Shortest Path Routing in Transportation Networks with Time-dependent Road Speeds

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Abstract: This paper deals with the subject of shortest (in terms of traveling time) path routing in transportation networks, where the speed in the network's roads is a function of the time interval. These networks are encountered in practice when the roads' speed has been measured for several time instants during a large period of time (e.g., an entire year). In this way, time-dependent speed patterns can be derived for the network's roads, that constitute an estimation of the network's future behavior. For shortest path routing in these networks, the traveling time on the network's roads must be calculated according to the time instant of departure. Conventional approaches perform this calculation under the assumption that the road's speed has a constant (possibly distinct) value inside each time interval. In the work presented here, the assumption that the road's speed is linear (possibly distinct) function of time inside each time interval, is considered. Under this assumption, a procedure is proposed that derives the traveling time on the network's roads according to the time instant of departure. It is combined with Dijkstra's algorithm, resulting in a practically applicable algorithm for optimal shortest path routing for the type of networks investigated in this work.

1 INTRODUCTION

The current paper deals with the subject of shortest path routing in time-dependent transportation networks, where the term "shortest path" refers to the path with the minimum traveling time. The time-dependent characteristic of the network is the road's speed, which is a function of the time interval. This is a category of networks that is encountered in practice when the speed on the network's roads has been measured for certain time instants (either directly measured, or derived from vehicle density measurements (May, 1990)), during a large time interval (e.g., an entire year). In this way, time-dependent speed patterns can be derived for the network's roads, that constitute an estimation of the network's future behavior.

For shortest path routing in these networks, the traveling time on a network's road depends on the time instant of departure from the origin of the road, and must be calculated during the derivation of the shortest path. Conventional approaches perform this calculation under the assumption that the road's speed has a constant (possibly distinct) value inside each time interval. However, this assumption leads to a discontinuous function of speed over time, and it is not in accordance with practical scenarios, as described in more detail in Section 2. In the work presented here, the assumption that the road's speed is a linear (possibly distinct) function of time inside each time interval, is considered. Such a function overcomes the aforementioned drawbacks of the conventional approaches. The main contribution of the current paper is a procedure that derives the traveling time on a network's road according to the time instant of departure from the origin of the road, under the aforementioned linearity assumption. The proposed procedure is combined with Dijkstra's algorithm, resulting to a practically applicable algorithm for optimal shortest path routing for the type of networks under investigation.

The proposed work can be easily utilized in navigation systems for the derivation of the fastest routes, considering that the network will demonstrate the same behavior in the future, as the one estimated by the speed measurements. Note that the proposed work refers to *optimal* routing, for the networks under investigation; therefore, numerical examples are not presented.

The outline of the remaining paper is as follows. In Section 2 the assumptions for this work are pre-

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sented and their validity examined under realistic scenarios. The modeling of the networks under investigation is also presented, as well as the relevant existing approach for optimal shortest path routing in these types of networks. Further, the drawbacks of this approach are described; these constitute the motivation for the work proposed in the paper. Section 3 describes alternative existing approaches to model a time-dependent network, and it is described why these approaches are not suitable for the networks investigated in this paper. The *contribution* of the paper can be found in Sections 4 and 5. Specifically, in Section 4, the case of the road's speed being a linear function of time inside each time interval is investigated. A procedure is proposed that derives the time needed to traverse a network's road according to the time instant of departure, under the aforementioned assumption. This procedure is then combined with Dijkstra's algorithm for optimal shortest path routing in the investigated networks. In addition, in Section 5, it is proven that the model utilized in this paper satisfies the First-In-First-Out (FIFO) property for the general case of the road's speed being an arbitrary function of time inside each time interval. Finally, in Section 6 the conclusions for this work are presented, as well as ongoing research.

2 UTILIZED MODEL

Throughout the paper it is considered that the speed in the network's roads has been measured for certain time instants (either directly measured or derived from vehicle density measurements (May, 1990)), during a large period of time (e.g., an entire year). In this way, time-dependent speed patterns can be derived for the network's roads, that constitute an estimation of the network's future behavior. More precisely, it is assumed that the network in the future demonstrates the same behavior (except for unpredictable events) as the one observed during the period of time where the roads' speed has been measured.

The network is modeled as a directed graph consisting of a set of N ($n = |N|$) nodes (i.e., road junctions) and a set of A ($m = |A|$) arcs (i.e., roads). The arc originating from node x and ending at node y is denoted by $< xy >$ and its length (i.e., actual length of the corresponding road) by d^{xy}. The speed in arc $< xy >$ is denoted by v^{xy}. A node y is considered to be adjacent to node x if arc $< xy >$ exists in the graph. The cost c^{xy} (or "traversal time" or "traveling time") of arc $< xy >$ is defined as the time needed to traverse it, i.e., to move from node x to node y. Consequently, the term "shortest path" from node a to node b refers to the minimum traveling time path from a to b.

