Intuitionistic De Morgan Verification and Falsification Logics

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Abstract: In this paper, two new logics called intuitionistic De Morgan verification logic DV and intuitionistic De Morgan falsification logic DF are introduced as a Gentzen-type sequent calculus. The logics DV and DF have De Morgan-like laws with respect to implication and co-implication. These laws are analogous to the well-known De Morgan laws with respect to conjunction and disjunction. On the one hand, DV can appropriately represent verification (or justification) of incomplete information, on the other hand DF can appropriately represent falsification (or refutation) of incomplete information. Some theorems for embedding DV into DF and vice versa are shown. The cut-elimination theorems for DV and DF are proved, and DV and DF are also shown to be paraconsistent and decidable.

1 INTRODUCTION

Intuitionistic logic is well-known to be useful for many computer science applications. For example, it is useful for representing and realizing proof-as-programs (Curry-Howard) paradigm, functional programing, logic programming, typed λ-calculus and program extraction. A reason for the usefulness of the intuitionistic logic is that it can appropriately represent verification of incomplete information. Various extensions and modifications of the intuitionistic logic, such as *dual-intuitionistic logic* (Czermak, 1977; Goodman, 1981; Urbas, 1996) and *Nelson's paraconsistent four-valued logic* (Almukdad and Nelson, 1984; Nelson, 1949), have been proposed for appropriately representing verification, falsification and inconsistency (or paraconsistency) of incomplete information.

In this paper, two new logics called *intuitionistic De Morgan verification logic* DV and *intuitionistic De Morgan falsification logic* DF are introduced as a Gentzen-type sequent calculus. The logics DV and DF have the following *De Morgan-like laws* with respect to the implication connective $\rightarrow$ and the co-implication (subtraction, exclusion or explication) connective $\leftarrow$:

1. $\sim(\alpha\rightarrow\beta) \leftrightarrow \sim\alpha\leftarrow\sim\beta$,
2. $\sim(\alpha\leftarrow\beta) \leftrightarrow \sim\alpha\rightarrow\sim\beta$.

These laws are analogous to the following well-known De Morgan laws with respect to the conjunction connective $\wedge$ and the disjunction connective $\vee$:

1. $\sim(\alpha\wedge\beta) \leftrightarrow \sim\alpha\vee\sim\beta$,
2. $\sim(\alpha\vee\beta) \leftrightarrow \sim\alpha\wedge\sim\beta$.

The following De Morgan-like laws, which are resemble to the above De Morgan-like laws, were originally introduced and studied in (Kamide and Wansing, 2010) for formalizing a duality principle for a classical paraconsistent logic called *symmetric paraconsistent logic*.

1. $\sim(\alpha\rightarrow\beta) \leftrightarrow \sim\beta\leftarrow\sim\alpha$,
2. $\sim(\alpha\leftarrow\beta) \leftrightarrow \sim\beta\rightarrow\sim\alpha$.

The De Morgan-like laws in DV and DF are required for showing a duality principle between DV and DF. The duality principle does not hold for the logics which are obtained from DV and DF by replacing the De Morgan-like laws with the other De Morgan-like laws introduced in (Kamide and Wansing, 2010), although the cut-elimination theorem holds for these modified logics.

DV is regarded as a variant of Nelson's paraconsistent four-valued logic N4 (Almukdad and Nelson, 1984; Nelson, 1949; Wansing, 1993; Kamide and Wansing, 2012; Kamide and Wansing, 2015), and DF is regarded as the dual version of DV in the sense that some theorems for embedding DV into DF and vice versa hold. These embedding theorems, which represent a duality principle between DV and DF, are regarded as a characteristic property of DV and DF, since similar embedding theorems do not hold

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for N4 and its straightforward dual-like counterpart. Such dual logics like DF were originally studied as the dual-intuitionistic logics (Czermak, 1977; Goodman, 1981; Urbas, 1996), which have a Gentzen-type sequent calculus in which sequents have the restriction that the antecedent contains at most one formula. DV and DF are, indeed, extensions of the positive intuitionistic logic and the positive dual-intuitionistic logic, respectively. Moreover, DV is regarded as a modified intuitionistic version of the symmetric paraconsistent logic (Kamide and Wansing, 2010).

