A Multi-period Vertex Cover Problem and Application to Fuel Management

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Keywords: Multi-period Vertex Cover, Wildfire, Fuel Management, Planar Graphs, Polynomial Approximation, Approximation Preserving Reductions.

Abstract: We consider a generalisation of MIN WEIGHTED VERTEX COVER motivated by a problem in wildfire prevention. The problem is defined for a fixed number of time periods and we have to choose, at each period, some vertices to be deleted such that we never have two adjacent remaining vertices. The specificity is that whenever a vertex is deleted it reappears after a given number of periods. Consequently we may need to delete a single vertex several times. The objective is to minimise the total weight (cost) of deleted vertices. The considered application motivates the case of planar graphs. While similar problems have been mainly solved using mixed integer linear models (MIP) we investigate a graph approach that allows to take into account the structure of the underlying graph. We use a reduction to the usual MIN WEIGHTED VERTEX COVER to devise efficient approximation algorithms and to raise some polynomial classes.

1 INTRODUCTION

Fuel management problems is one of the main techniques used for reducing the impact of wildfires (Boer et al., 2015). The main idea consists in dividing a landscape into a number of cells representing candidate locations for fuel treatment like harvesting (EU, North America) or burning (USA, Australia). Treatments are generally applied on multiple periods (several years) and key decisions are to determine which cells should be treated during each time period (i.e., each year).

Spatially explicit multi-period fuel treatment scheduling is a very complex problem (Chung, 2015). A particularity to take into account is the transient effect since vegetation begins to re-grow after it has been treated. The risk of fire spread from one cell to another one becomes important if the ages of the vegetation in the two adjacent cells achieve a specific threshold and does not significantly increase for older vegetation (Boer et al., 2015).

Most of the known optimisation solutions for this problem involve a Mixed Integer Linear Programming approach (MIP) for locating fuel treatments (see, e.g., (Hof et al., 2000; Kim et al., 2009; Minas et al., 2014; Rachmawati et al., 2015; Wei et al., 2008)) with various objectives. For further details on this area, the reader is referred to (Ager et al., 2010; Chung, 2015; Minas et al., 2012).

Due to computational constraints, many of the modelling efforts have considered very small landscapes in a single period problem (see e.g., (Hof et al., 2000; Wei et al., 2008)). Multi-year spatial fuel treatment planning has been considered in (Kim et al., 2009; Minas et al., 2014). In the latter paper, the underlying adjacency graph structure is explicitly used in the model and computational results are proposed on small grid-like partition of the map. However, to our knowledge, the adjacency graph of cells has never been used for algorithmic purpose.

MIP has the advantage of flexibility and of easy representation of diverse constraints and objectives as well as the possibility to use efficient solvers. However, as an almost universal model, a well known drawback is that it makes it difficult to take into account the true nature of the problem and the specificity of the instances appearing in applications. In particular, the MIP approaches that have been developed so far for fuel management problems do not use any specific property of the underlying matrix and thus, induce exponential time algorithms that become very quickly intractable (Chung, 2015). As a consequence, their use is limited to small instances and is justified only for problems that are hard, even for the considered classes of instances appearing in practice. Therefore, preliminary theoretical complex-

Demange, M. and Tanasescu, C.
A Multi-period Vertex Cover Problem and Application to Fuel Management.
DOI: 10.5220/0005708900510057
In *Proceedings of 5th the International Conference on Operations Research and Enterprise Systems (ICORES 2016)*, pages 51-57
ISBN: 978-989-758-171-7

ity considerations are important to justify these approaches. The last drawback of the current models is that they make it very hard to detect and use links with other problems studied in other areas of applications. The notion of *approximation preserving reductions* gives a suitable theoretical framework for establishing such links and using them in practice. The reader is referred to (Ausiello et al., 1999) for basic definitions in polynomial approximation theory.