For the case of time-independent networks, the cost of an arc $< xy >$ is constant over time, equal to d^{xy}/v^{xy}. For time-dependent networks, it is a function of the time instant of departure, τ, from node x, and it is denoted by $c^{xy}(\tau)$. Therefore, for shortest path routing in time-dependent networks an additional calculation must be performed, compared to time-independent networks; that is, the calculation of $c^{xy}(\tau)$ according to the time instant of departure from node x. At this point, it must be stated that the time instant of departure τ is permitted to have any real value, rather than being a discrete-value variable. This is in accordance with practical scenarios.

Consider that the speed in a certain arc (i.e., road) is measured at several time instants, τ_k, with $k \in \mathcal{N}$ and $0 \leq k \leq K$; the notation $v^{xy}(\tau_k) = v_k^{xy}$ stands for the measurement of speed at time instant τ_k. Then, for arc $< xy >$ the time horizon is partitioned into (in general not equal) non-overlapped time intervals, where the partitioning points are the time instants where the speed has been measured. The $(k+1)^{th}$ time interval is denoted by $[\tau_k^{xy}, \tau_{k+1}^{xy})$. According to the aforementioned notation, the number of time intervals is equal to $K+1$.

Let the set of speeds v_k^{xy} and time intervals $[\tau_k^{xy}, \tau_{k+1}^{xy})$, $0 \leq k \leq K$, for arc $< xy >$ be denoted by V^{xy} and T^{xy} respectively. Consider also that

$$V = \bigcup_{\forall < xy > \in A} V^{xy} \tag{1}$$

$$T = \bigcup_{\forall < xy > \in A} T^{xy} \tag{2}$$

Then, the network graph is denoted by $G = (N, A, T, V)$.

This modeling of the time-dependent network was proposed in (Sung et al., 2000), and it is called *Flow Speed Model (FSM)*. One aspect of this modeling of a time-dependent network that is not described yet, is the value of an arc's speed for time instants where the speed has not been measured. In (Sung et al., 2000), it is assumed that the speed is *constant* inside each time interval, equal to the value measured at the left bound of the time interval. More precisely, as described previously, the arbitrary time interval of arc $< xy >$ (denoted by $[\tau_k^{xy}, \tau_{k+1}^{xy})$) was determined based on the fact that the speed has been measured at time instants τ_k^{xy} and τ_{k+1}^{xy}. Therefore, in (Sung et al., 2000) it is considered that the speed in arc $< xy >$ is equal to v_k^{xy} for all time instants τ such that $\tau_k^{xy} \leq \tau < \tau_{k+1}^{xy}$. In other words, it is considered that for time instant $\tau = \tau_{k+1}^{xy} - \varepsilon$, where ε is a positive, arbitrarily small constant, the speed in $< xy >$ is equal to v_k^{xy} (i.e., equal to the one measured at time instant τ_k^{xy}), whereas for time instant $\tau = \tau_{k+1}^{xy}$ the speed in $< xy >$ is equal to v_{k+1}^{xy} (i.e., equal to the one measured at time instant τ_{k+1}^{xy}). This assumption has the following potential drawbacks:

i) It leads to a discontinuous function of speed over time, since

$$\lim_{\varepsilon \to 0} \{v^{xy}(\tau_{k+1}^{xy} - \varepsilon)\} \neq v^{xy}(\tau_{k+1}^{xy}) \qquad (3)$$

(Note that in equation 3 the expression $v^{xy}(\tau_{k+1}^{xy} - \varepsilon)$, as stated previously, refers to the speed at time instant $\tau_{k+1}^{xy} - \varepsilon$, i.e., it does not denote the multiplication of v^{xy} with the latter.)

ii) It is not in accordance with practical scenarios, since for time instants τ very close to τ_{k+1}^{xy}, this assumption still gives a speed value equal to v_k^{xy}, rather than a value close to v_{k+1}^{xy}.

The work presented in the current paper is based on the assumption that the speed *inside* a time interval must be considered as a *linear function of time*, where this function must lead to speed values equal to v_k^{xy}, v_{k+1}^{xy} for time instants τ_k^{xy}, τ_{k+1}^{xy} respectively. Such a function would overcome the aforementioned drawbacks of the conventional approach. This is exactly the contribution of the current paper, presented in detail in Section 4.