Since DV and DF do not have the law $\alpha \vee \sim\alpha$ of excluded middle, these logics are appropriate for handling *incomplete information*. Moreover, since DV and DF do not have the law $(\alpha \wedge \sim\alpha) \rightarrow \beta$ of explosion, these logics are regarded as *paraconsistent logics* (Priest, 2002). Since the base-logics of DV and DF, i.e., the positive intuitionistic logic and the positive dual-intuitionistic logic, are known to be appropriate for representing "verification (or justification)" and "falsification (or refutation)", respectively (Shramko, 2005), DV and DF are regarded also as suitable for representing verification and falsification, respectively. Thus, on the one hand, DV is suitable for representing verification of incomplete information, on the other hand DF is suitable for representing falsification of incomplete information.

The contents of this paper are then summarized as follows. In Section 2, the logic DV is introduced as a Gentzen-type sequent calculus, and some theorems for embedding DV and its negation-free fragment IC are proved. By using these embedding theorems, the cut-elimination theorem for DV is shown, and DV is also shown to be paraconsistent and decidable. In Section 3, the logic DF is introduced as a Gentzen-type sequent calculus, and some theorems for embedding DF and its negation-free fragment DC are shown. By using these embedding theorems, the cut-elimination theorem for DF is obtained, and DF is shown to be paraconsistent and decidable. In Section 4, some theorems for embedding DV into DF and vice versa, which represent a duality principle for them, are shown. In Section 5, this paper is concluded.

2 INTUITIONISTIC DE MORGAN VERIFICATION LOGIC

The language of *intuitionistic De Morgan verification logic* consists of logical connectives $\wedge_t$ (conjunction), $\vee_t$ (disjunction), $\rightarrow_t$ (implication), $\leftarrow_t$ (co-implication) and $\sim_t$ (paraconsistent negation). Lower case letters $p, q, ...$ are used for propositional variables, lower case Greek letters $\alpha, \beta, ...$ are used for formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used for finite (possibly empty) multisets of formulas. These letters are also used for other logics discussed in this paper. A *positive sequent* is an expression of the form $\Gamma \Rightarrow \gamma$ where γ denotes a single formula or the empty sequence. A negative sequent will also be defined.

An expression $L \vdash S$ is used to denote the fact that a (positive/negative) sequent S is provable in a sequent calculus L. An expression of the form $\alpha \Leftrightarrow \beta$ is used to represent both $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: for any instance

$$\frac{S_1 \ \cdots \ S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$. Since all logics discussed in this paper are formulated as sequent calculi, we will frequently identify a sequent calculus with the logic determined by it.

A Gentzen-type sequent calculus DV for intuitionistic De Morgan verification logic is defined as follows based on positive sequents.

Definition 2.1 (DV). *The initial sequents of* DV *are of the following form, for any propositional variable p:*

$$p \Rightarrow p \qquad \sim_t p \Rightarrow \sim_t p.$$

The structural rules of DV *are of the form:*

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma}\ (\text{t-cut}) \qquad \frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{t-co-l})$$

$$\frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{t-we-l}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (\text{t-we-r}).$$