In this work we propose to use a graph model as a complementary approach for some fuel management problems. We consider the example of the model developed in (Minas et al., 2014) and give first results in this direction that motivate such though process and illustrate how it can be used for better understanding the problem, studying its complexity [1], designing efficient algorithms in some specific graph instances and identifying some links with extensively studied problems. As a preliminary work, this paper remains essentially theoretical and considers a simplified version of the practical problem. However, our example illustrates how such though process can raise some new research questions as well as particular kind of solutions that are not naturally produced by a usual MIP model. We strongly believe that such approach is interesting, as a complement of MIP solutions, and we hope it will foster further works in this direction.

2 THE MODEL

In this study we will mainly follow the model proposed in (Minas et al., 2014). We assume that the landscape is already partitioned into zones (called *cells*), each having a specific predominant fuel pattern. Each cell is associated with a discrete time step function representing the age of the fuel that is increased by one each year if the cell is untreated and reset to zero if it is treated; however, up to some threshold the age of the cell is considered as *old* and does not need to be incremented since the vegetation's behaviour is essentially the same up to the old-vegetation threshold. From this threshold the risk of fire spread becomes high and a critical situation occurs when two such cells are adjacent (i.e. share a boundary line). This age function is characterised by two parameters supposed to be known:

1. the initial fuel age;
2. the old-vegetation threshold.

The model is formulated with the following notations:

[1] Thus, eventually motivating some previous MIP approaches.

- $I = \{1,\ldots,n\}$ will denote the set of all cells in the landscape and we will denote each cell with index $i \in I$;
- $V = \{1,\ldots,m\}$ is a finite set of vegetation types and $v_i \in V$, $i = 1,\ldots,n$ is the predominant vegetation type of cell $i \in I$;
- Each vegetation type $v \in V$ is associated with an old-vegetation threshold o_v : the vegetation will be called *old* if its old-vegetation threshold is reached; a cell with old vegetation is called old;
- T is the number of time periods in the planning horizon and $t = 1,\ldots,T$ is the time parameter;
- $a_{i,0}$, $i \in I$ is the initial fuel age of cell i and $a_{i,t}$, $t = 1,\ldots,T+1$ will denote the age of cell i at the beginning of time period t where $t = T+1$ corresponds to the end of the last time period;
- For every cell $i \in I$, $c_i(t, a_{i,t}), t = 1\ldots,T$ is the cost for treating the cell i at time t if its age vegetation is $a_{i,t}$

The risk of fire spread from one cell to another one is represented by a graph $G_I = (I,E)$ with the cells as vertex set and E as edges. Two cells $i, j \in I$ are connected if there is a risk of fire spread from one to the other when both cell's vegetations are old ($a_{i,t} = o_{v_i}$ and $a_{j,t} = o_{v_j}$). In this work we consider G_I as non-directed and we will mainly consider planar G_Is corresponding to the case where the edge set E corresponds to the usual spatial adjacency of cells.

The aim is to decide, for each time period t, which set $C_t \subset I$ to treat. If a cell i is treated during the time period t then its vegetation age is set to 0 at the next period while is is just increased by one - up to the old-vegetation threshold - if it is not selected:

$$\forall t = 1,\ldots,T, \forall i = 1,\ldots,n,$$
$$a_{i,t+1} = \begin{cases} \min(a_{i,t}+1, o_i) & \text{if} \quad i \notin C_t \\ 0 & \text{if} \quad i \in C_t \end{cases}$$

The following combinatorial problem, called *minimum Zero Risk Fuel Treatment Scheduling*($ZFTS$), is a simplified abstraction of the real problem to be solved in practice. It provides a theoretical framework that allows us to understand the combinatorial structure of the problem.