Another characteristic of the FSM that must be stated, is that it always satisfies the First-In-First-Out (FIFO) property, as proven in (Sung et al., 2000). In simple words, the FIFO property has the following meaning: Consider a vehicle A that can depart from node x and traverse arc $< xy >$, having two possible time instants of departure, τ_1 and τ_2, with $\tau_1 < \tau_2$, and let τ_1' and τ_2' be the corresponding time instants of arrival at node y. Then, if the FIFO property is valid, $\tau_1' < \tau_2'$ (the traveling time $\tau_2' - \tau_2$, though, may be smaller than traveling time $\tau_1' - \tau_1$). The reader should note that this property is necessary for a model in order for it to objectively represent real networks, since the scenario of departing later and arriving earlier deviates from reality. Section 5 addresses precisely this point, proving that the model utilized in this paper satisfies the First-In-First-Out (FIFO) property for the general case of the road's speed being an arbitrary function of time inside each time interval.

As stated previously, for shortest path routing in time-dependent networks an additional calculation must be performed compared to time-independent networks, that is the calculation of $c^{xy}(\tau)$, i.e., the traversal time on arc $< xy >$ according to the time instant of departure, τ, from node x. In the FSM, this is performed as described below.

2.1 *ATT* Procedure

In the FSM, the traversal time $c^{xy}(\tau)$ on arc $< xy >$, if τ is the time instant of departure from x, is derived using the following procedure, called *Arc Traversal Time* Procedure ($ATT^{xy}(\tau)$, or simply ATT). This procedure has been proposed in (Sung et al., 2000) (in this paper it is presented in an equivalent, slightly different way). The reader should note that in the steps of the ATT, as well as in the succeeding illustrative example, for simplicity index $*^{xy}$ is omitted from the relevant variables.

Steps of *ATT* Procedure

1. Locate index k such that $\tau_k \leq \tau < \tau_{k+1}$;
2. If $(v_k \cdot (\tau_{k+1} - \tau) \geq d)$
   $c(\tau) = \frac{d}{v_k}$;
   Else
   {
3. (a) i. $a = d - v_k \cdot (\tau_{k+1} - \tau)$;
   ii. $k^* = k + 1$;
   (b) While $(v_{k^*} \cdot (\tau_{k^*+1} - \tau_{k^*}) < a)$
   {
   i. $a \leftarrow a - v_{k^*} \cdot (\tau_{k^*+1} - \tau_{k^*})$;
   ii. $k^* \leftarrow k^* + 1$;
   }
4. $c(\tau) = (\tau_{k^*} - \tau) + \frac{a}{v_{k^*}}$;
   }

The necessity of the aforementioned procedure lies on the fact that a single time interval may not be enough for the traversal of the whole arc. This occurs when the distance that can be traversed from the time instant of departure till the end of the corresponding time interval is less than the length of the arc.

Example of *ATT*

The example of Figure 1 illustrates the operation of the ATT procedure. Here, the length d of arc $< xy >$ is equal to $170m$ and the departure time τ from x is equal to $6s$. The first five time intervals are $[\tau_0, \tau_1) = [0, 10)$, $[\tau_1, \tau_2) = [10, 15)$, $[\tau_2, \tau_3) = [15, 30)$, $[\tau_3, \tau_4) = [30, 40)$, $[\tau_4, \tau_5) = [40, 50)$. These time intervals, along with their corresponding speeds, are shown in Figure 1.

For this example, the ATT procedure performs as follows:

1. The departure time $\tau = 6s$ lies in time interval $[\tau_0, \tau_1) = [0, 10) \Rightarrow k = 0$

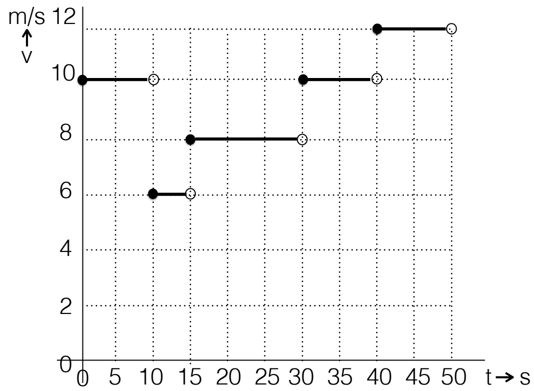

Figure 1: Example for the derivation of arc traversal time ($d = 170m$, $\tau = 6s$).

2. The distance that can be traversed from $\tau = 6s$ until the end of this time interval is equal to $v_0 \cdot (\tau_1 - \tau) = 10 \cdot (10 - 6) = 40m$. Since it is smaller than the length of the arc, the procedure continues.