The positive logical inference rules of DV *are of the form:*

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\alpha \wedge_t \beta, \Gamma \Rightarrow \gamma}\ (\wedge_t\text{l1}) \qquad \frac{\beta, \Gamma \Rightarrow \gamma}{\alpha \wedge_t \beta, \Gamma \Rightarrow \gamma}\ (\wedge_t\text{l2})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge_t \beta}\ (\wedge_t\text{r}) \qquad \frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha \vee_t \beta, \Gamma \Rightarrow \gamma}\ (\vee_t\text{l})$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee_t \beta}\ (\vee_t\text{r1}) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee_t \beta}\ (\vee_t\text{r2})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha \rightarrow_t \beta, \Gamma, \Delta \Rightarrow \gamma}\ (\rightarrow_t\text{l}) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow_t \beta}\ (\rightarrow_t\text{r})$$

$$\frac{\beta, \Gamma \Rightarrow \gamma}{\alpha \leftarrow_t \beta, \Gamma \Rightarrow \gamma}\ (\leftarrow_t\text{l1}) \qquad \frac{\Gamma \Rightarrow \alpha}{\alpha \leftarrow_t \beta, \Gamma \Rightarrow}\ (\leftarrow_t\text{l2})$$

$$\frac{\alpha, \Gamma \Rightarrow \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \leftarrow_t \beta}\ (\leftarrow_t\text{r}).$$

The negative logical inference rules of DV *are of the form:*

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\sim_t \sim_t \alpha, \Gamma \Rightarrow \gamma}\ (\sim_t\text{l}) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \sim_t \sim_t \alpha}\ (\sim_t\text{r})$$

$$\dfrac{\sim_t\alpha, \Gamma \Rightarrow \gamma \quad \sim_t\beta, \Gamma \Rightarrow \gamma}{\sim_t(\alpha\wedge_t\beta), \Gamma \Rightarrow \gamma}\ (\sim_t\wedge_t\mathrm{l})$$

$$\dfrac{\Gamma \Rightarrow \sim_t\alpha}{\Gamma \Rightarrow \sim_t(\alpha\wedge_t\beta)}\ (\sim_t\wedge_t\mathrm{r1}) \quad \dfrac{\Gamma \Rightarrow \sim_t\beta}{\Gamma \Rightarrow \sim_t(\alpha\wedge_t\beta)}\ (\sim_t\wedge_t\mathrm{r2})$$

$$\dfrac{\sim_t\alpha, \Gamma \Rightarrow \gamma}{\sim_t(\alpha\vee_t\beta), \Gamma \Rightarrow \gamma}\ (\sim_t\vee_t\mathrm{l1})$$

$$\dfrac{\sim_t\beta, \Gamma \Rightarrow \gamma}{\sim_t(\alpha\vee_t\beta), \Gamma \Rightarrow \gamma}\ (\sim_t\vee_t\mathrm{l2})$$

$$\dfrac{\Gamma \Rightarrow \sim_t\alpha \quad \Gamma \Rightarrow \sim_t\beta}{\Gamma \Rightarrow \sim_t(\alpha\vee_t\beta)}\ (\sim_t\vee_t\mathrm{r})$$

$$\dfrac{\sim_t\beta, \Gamma \Rightarrow \gamma}{\sim_t(\alpha\rightarrow_t\beta), \Gamma \Rightarrow \gamma}\ (\sim_t\rightarrow_t\mathrm{l1})$$

$$\dfrac{\Gamma \Rightarrow \sim_t\alpha}{\sim_t(\alpha\rightarrow_t\beta), \Gamma \Rightarrow}\ (\sim_t\rightarrow_t\mathrm{l2})$$

$$\dfrac{\sim_t\alpha, \Gamma \Rightarrow \quad \Delta \Rightarrow \sim_t\beta}{\Gamma, \Delta \Rightarrow \sim_t(\alpha\rightarrow_t\beta)}\ (\sim_t\rightarrow_t\mathrm{r})$$

$$\dfrac{\Gamma \Rightarrow \sim_t\alpha \quad \sim_t\beta, \Delta \Rightarrow \gamma}{\sim_t(\alpha\leftarrow_t\beta), \Gamma, \Delta \Rightarrow \gamma}\ (\sim_t\leftarrow_t\mathrm{l})$$

$$\dfrac{\sim_t\alpha, \Gamma \Rightarrow \sim_t\beta}{\Gamma \Rightarrow \sim_t(\alpha\leftarrow_t\beta)}\ (\sim_t\leftarrow_t\mathrm{r}).$$

Some remarks are given as follows.

1. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in DV. This fact can be shown by induction on α.
2. A sequent calculus for *Nelson's paraconsistent four-valued logic* N4 (Almukdad and Nelson, 1984; Nelson, 1949) is obtained from the $\leftarrow_t$-free fragment of DV by replacing $\{(\sim_t\rightarrow_t\mathrm{l1}), (\sim_t\rightarrow_t\mathrm{l2}), (\sim_t\rightarrow_t\mathrm{r})\}$ with the negative inference rules of the form:

$$\dfrac{\alpha, \Gamma \Rightarrow \gamma}{\sim_t(\alpha\rightarrow_t\beta), \Gamma \Rightarrow \gamma}\ (\mathrm{n}\sim_t\rightarrow_t\mathrm{l1})$$

$$\dfrac{\sim_t\beta, \Gamma \Rightarrow \gamma}{\sim_t(\alpha\rightarrow_t\beta), \Gamma \Rightarrow \gamma}\ (\mathrm{n}\sim_t\rightarrow_t\mathrm{l2})$$

$$\dfrac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \sim_t\beta}{\Gamma \Rightarrow \sim_t(\alpha\rightarrow_t\beta)}\ (\mathrm{n}\sim_t\rightarrow_t\mathrm{r})$$

   which correspond to the axiom scheme: $\sim_t(\alpha\rightarrow_t\beta) \leftrightarrow (\alpha\wedge_t\sim_t\beta)$.

Proposition 2.2. *The following sequents are provable in* DV*: for any formulas* α *and* β*,*

1. $\sim_t\sim_t\alpha \Leftrightarrow \alpha$,
2. $\sim_t(\alpha\wedge_t\beta) \Leftrightarrow (\sim_t\alpha\vee_t\sim_t\beta)$,
3. $\sim_t(\alpha\vee_t\beta) \Leftrightarrow (\sim_t\alpha\wedge_t\sim_t\beta)$,
4. $\sim_t(\alpha\rightarrow_t\beta) \Leftrightarrow (\sim_t\alpha\leftarrow_t\sim_t\beta)$,
5. $\sim_t(\alpha\leftarrow_t\beta) \Leftrightarrow (\sim_t\alpha\rightarrow_t\sim_t\beta)$.

Proof. We show only the following case. (4):

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \sim_t\alpha \Rightarrow \sim_t\alpha\end{array}}{\sim_t(\alpha\rightarrow_t\beta), \sim_t\alpha \Rightarrow}\ (\sim_t\rightarrow_t\mathrm{l2}) \quad \dfrac{\begin{array}{c}\vdots\\ \sim_t\beta \Rightarrow \sim_t\beta\end{array}}{\sim_t(\alpha\rightarrow_t\beta) \Rightarrow \sim_t\beta}\ (\sim_t\rightarrow_t\mathrm{l1})}{\sim_t(\alpha\rightarrow_t\beta), \sim_t(\alpha\rightarrow_t\beta) \Rightarrow \sim_t\alpha\leftarrow_t\sim_t\beta}\ (\leftarrow_t\mathrm{r})}{\sim_t(\alpha\rightarrow_t\beta) \Rightarrow \sim_t\alpha\leftarrow_t\sim_t\beta}\ (\text{t-co-r})$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \sim_t\alpha \Rightarrow \sim_t\alpha\end{array}}{\sim_t\alpha\leftarrow_t\sim_t\beta, \sim_t\alpha \Rightarrow}\ (\leftarrow_t\mathrm{l2}) \quad \dfrac{\begin{array}{c}\vdots\\ \sim_t\beta \Rightarrow \sim_t\beta\end{array}}{\sim_t\alpha\leftarrow_t\sim_t\beta \Rightarrow \sim_t\beta}\ (\leftarrow_t\mathrm{l1})}{\sim_t\alpha\leftarrow_t\sim_t\beta, \sim_t\alpha\leftarrow_t\sim_t\beta \Rightarrow \sim_t(\alpha\rightarrow_t\beta)}\ (\sim_t\rightarrow_t\mathrm{r})}{\sim_t\alpha\leftarrow_t\sim_t\beta \Rightarrow \sim_t(\alpha\rightarrow_t\beta)}\ (\text{t-co-r}).$$