$ZFTS$'s objective is to minimise the total cost over the time period $[1,T]$ in order to never have two cells with old vegetation that are connected in G_I. In the conclusion we will also mention the version where we are given some a budget b_t for each period t and the objective is to minimise over the time period $[1,T]$ the total number of connections between two old vegetation cells such that the total budget used during each time period t does not exceed b_t. This problem, called *minimum Budget constrained Fuel Treatment Scheduling* ($BFTS$) is exactly the one studied in (Minas et al., 2014).

2.1 Some Graph Theory Related Notations

For all graph notions not defined here, the reader is referred to (Diestel, 2012) and for all complexity notions he is referred to (Garey and Johnson, 1979).

All graphs considered in this work are simple undirected graphs. Given a graph $G = (I,E)$, I denotes the set of vertices and E the set of edges. Given $I' \subset I$, we denote by $G[I']$ the subgraph of G induced by I'. A *vertex cover* is a set of vertices such that every edge has at least one extremity in this set. The problem of deciding, for any graph G and integer K whether G contains a vertex cover of size at most K is a well-known NP-complete problem (Garey and Johnson, 1979). MIN VERTEX COVER is the problem of determining in any graph instance G a vertex cover of minimum size; since its decision version is NP-complete in general graphs, it is itself NP-hard. In the weighted version, non negative weight ω_i is associated to vertex $i \in I$ and the weight of a subset $I' \subset I$ is $\omega(I') = \sum_{i \in I'} \omega_i$. MIN WEIGHTED VERTEX COVER is to determine, for any weighted graph a vertex cover of minimum weight.

A graph $G = (V,E)$ is called *planar* if it can be embedded in the two dimension plane without cross edges. In particular, given a set of connected areas in the plane, the adjacency graph with vertex set these areas and two areas linked by an edge if they share a common bounder-line of positive length is planar. A basic example of planar graph is the case of *grid graph*. A grid graph of size $n \times m$ is defined as follows: its vertex set is $\{1,\dots,n\} \times \{1,\dots,m\}$ with edges between (i, j) and (i', j') if and only if $|i-i'|+|j-j'| = 1$. Grid graphs are planar and *bipartite*, which means that their vertex set can be divided into two stable sets (set of vertices pairwise not linked by an edge).

3 A VERTEX COVER APPROACH FOR $ZFTS$

We will say that a treatment cost $c(t,a)$ is *not increasing* if $\forall (t,a) \in \mathbb{N}^2, c(t+1,a+1) \leq c(t,a)$.

Let us first note that, for any instance of $ZFTS$ on a graph G_I, if there is only one single period, then the problem is to find a minimum vertex cover of the graph induced by the vertices associated with old vegetation cells: the vertices in a vertex cover correspond to the cells to be treated. So, we immediately get the following result:

Remark 1. *$ZFTS$ is at least as hard as* MIN VERTEX COVER *and in particular it is NP-hard in planar graphs (Garey and Johnson, 1979).*

This result holds even for the case with only one vegetation type and all costs are equal to one. It motivates us working in two different research directions: either finding polynomial classes of instances or devising good approximations for NP-hard cases.

For a minimisation combinatorial problem, a polynomial-time algorithm (Ausiello et al., 1999) is said to guarantee the approximation ratio ρ if, for each instance H of optimal value β_H it computes a feasible solution of value λ_H such that $\lambda_H \leq \rho\beta_H$. Moreover, a family of polynomial algorithms indexed by $\varepsilon > 0$ and guaranteeing the ratio $1+\varepsilon$ is called a polynomial approximation scheme (PTAS).

Proposition 1. *For any instance of $ZFTS$ with not increasing treatment costs, for every feasible solution S, there is a feasible solution $\widetilde{S}$, with an objective value not greater than S for which every treatment occurs on old vegetation cells.*

Proof. (sketch) We iteratively transform S as follows. Consider the first time t a cell $i \in I$ is treated in S with a vegetation age a_i verifying $a_i < o_i$. If $T-t < o_i - a_i$, then we just withdraw the related treatment, else we delay it until $t+o_i-a_i$. All other treatments are kept unchanged. We denote by S_1 the transformed solution. We can easily verify that S_1 is still feasible (consider the two cases, dates up to $t+o_i-a_i$ and later dates).