3.(a)i. $a = d - v_0 \cdot (\tau_1 - \tau) = 170 - 40 = 130m$
   ii. $k^* = 1$
  (b)• $v_1 \cdot (\tau_2 - \tau_1) = 6 \cdot (15 - 10) = 30m < a = 130m \Rightarrow$
   i. $a = 130 - 30 = 100m$
   ii. $k^* = 2$
  • $v_2 \cdot (\tau_3 - \tau_2) = 8 \cdot (30 - 15) = 120m > a = 100m \Rightarrow$ exit from the while loop

4. $c(6) = (\tau_2 - \tau) + a/v_2 = (15 - 6) + 100/8 = 9 + 12.5 = 21.5s$

The traversal time is equal to $21.5s$ and the time instant of arrival at node y is equal to $6 + 21.5 = 27.5s$.

Complexity of ATT

Step 1 of the ATT procedure needs $O(K)$ time, if the time intervals are checked sequentially. The order of the number of the time intervals that are checked during the *while* loop (step 3b) is $O(K)$. Therefore, the order of the computational complexity of the ATT procedure is $O(K)$.

In (Sung et al., 2000), the ATT procedure is combined with Dijkstra's algorithm (Dijkstra, 1959), and the resulting algorithm can provide optimal shortest path routing for the networks under investigation. For completeness, this algorithm is presented in the succeeding section. Hereafter, it is called *Time-Dependent-Dijkstra (TD-Dijkstra)*.

2.2 TD-Dijkstra

The input of TD-Dijkstra is the network graph $G = (N, A, T, V)$, the source node s, and the time instant of departure τ, from the source. The output is the shortest path from the source to every other network node. For the execution of TD-Dijkstra, the following are used.

- $G = (N, A, T, V)$: Network graph
- s: Source
- τ: Time instant of departure from s
- $W(x)$: Label of node x
- $p(x)$: Predecessor of node x
- G_x: Set of nodes adjacent to node x
- g_x: Number of nodes adjacent to node x (i.e., $g_x = |G_x|$)

The exact steps of TD-Dijkstra are:

1.(a) $W(s) = \tau$;
  (b) $p(s) = 0$;
  (c) $N^* = N - \{s\}$;
  (d) $\forall x \in N^*$:
    i. If $(x \in G_s)$
      {
      Run $ATT^{sx}(\tau)$;
      $W(x) = \tau + c^{sx}(\tau)$;
      $p(x) = s$;
      }
    ii. Else
      {
      $W(x) = \infty$;
      $p(x) = 0$;
      }

2. While $(N^* \neq \emptyset)$
  {
  (a) Find $x \in N^*$ such that $\forall x' \in N^*$: $W(x) \leq W(x')$;
  (b) $N^* \leftarrow N^* - \{x\}$;
  (c) $\forall x' \in (N^* \cap G_x)$:
    i. Run $ATT^{xx'}(W(x))$;
    ii. If $(W(x) + c^{xx'}(W(x)) < W(x'))$
      {
      $W(x') = W(x) + c^{xx'}(W(x))$;
      $p(x') = x$;
      }
  }

The TD-Dijkstra algorithm functions as the classical Dijkstra's algorithm, with the difference that the ATT procedure is used in steps 1(d)i and 2(c)i for the derivation of the cost of the arbitrary arc $< xy >$, according to the time instant of departure from node x.

On termination of the algorithm, the label $W(x)$ of the arbitrary node x gives the time instant of arrival at x. Therefore, the cost of the shortest path from the source s to node x is equal to $W(x) - \tau$. The variable $p(x)$ gives the predecessor of x in this path.

Since the computational complexity of the ATT procedure is of order $O(K)$, step 1 requires $O(g_sK + n)$ time and each iteration of step 2 requires $O(n + g_xK)$ time. Since the number of repetitions of step 2 is of order $O(n)$, and $\sum_{x=1}^{n} g_x = m$, the complexity of TD-Dijkstra is of order $O(g_sK + n + n^2 + mK) = O(n^2 + mK)$. The reader should note that although the aforementioned, simple implementation is presented in (Sung et al., 2000) where this algorithm was proposed, a faster implementation can be performed if Fibonacci heaps are utilized (Fredman and Tarjan, 1987), (Ahuja et al., 1993). Under this data structure, the computational complexity of TD-Dijkstra would be of order $O(n \log n + mK)$.

In the section that follows, existing alternative approaches to model a time-dependent transportation network are presented, and it is explained why they are unsuitable for optimal shortest path routing in the networks under investigation.

