^k}{\partial t} + g'(x) \left( \frac{\partial V^k}{\partial x} \right) + \frac{1}{4} \left( \frac{\partial V^k}{\partial x} \right)' B_k R_k^{-1} B_k' \left( \frac{\partial V^k}{\partial x} \right) - \frac{1}{2} \left( \frac{\partial V^k}{\partial x} \right) + \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) B_1 R_1^{-1} B_1' \left( \frac{\partial V^k}{\partial x} \right) \]

\(- \frac{1}{2} \left( \frac{\partial V^k}{\partial x} \right) + \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) B_2 R_2^{-1} B_2' \left( \frac{\partial V^k}{\partial x} \right) - \frac{1}{2} \left( \frac{\partial V^k}{\partial x} \right) \gamma \frac{1}{2p^2} B_3 B_3' \left( \frac{\partial V^k}{\partial x} \right) - \frac{\gamma V^k}{2} \left( \frac{\partial V^k}{\partial x} \right) \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) + x' Q x + \frac{1}{2} r (\sigma W \sigma' \frac{\partial V^k}{\partial x^2}) = 0.
\]

(28)

Also, substituting for \( \mu^*_1 \) in (19) or (20) for the second cumulant HJB equation (second line of (19) or (20)) gives

\[
\frac{\partial V^k}{\partial t} + g'(x) \left( \frac{\partial V^k}{\partial x} \right) - \frac{1}{2} \left( \frac{\partial V^k}{\partial x} \right) \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) B_3 B_3' \left( \frac{\partial V^k}{\partial x} \right) \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) + \frac{\gamma V^k}{2} \left( \frac{\partial V^k}{\partial x} \right) \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) \gamma^2 \left( \frac{\partial V^k}{\partial x} \right) + x' Q x + \frac{1}{2} r (\sigma W \sigma' \frac{\partial V^k}{\partial x^2}) = 0.
\]

(29)

Similarly, substituting \( \nu^* \) in (21) for the first and second cumulant HJB equations will yield closed-loop equations as in (28) and (29). Thus, the resulting six (6) coupled HJB equations are solved for the value functions \( V^k, \bar{V}_1, \bar{V}_2 \).

**Remark:** The minimal second cumulant strategies are found under constrained first cumulant at constrained worst case disturbance related by the cost function (4).

## 5 APPROXIMATE SOLUTION

The analytical solutions of HJB equations (19), (20), (21) are difficult to find for nonlinear systems. Several approximate methods such as power series, spectral and pseudo-spectral, wavelength, path integral and neural network methods have been utilized to solve coupled HJB equations (Al’brekht, 1961), (Beard et al., 1998), (Song and Dyke, 2011), (Kappen, 2005), (Chen et al., 2007). In this paper, neural network approximate method is applied to solve the HJB equation. A polynomial series function is utilized to approximate the value function using the method of least squares on a pre-defined region. The value functions \( V^k, \bar{V}_1, \bar{V}_2 \) in (19), (20), (21) can be approximated as

\( V^k(t,x) = \bar{V}_k(t,x) = \bar{V}(t,x) = \bar{V}(t,x) = \sum_{i=1}^{L} w_i(t) \eta_i(x) \) on \( t \) over a compact set \( \Omega \rightarrow \mathbb{R}^n \). Using the approximated value functions \( V^k(t,x) \) in the HJB equations result in residual error equations. Then weighted residual method (Finlayson, 1972) is applied to minimize the residual error equations and then numerically solve for the least square \( w_i(t) \) weights. See (Chen et al., 2007) for details.

## 6 SIMULATION RESULTS

Consider a 3-player nonlinear stochastic system with full-state feedback information. The stochastic system is represented as

\[
dx(t) = \left[ 5x(t) + x^3(t) + 3u_1(t) + 2u_2(t) + 1.5v(t) \right] dt + x(t) dw(t),
\]

(30)

with the state variable defined as \( x(t) \). The three players are \( u_1(t), u_2(t), v(t) \), where \( u_1(t), u_2(t) \) are the controls while \( v(t) \) is the external disturbance. The initial state condition is given as \( x(0) = 0.5 \) and \( dw(t) \) in (30) is a Gaussian process with mean \( E[\,dw(t)\,] = 0 \), and covariance \( E[\,dw(t)\,dw(t)'\,] = 0.01 \). The first player cost function \( J_1 \) is

\[
J_1(t_0, x(t), u_1(t)) = \int_{t_0}^{t} \left[ \frac{1}{2} x^2(t) + u_1^2(t) \right] dt + \psi_1(x(t_F)),
\]

(31)
where $\psi_1(x(t_F)) = 0$ is the terminal cost and the second player cost function $J_2$ is

$$J_2(t_0, x(t), u_2(t)) = \int_{t_0}^{t_F} \left[ x^2(t) + u_2^2(t) \right] \, dt + \psi_2(x(t_F)),$$

where $\psi_2(x(t_F)) = 0$ is the terminal cost and the third player cost function $J$ is

$$J(t_0, x(t), u_1(t), u_2(t), v(t)) = \int_{t_0}^{t_F} \left[ \rho^2 + \left( x^2(t) + u_1^2(t) + u_2^2(t) \right) \right] \, dt + \psi(x(t_F)),$$

where $\psi(x(t_F)) = 0$ is the terminal cost. The attenuation level is set at $\rho = 1$. In the simulation, the asymptotic stability region for state was arbitrarily chosen as $-1 \leq x \leq 1$. The final time $t_F$ was 5 seconds and external disturbance was $v(t) = 0.5 \cos(t) \exp(-t)$.

Fig. 1(a) shows the state trajectory for noise influence with variance $\sigma^2 = 0.01$ for the $2^{nd}$ cumulant $H_{2c}$ game control. The state is bounded and converged to value close to the origin. It should be noted from Fig. 1(b), that the Nash equilibrium controls for the two player is solved by selecting $\gamma = \gamma_1^2 = \gamma_2^2$ where the value functions are minimum which in our case were $\gamma = 10$, $\gamma_1 = 10$ and $\gamma_2 = 10$. In addition, we have the design freedom in $\gamma_1^2$ and $\gamma$ values selection to enhance system performance at the chosen attenuation level.

Remark: The second cumulant Nash strategy is found within all admissible first cumulant strategy. A closer look at the state trajectory 1(a) and players trajectory 1(b) show that convergence to the origin is gradual. Additional investigation is required to verify convergence rate at different attenuation levels.

7 CONCLUSION

In this paper, finite-time higher-order control with multiple players was investigated for a nonlinear stochastic system. The second cumulant $H_{2c}$ control problem which is a generalization of higher-order multi-objective control problem was analyzed and the necessary condition for the existence of Nash equilibrium solution was given. A 3-player optimal strategy was derived where a Nash game approach was taken to minimize the different orders of the cost cumulants of the players. A nonlinear example problem was solved to evaluate the theoretical concepts. As a future work, a more practical system example and improved numerical approaches for fast convergence will be explored.

REFERENCES