We then define a *measure* m of a solution σ as the sum $\sum_{i \in I} d_i$, where d_i is either the first date when i is treated with an age less than its old-vegetation threshold in σ or $T+1$ if such a date does not exist. $m(\sigma)$ is an integer that ranges between 0 and $(T+1)|I|$ and moreover $m(\sigma) = (T+1)|I|$ if and only if all treatments in solution σ occur on old vegetation cells. It is straightforward to verify that if $m(S) < (T+1)|I|$, then $m(S_1) > m(S)$. As a consequence, by repeating the previous process, we iteratively build instances $S_2, S_3, \dots$ until obtaining an instance S_k such that $m(S_k) = (T+1)|I|$. $\widetilde{S} = S_k$ satisfies the required constraints. $\square$

In what follows, we show how to transform any algorithm for MIN WEIGHTED VERTEX COVER into an algorithm for $ZFTS$ with some performance guaranties (approximation preserving reduction). Proposition 2 corresponds to short time periods and multi-vegetation while Proposition 3 deals with the single vegetation case and any time period. Both results lead to polynomial cases. Finally, our main result is Proposition 4 that leads to interesting asymptotical results for long time periods and multi-vegetation.

Proposition 2. *Let $\mathcal{C}$ be a hereditary class of graphs for which* MIN WEIGHTED VERTEX COVER *can be approximated in polynomial time within the ratio* ρ. *The problem ZFTS on the class $\mathcal{C}$ with $T \leq o_v$ for all vegetation types v and with constant treatment costs can be approximated in polynomial time within the same approximation ratio* ρ.

Proof. We consider an instance satisfying the hypotheses and denote by G_I the related graph with cells set I. We then associate each vertex (cell) i with the weight c_i corresponding to the treatment cost.

Using Proposition 1 we can restrict ourselves to feasible solutions for which only cells with old vegetation are treated. Moreover, since $T \leq o_v$, whenever a cell is treated during the period, it will never achieve its old-vegetation threshold and will not be treated again. We call *nice* a solution for which each cell is treated no more than once and every treated cell is treated at the date it achieves its old-vegetation threshold. The above arguments show that there is an optimal solution that is nice.

Let I' denote the set of cells i with an initial age a_i satisfying $a_i \geq o_i - T$, then the set of nice optimal solutions is in one-to-one correspondence with vertex covers of the graph $G_I[I']$ and moreover, the cost of such solutions is exactly the weight of the related vertex cover. Consider indeed an edge xy of $G_I[I']$, since $x, y \in I'$ they both achieve their old-vegetation threshold during the period and consequently one of them needs to be treated at least once. Conversely, for any vertex cover of $G_I[I']$ with weight W, treating each of its vertices when it achieves its old-vegetation threshold constitutes a nice solution with a cost equal to W. This concludes the proof. ☐

In (Baker, 1994), a polynomial time approximation scheme for Min Vertex Cover in planar graphs is devised and the dynamic programming approach also holds for the weighted case. Another interesting particular case is the bipartite case since a landscape with square cells, leading to a subgrid as the fire spread graph, is considered in different papers (see e.g. (Minas et al., 2014)). In this case the weighted vertex cover is known to be polynomially solvable. We deduce the following corollary.

Corollary 1. *The problem ZFTS with $T \leq o_v$ for all vegetation types v and with constant treatment costs has:*

1. *a polynomial time algorithm in bipartite graphs and in particular in sub-grids.*
2. *a polynomial time approximation scheme in planar graphs.*

For real applications however, it is natural to consider long time horizons with, possibly, a re-optimisation with a smaller periodicity. We then propose the following strategy built from a weighted vertex cover V' for some weight system.

> At each time period one treats all the cells in V' which have achieved their old-vegetation threshold together with at least one adjacent cell.