3 ALTERNATIVE APPROACHES

Alternative existing approaches to model a time-dependent network can be found, among others, in (Cooke and Halsey, 1966), (Delling, 2011), (Nannicini et al., 2012), (Delling and Nannicini, 2012), (Delling et al., 2009), (Delling and Wagner, 2009), (Ding et al., 2008), (Batz et al., 2013), (Chabini and Lan, 2002), and (Chabini, 1998). All the aforementioned papers assume that for each network's arc $< xy >$, the traversal time is available for certain time instants of departure from node x. Let this arc traversal time, which is a function of the time instant of departure, be called as *traversal time function* and be denoted by $f^{xy}(\tau)$, for the arbitrary arc $< xy >$ and for τ as the time instant of departure. In the aforementioned papers, the traversal time function is assumed to be known for some values of τ and these values are utilized for the derivation of the arc traversal time for an arbitrary time instant of departure.

However, as stated previously, in the networks investigated in the current paper, it is (realistically) assumed that the time instant of departure may have any arbitrary value. Therefore, the work presented in (Chabini and Lan, 2002), (Chabini, 1998), which is concentrated on discrete-time networks, cannot lead to optimal shortest path routing in the investigated networks. Furthermore, in the networks under investigation, the traversal time function is not directly available, since, as described previously, the speed rather than the arc traversal time, has been measured. Consequently, the work found in the rest of the aforementioned papers (which are concentrated on continuous-time networks) cannot be directly applied for optimal shortest path routing in the networks under investigation.

Consider the case where the traversal time function is derived from the available data (i.e., the speed measurements) for certain time instants (e.g., for the time instants that the speed has been measured). This would be a preprocessing step, performed using the ATT procedure. Even under this scenario, the work found in these papers would not lead to optimal shortest path routing in the investigated networks. The reason is that in all these works, it is assumed that the traversal time function is a *piecewise linear function of time*, having the time instants where it is known, as breakpoints. Under this assumption, for a time instant τ for which $f^{xy}(\tau)$ is unknown, it can be derived by linear interpolation between the consecutive breakpoints τ_k, τ_{k+1} such that $\tau_k < \tau < \tau_{k+1}$, using the known values $f^{xy}(\tau_k)$ and $f^{xy}(\tau_{k+1})$, as shown in equation 5 below.

$$\frac{f^{xy}(\tau) - f^{xy}(\tau_k)}{\tau - \tau_k} = \frac{f^{xy}(\tau_{k+1}) - f^{xy}(\tau_k)}{\tau_{k+1} - \tau_k} \tag{4}$$

$$\Rightarrow \quad f^{xy}(\tau) = \frac{f^{xy}(\tau_{k+1}) - f^{xy}(\tau_k)}{\tau_{k+1} - \tau_k} \cdot (\tau - \tau_k) + f^{xy}(\tau_k) \tag{5}$$

In the networks investigated in the current paper, though, the assumption that the traversal time function is a piecewise linear function of time, *is, in general, not valid*. This can be proven theoretically; it is not presented here, since it deviates from the purpose of the current paper. Nevertheless, this can be easily verified from the example of Figure 1. Here, if the arc traversal time is derived for the time instants that the speed has been measured (this must be performed using the ATT procedure), then the derived traversal time function $f^{xy}(\tau)$ for time instants equal to $0s$, $10s$ is equal to $20s$, $22s$, respectively. Using equation 5, $f^{xy}(6) = 21.2s$, i.e., the assumption that $f^{xy}(\tau)$ is piecewise linear function of time, leads to arc traversal time equal to $21,2s$, for $\tau = 6s$ as the time instant of departure. However, the correct value is equal to $21.5s$, as derived in Section 2. Therefore, using this example, it is shown that this assumption, in general, is not valid.

Thus, according to the aforementioned analysis, the alternative approaches for modeling a time-dependent network cannot be applied for optimal shortest path routing in the networks under investigation, since for these networks they give suboptimal solutions. Further analysis of the advantages of the

FSM model, compared to other approaches, can also be found in (Sung et al., 2000), where this model was proposed.

In the section that follows, the proposed model is presented, along with the corresponding procedure that derives the traveling time on the network's roads according to the time instant of departure.

4 SPEED AS LINEAR FUNCTION OF TIME, INSIDE THE TIME INTERVAL

As stated previously, the conventional approach (Sung et al., 2000) assumes that the speed is considered to have *constant value* inside each time interval. In this section, the case of speed being a *linear function of time* inside each time interval, is proposed. The reader should note that the linearity here refers to the function of speed over time, inside each time interval; it must not be confused with the linearity of the traversal time function that was described in Section 3.