Such strategy is very easy to be implemented once V' is determined and moreover it has the property to be *periodic*, which means that, once a cell i is treated it will be treated again every o_i years. We first consider the single vegetation case and then turn to the multi-vegetation case.

Proposition 3. *Let $\mathcal{C}$ be a hereditary class of graphs for which* MIN WEIGHTED VERTEX COVER *can be approximated in polynomial time within the ratio* ρ. *The problem ZFTS on the class $\mathcal{C}$ with a single vegetation type of old-vegetation threshold o and with constant treatment costs can be approximated in polynomial time within:*

(1) the same approximation ratio ρ if $T \equiv 0 (\bmod\, o)$.

(2) the approximation ration $\left(1 + \frac{1}{\lfloor \frac{T}{o} \rfloor}\right) \rho$ else.

Moreover, these ratios can be guaranteed by a periodic solution of period o: the same treatment program is proposed during each time interval $[ko, (k+1)o), k \in \mathbb{N}$.

Proof. As in the proof of Proposition 2, we consider an instance satisfying the hypotheses and denote by G_I the related graph with cells set I. We then associate each vertex (cell) i with the weight c_i corresponding to the treatment cost. Using similar arguments we get that for any edge xy, either x or y should be treated during any time period of length o. As a consequence, the optimal value β satisfies:

$$\beta \geq \left\lfloor \frac{T}{o} \right\rfloor \tau_c \tag{1}$$

where τ_c denotes the minimum weight of a vertex cover of G_I.

As approximated solution, we then consider a vertex cover $I' \subset I$ computed by the considered vertex cover approximation algorithm. Denoting by τ' the cost of I', we have $\tau' \leq \rho\tau_c$. The solution we take for the whole instance just consists in treating during each time period $[ko, (k+1)o), k \in \mathbb{N}$ vertices in I' at the time they achieve their old-vegetation threshold. During the last period $\left]\lfloor \frac{T}{o} \rfloor o; T\right]$, one does not treat vertices that do not achieve their old-vegetation threshold. Whenever a vertex is treated, at time t, it will be treated again at time $t+o$ and not before; this shows that the solution is o-periodic.

Since treatment costs are not time-dependent, the total cost λ' of this solution satisfies:

1. if $T \equiv 0(\bmod o)$, $\lambda' = \frac{T}{o}\tau'$ and as a consequence, using the relation 1 we immediately get $\lambda' \leq \rho\beta$.
2. else, we have $\lambda' \leq \left(\lfloor \frac{T}{o} \rfloor + 1\right)\tau' \leq \left(\lfloor \frac{T}{o} \rfloor + 1\right)\rho\tau_c$ and using the relation 1 we get $\lambda' \leq \left(1 + \frac{1}{\lfloor \frac{T}{o} \rfloor}\right)\rho\beta$

This completes the proof. □

Note that the strategy we have used is less restrictive than the previous quoted strategy since we treat a vertex in the pre-determined solution as soon as it becomes old, even if it does not have an old neighbour. However the more precise strategy that would be used in practice does not allow to improve the worst case approximation ratio.

Corollary 2. *The problem $ZFTS$ with a single vegetation type of old-vegetation threshold o and with constant treatment costs has:*

1. *a periodic polynomial time algorithm (resp., asymptotically optimal) in bipartite graphs if $T \equiv 0(\bmod o)$ (resp., if $T \gg 0$).*
2. *a periodic polynomial time approximation scheme (resp., asymptotic PTAS) in planar graphs if $T \equiv 0(\bmod o)$ (resp. if $T \gg 0$).*

We now present our main result.