The proposed approach overcomes the drawbacks of the conventional one, since it leads to a continuous function of speed over time, i.e.,

$$\lim_{\varepsilon\to 0}\{v^{xy}(\tau^{xy}_{k+1}-\varepsilon)\} = v^{xy}(\tau^{xy}_{k+1}) \quad (6)$$

It is in accordance with practical scenarios as well, since for time instants τ very close to τ^{xy}_{k+1}, this approach leads to a speed value close to v^{xy}_{k+1}.

Let this linear function of speed be denoted by $g^{xy}_k(t)$ for arc $< xy >$ and for time instant of departure t belonging to time interval $[\tau_k, \tau_{k+1})$, and let $G^{xy}_k(t_1,t_2) = \int_{t_1}^{t_2} g^{xy}_k(t)dt$ (with $\tau_k \le t_1 \le t_2 \le \tau_{k+1}$). The latter gives the value of the distance traversed on arc $< xy >$ from time instant t_1 to t_2 (as long as t_1 and t_2 belong to the same time interval).

Consider that the speed has been measured at time instants τ_k and τ_{k+1} with measured values equal to v^{xy}_k and v^{xy}_{k+1} respectively. Then, function $g^{xy}_k(t)$ is considered to be a linear function of time t inside the time interval $[\tau_k, \tau_{k+1})$, taking the values $g^{xy}_k(\tau_k) = v^{xy}_k$ and $g^{xy}_k(\tau_{k+1}) = v^{xy}_{k+1}$ at the bounds of this time interval. As stated previously, in the conventional approach it is assumed that this function is constant, equal to v^{xy}_k for the entire time interval $[\tau_k, \tau_{k+1})$.

Since function $g^{xy}_k(t)$ is considered to be linear inside each time interval, the following hold ($\tau_k \le t < \tau_{k+1}$, and index $*^{xy}$ is omitted for simplicity):

$$\frac{g_k(t)-v_k}{t-\tau_k} = \frac{v_{k+1}-v_k}{\tau_{k+1}-\tau_k} \quad (7)$$

$$g_k(t)-v_k = \frac{v_{k+1}-v_k}{\tau_{k+1}-\tau_k}\cdot(t-\tau_k) \quad (8)$$

$$g_k(t) = \frac{v_{k+1}-v_k}{\tau_{k+1}-\tau_k}t + \left(v_k - \frac{v_{k+1}-v_k}{\tau_{k+1}-\tau_k}\tau_k\right) \quad (9)$$

$$g_k(t) = \frac{v_{k+1}-v_k}{\tau_{k+1}-\tau_k}t + \frac{v_k\tau_{k+1}-v_{k+1}\tau_k}{\tau_{k+1}-\tau_k} \quad (10)$$

$$G_k(t_1,t_2) = \left(\frac{v_{k+1}-v_k}{\tau_{k+1}-\tau_k}\frac{t_2{}^2}{2} + \frac{v_k\tau_{k+1}-v_{k+1}\tau_k}{\tau_{k+1}-\tau_k}t_2\right) - \\ -\left(\frac{v_{k+1}-v_k}{\tau_{k+1}-\tau_k}\frac{t_1{}^2}{2} + \frac{v_k\tau_{k+1}-v_{k+1}\tau_k}{\tau_{k+1}-\tau_k}t_1\right) \quad (11)$$

$$G_k(t_1,t_2) = \\ \frac{v_{k+1}-v_k}{\tau_{k+1}-\tau_k}\frac{t_2{}^2-t_1{}^2}{2} + \frac{v_k\tau_{k+1}-v_{k+1}\tau_k}{\tau_{k+1}-\tau_k}(t_2-t_1) \quad (12)$$

$$G_k(t_1,t_2) = R_k\frac{t_2{}^2-t_1{}^2}{2} + S_k(t_2-t_1) \quad (13)$$

where equation 13 is derived if we set:

$$\frac{v_{k+1}-v_k}{\tau_{k+1}-\tau_k} = R_k \quad (14)$$

$$\frac{v_k\tau_{k+1}-v_{k+1}\tau_k}{\tau_{k+1}-\tau_k} = S_k \quad (15)$$

Let the ATT procedure under the linearity assumption be denoted as *Linear ATT (ATT_L)*. Then, the proposed ATT_L is derived from the existing ATT as follows.

- Step 1 remains the same as in ATT.
- In step 2 of the ATT, $v_k\cdot(\tau_{k+1}-\tau)$ is equal to the distance traversed from τ to τ_{k+1}. This distance in ATT_L is given by

$$G_k(\tau,\tau_{k+1}) = R_k\frac{\tau_{k+1}{}^2-\tau^2}{2} + S_k(\tau_{k+1}-\tau) \quad (16)$$

  - If this distance is equal to or larger than d (i.e., if $G_k(\tau,\tau_{k+1}) \ge d$), then the traversal time $c(\tau)$ is given by the solution of equation 17, as follows.

$$G_k(\tau,\tau+c(\tau)) = d \quad (17)$$

$$R_k\frac{(\tau+c(\tau))^2-\tau^2}{2} + S_k(\tau+c(\tau)-\tau) = d \quad (18)$$

$$R_k((\tau+c(\tau))^2-\tau^2) + 2S_k(\tau+c(\tau)-\tau) = \\ = 2d \quad (19)$$

$$R_k(c(\tau)^2+2\tau c(\tau)) + 2S_k c(\tau) = 2d \quad (20)$$

$$R_k c(\tau)^2 + 2(R_k\tau+S_k)c(\tau) - 2d = 0 \quad (21)$$

$$c(\tau) = \frac{-2(R_k\tau+S_k)\pm\sqrt{4(R_k\tau+S_k)^2+8R_kd}}{2R_k} \quad (22)$$

$$c(\tau) = \frac{-(R_k\tau+S_k)\pm\sqrt{(R_k\tau+S_k)^2+2R_kd}}{R_k} \quad (23)$$

  - If this distance is less than d (i.e., if $G_k(\tau,\tau_{k+1}) < d$), the procedure continues to step 3.

- In step 3(a)i, a is set to $d - G_k(\tau, \tau_{k+1})$. Step3(a)ii remains the same.
- In step 3(b), the statement in the *while* loop would be $G_{k^*}(\tau_{k^*}, \tau_{k^*+1}) < a$; every time this is valid, a takes the value $a - G_{k^*}(\tau_{k^*}, \tau_{k^*+1})$, and $k^* \leftarrow k^* + 1$.
- When the procedure exits from the *while* loop, the value of the arrival time, τ', will lie in time interval $[\tau_{k^*}, \tau_{k^*+1})$. let $c'(\tau) = \tau' - \tau_{k^*}$; this is the time needed to traverse the last part of the arc, i.e., while time interval $[\tau_{k^*}, \tau_{k^*+1})$ is under consideration. Therefore, the arc traversal time will be equal to $(\tau_{k^*} - \tau) + c'(\tau)$. The value of $c'(\tau)$ is derived as follows.

$$G_{k^*}(\tau_{k^*}, \tau') = a \tag{24}$$
$$G_{k^*}(\tau_{k^*}, \tau_{k^*} + c'(\tau)) = a \tag{25}$$
$$R_{k^*}\frac{(\tau_{k^*}+c'(\tau))^2 - \tau_{k^*}{}^2}{2} + +S_k(\tau_{k^*} + c'(\tau) - \tau_{k^*}) = a \tag{26}$$
$$R_{k^*}((\tau_{k^*} + c'(\tau_{k^*}))^2 - \tau_{k^*}{}^2) + +2S_{k^*}(\tau_{k^*} + c'(\tau) - \tau_{k^*}) = 2a \tag{27}$$
$$R_{k^*}(c'(\tau)^2 + 2\tau_{k^*}c'(\tau)) + 2S_{k^*}c'(\tau) = 2a \tag{28}$$
$$R_{k^*}c'(\tau)^2 + 2(R_{k^*}\tau_{k^*} + S_{k^*})c'(\tau) - 2a = 0 \tag{29}$$
$$c'(\tau) = = \frac{-2(R_{k^*}\tau_{k^*}+S_{k^*}) \pm \sqrt{4(R_{k^*}\tau_{k^*}+S_{k^*})^2 + 8R_{k^*}a}}{2R_{k^*}} \tag{30}$$
$$c'(\tau) = = \frac{-(R_{k^*}\tau_{k^*}+S_{k^*}) \pm \sqrt{(R_{k^*}\tau_{k^*}+S_{k^*})^2 + 2R_{k^*}a}}{R_{k^*}} \tag{31}$$

Equation 31 is used in step 4 of the ATT_L, for the derivation of the arc traversal time after the procedure exits from the *while* loop of step 3, i.e.,

$$c(\tau) = \tau' - \tau \tag{32}$$
$$c(\tau) = (\tau_{k^*} + c'(\tau)) - \tau \tag{33}$$
$$c(\tau) = (\tau_{k^*} - \tau) + + \frac{-(R_{k^*}\tau_{k^*}+S_{k^*}) \pm \sqrt{(R_{k^*}\tau_{k^*}+S_{k^*})^2 + 2R_{k^*}a}}{R_{k^*}} \tag{34}$$

Considering the analysis above, the exact steps of the proposed ATT_L procedure are as follows.