Proposition 4. *Let $\mathcal{C}$ be a hereditary class of graphs for which* MIN WEIGHTED VERTEX COVER *can be approximated in polynomial time within the ratio ρ. The problem $ZFTS$ on the class $\mathcal{C}$ with constant treatment costs, multi-vegetation type (we denote respectively by o_i and a_i the old-vegetation threshold and the initial vegetation age in cell i) can be approximated in polynomial time within*

$$\frac{T+a}{T-\ell}\rho \leq (1+\frac{2o}{T-o})\rho$$

on a time period $T \geq \ell$, where $a = \max_{i\in I} a_i$ and $\ell = \max_{i\in I}(o_i - a_i)$.

Proof. (sketch) Consider an instance of $ZFTS$ with $G_I = (I,E)$ the associated graph. We denote by c_i the treatment cost of vertex (cell) i and define the following weight system on I: $\forall i \in V, \omega_i = \frac{c_i}{o_i}$. We denote by $\tau_\omega(G_I)$ the minimum weight of a vertex cover in G_I. Let $I' \subset I$ be a ρ-approximated weighted vertex cover for this weight system:

$$\omega(I') = \sum_{i\in I'} \omega_i \leq \rho\tau_\omega(G_I) \tag{2}$$

We denote by $S_{I'}$ the $ZFTS$ solution constructed from I' and by S^* an optimal solution. According to Proposition 1 we assume that only old vegetation cells are treated in this optimal solution. The cost of $S_{I'}$ and S^* are denoted by $c(S_{I'})$ and $c(S^*)$, respectively. We have $c(S^*) \leq c(S_{I'})$.

By definition of the solution $S_{I'}$ a vertex $i \in I'$ is treated at most $\left\lfloor \frac{T+a_i}{o_i} \right\rfloor$ times and consequently, using Relation 2:

$$c(S_{I'}) \leq \sum_{i\in I'} c_i \left\lfloor \frac{T+a_i}{o_i} \right\rfloor \leq (T+a)\rho\tau_\omega(G_I) \tag{3}$$

For each time period $t = 1,\ldots,T$ we define the set $I^t = \{i \in I, a_i + t - 1 \geq o_i\}$ that can be seen as the set of cells that would have an old vegetation at the beginning of period t if no treatment was applied at all. Sets $I^t, t = 1,\ldots,T$, are nested ($\forall t < T, I^t \subseteq I^{t+1}$) and for $t \geq \ell+1$, we have $I^t = I$. For $t = 1,\ldots,T$, we denote by $G_I^t = G_I[I^t]$.

Consider now the optimal solution S^*. We associate to S^* as subset U of $\mathcal{U} = \{(i,t), t = 1,\ldots,T, i \in I^t\}$ as follows: whenever a vertex $i \in I$ is treated during the period t, we add in U the vertices $(i,t+k), k = 0,\ldots,\min\{o_i - 1, T-t\}$, and assign to each one the weight $\frac{c_i}{o_i}$. Roughly speaking this corresponds to spread the treatment cost over the whole treatment and regrowing period. For each period t we define $U^t = \{i,(i,t) \in U\}$. We have:

$$c(V^*) \geq \sum_{t=1}^{T} \omega(U^t) \tag{4}$$

Since the sets $I^t, t = 1,\ldots,T$, are nested and only vertices in I^t are treated during period t, we have $\forall t = 1,\ldots,T, U^t \subset I^t$. Now, it is easy to see that for every $t = 1,\ldots,T$, U^t is a vertex cover of G_I^t. Since the weights of vertices in U^t correspond to the weights in G_I, we deduce from Relation 4:

$$c(V^*) \geq \sum_{i=1}^{T} \tau_\omega(G_I^t) \geq (T-\ell)\tau_\omega(G_I) \tag{5}$$

Relations 3 and 5 immediately conclude the proof:

$$c(V') \leq \frac{T+a}{T-\ell} c(V^*) \tag{6}$$

□

We emphasise some interesting particular cases:

Remark 2. *Note that the result still holds if we replace a by $a_{V'} = \max_{i \in V'} a_i$.*
Then, if $a_{V'} = 0$, meaning that the computed vertex cover only includes cells with new vegetation, the related ratio will be

$$\frac{T}{T-\ell}\rho \leq \frac{T}{T-o}\rho$$

with $o = \max_{i \in I} o_i$.
If, on the contrary, all cells are old at the beginning of the process ($\ell = 0$), the ratio will be

$$\frac{T+a}{T}\rho = \frac{T+o}{T}\rho$$

Remark 3. *For large T, in particular if $T \geq \frac{o}{\varepsilon}$, $\varepsilon > 0$, we get the ratio $(1 + \frac{2\varepsilon}{1-\varepsilon})\rho$*

Corollary 3. *The problem $ZFTS$ with constant treatment costs, multi-vegetation type has:*

1. *a periodic asymptotical optimal polynomial time algorithm in bipartite graphs and in chordal graphs if $T \gg 0$.*
2. *a periodic asymptotic polynomial time approximation scheme in planar graphs if $T \gg 0$.*

4 DISCUSSION AND FUTURE DIRECTIONS

In this work we have studied the problem $ZFTS$ that can be seen as a multi-period vertex cover problem: how to remove a set of vertices of minimum weight so as to eliminate all edges with the particularity that a removed vertex reappears after some time periods (regrowing process). We have shown how to build efficient solutions by using efficient algorithms for MIN WEIGHTED VERTEX COVER. This example illustrates the potential advantage of such a graph approach:

1. It is suitable to understand links with well known problems - MIN WEIGHTED VERTEX COVER in our case - that has been extensively studied. The proposed reduction allows even to transform good heuristics for the latter into heuristics for the former. Since the reduction preserves good approximated values we can expect the resulting heuristic to be good.
2. It is suitable to derive some properties and algorithms in specific graph classes from the known properties of MIN WEIGHTED VERTEX COVER in these classes.
3. It also allows to better understand the hardness of the problem in some particular cases using their structural properties.

The continuation of this work includes numerical tests on real instances and the analysis of the solutions with final users in Victoria State.

Our results emphasised the notion of periodic solutions that has some advantages. Once the original vertex cover is computed, the treatment schedule can be computed in linear time and has a better behaviour for long time periods contrary to other methods. It is also very simple to implement in practice. This raises mainly two research questions. The first one is to determine the best periodic solution and we conjecture that our approach is asymptotically optimal. On the contrary if periodicity is not desirable since the same cells are always treated, it motivates an opposite notion for solutions that spread uniformly the treatments over all cells for long time periods.

The problem $ZFTS$ is a very simplified abstraction of the real problem. We plan to generalise the results obtained here to some generalisations of vertex cover that are more relevant for fuel management. The notion of fragmenting a graph (Edwards and Farr, 2001) is particularly interesting in this context. It corresponds to remove as few vertices as possible so that the components of the resulting graph do not have more than k vertices, for a fixed k. The case $k = 1$ corresponds to the notion of vertex cover.

A second research direction is to investigate the problem with fixed budgets at each time period ($BFTS$). Both complexity results and approximation results similar to the ones presented here would be interesting. So far, using a general though process described in (Demange and Ekim, 2013), we have already proved it is NP-hard even in grid graphs, with unitary treatment costs, single vegetation type and $T < o$. This hardness result emphasises an important difference with the problem addressed in this paper and, in some way, justifies a posteriori the MIP approach proposed in (Minas et al., 2014). Our proof however still does not prove whether the same problem remains hard with a constant number of periods and constant old-vegetation threshold. Such requirements are natural from the application point of view. We conjecture that this case is still hard in grids.

ACKNOWLEDGEMENTS

Cerasela Tanasescu was supported by the Grant *Probability of Fire Ignition and Escalation*, Schedule 7 - Bushfire and Natural Hazard CRC, for the Department of Environment, Land, Water and Planning of Victoria State, Australia. This support is greatly acknowledged.

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