Steps of ATT_L Procedure

1. Locate index k such that $\tau_k \leq \tau < \tau_{k+1}$;
2. If $(G_k(\tau, \tau_{k+1}) \geq d)$

$$c(\tau) = \frac{-(R_k\tau+S_k) \pm \sqrt{(R_k\tau+S_k)^2 + 2R_k d}}{R_k};$$

Else

{

3.(a)i. $a = d - G_k(\tau, \tau_{k+1})$;

ii. $k^* = k + 1$;

(b) While $(G_{k^*}(\tau_{k^*}, \tau_{k^*+1}) < a)$

{

i. $a \leftarrow a - G_{k^*}(\tau_{k^*}, \tau_{k^*+1})$;

ii. $k^* \leftarrow k^* + 1$;

}

4.

$$c(\tau) = (\tau_{k^*} - \tau) + + \frac{-(R_{k^*}\tau_{k^*}+S_{k^*}) \pm \sqrt{(R_{k^*}\tau_{k^*}+S_{k^*})^2 + 2R_{k^*}a}}{R_{k^*}};$$

}

The computational complexity of the proposed ATT_L procedure is $O(K)$ (i.e, the same as the one of ATT). The optimal shortest path routing for this case can be performed by TD-Dijkstra, (Section 2.2) just by substituting the ATT procedure with the ATT_L procedure in the description of the algorithm.

In Section 3 it was shown that the alternative approaches give suboptimal solutions for the networks investigated in the current paper, under the assumption that the speed has constant value inside each time interval. These approaches also give suboptimal solutions for the investigated networks, under the assumption that the speed is linear function of time inside each time interval. This can be verified using the example of Figure 1, modified under the linearity assumption.

5 FSM AND FIFO PROPERTY FOR THE GENERALISED CASE

In (Sung et al., 2000), it was proven that the FSM satisfies the FIFO property for the case of constant speed inside each time interval. In this section, it is proven that the FIFO property also holds for the general case of the speed being an arbitrary function of time ($g_k(\tau)$) inside the time interval. A direct consequence of it is that the FSM satisfies the FIFO property under the linearity assumption utilized in Section 4.

Consider that for an arbitrary arc $< xy >$, two vehicles 1, 2 depart from node x at time instants τ_1, τ_2 respectively, with $\tau_2 > \tau_1$, and arrive at node y at time instants τ_1', τ_2' respectively. Obviously, $\tau_1' > \tau_1$ and $\tau_2' > \tau_2$.

To prove that the FIFO property is valid, it must be proven that $\tau_2' > \tau_1'$. The following cases are possible:

- $\tau_2 \geq \tau'_1 \xrightarrow{\tau'_2 > \tau_2} \tau'_2 > \tau'_1$
- $\tau_1 < \tau_2 < \tau'_1$. The distance d is traversed from time instant τ_1 to τ'_1, and it can be split into distances a and b ($a+b=d$), where a is the distance traversed from τ_1 to τ_2 and b is the distance traversed from τ_2 to τ'_1. Then,

$$\tau_2 > \tau_1 \Rightarrow a > 0 \Rightarrow b < d \qquad (35)$$

From equation 35, it is concluded that the time that elapses from τ_2 to τ'_1 is not enough to traverse the whole arc $< xy >$. Therefore, vehicle 2 that departs from node x at time instant τ_2 will arrive at node y at $\tau'_2 > \tau'_1$.

6 CONCLUSIONS

This work focused on the subject of shortest (in terms of traveling time) path routing in transportation networks, where the speed in the network's roads is a function of the time interval. A procedure was proposed that derives the road's traversal time in these networks, according to the time instant of departure, under the assumption that the road's speed is a linear (possibly distinct) function of time inside each time interval. This procedure can then be combined with Dijkstra's algorithm to obtain optimal shortest paths for the networks under investigation. Further, it was also proven that the approach utilized in this work for modeling the investigated networks satisfies the FIFO property for the general case of the speed being an arbitrary function of time inside each time interval. The proposed approach is more practical and readily addresses the limitations of the conventional existing approaches that assume that the speed is constant inside each time interval.

On going work focuses on the combination of the proposed procedure with algorithms that provide practically fast optimal shortest path routing in time-independent networks, so as to develop their time-dependent versions that will lead to optimal solutions for the networks under investigation.

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