▮

Definition 2.3 (IC)**.** *A sequent calculus* IC *for positive intuitionistic logic with co-implication is defined as the* $\sim_t$*-free fragment of* DV*, i.e., it is obtained from* DV *by deleting the negative initial sequents* $\sim_t p \Rightarrow \sim_t p$ *and the negative logical inference rules concerning* $\sim_t$*.*

The following result is known (Urbas, 1996).

Proposition 2.4 (Cut-elimination and decidability for IC)**.** *We have:*

1. *The rule* (t-cut) *is admissible in cut-free* IC*.*
2. IC *is decidable.*

Next, we introduce a translation of DV into IC, and by using this translation, we show some theorems for embedding DV into IC. A similar translation has been used by Gurevich (Gurevich, 1977), Rautenberg (Rautenberg, 1979) and Vorob'ev (Vorob'ev, 1952) to embed Nelson's constructive logic (Almukdad and Nelson, 1984; Nelson, 1949) into intuitionistic logic.

Definition 2.5. *We fix a set* Φ *of propositional variables and define the set* $\Phi' := \{p' \mid p \in \Phi\}$ *of propositional variables. The language* $\mathcal{L}_{\mathrm{DV}}$ *of* DV *is defined using* Φ, $\wedge_t, \vee_t, \rightarrow_t, \leftarrow_t$ *and* $\sim_t$*. The language* $\mathcal{L}_{\mathrm{IC}}$ *of* IC *is obtained from* $\mathcal{L}_{\mathrm{DV}}$ *by adding* Φ' *and deleting* $\sim_t$*.*

A mapping f *from* $\mathcal{L}_{\mathrm{DV}}$ *to* $\mathcal{L}_{\mathrm{IC}}$ *is defined inductively by:*

1. *for any* $p \in \Phi$, $f(p) := p$ *and* $f(\sim_t p) := p' \in \Phi'$,
2. $f(\alpha \circ \beta) := f(\alpha) \circ f(\beta)$ *where* $\circ \in \{\wedge_t, \vee_t, \rightarrow_t, \leftarrow_t\}$,
3. $f(\sim_t\sim_t\alpha) := f(\alpha)$,
4. $f(\sim_t(\alpha\wedge_t\beta)) := f(\sim_t\alpha)\vee_t f(\sim_t\beta)$,
5. $f(\sim_t(\alpha\vee_t\beta)) := f(\sim_t\alpha)\wedge_t f(\sim_t\beta)$,
6. $f(\sim_t(\alpha\rightarrow_t\beta)) := f(\sim_t\alpha)\leftarrow_t f(\sim_t\beta)$,
7. $f(\sim_t(\alpha\leftarrow_t\beta)) := f(\sim_t\alpha)\rightarrow_t f(\sim_t\beta)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$. The same notation is used for other mappings discussed in this paper.

We then obtain a weak theorem for syntactically embedding DV into IC.

Theorem 2.6 (Weak embedding from DV into IC)**.** *Let Γ be a set of formulas in $\mathcal{L}_{\mathrm{DV}}$, γ be a formula in $\mathcal{L}_{\mathrm{DV}}$ or the empty sequence, and f be the mapping defined in Definition 2.5.*

1. *If* DV $\vdash \Gamma \Rightarrow \gamma$, *then* IC $\vdash f(\Gamma) \Rightarrow f(\gamma)$.
2. *If* IC $-$ (t-cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$, *then* DV $-$ (t-cut) $\vdash \Gamma \Rightarrow \gamma$.

Proof. (1): By induction on the proofs P of $\Gamma \Rightarrow \gamma$ in DV. We distinguish the cases according to the last inference of P, and show some cases.

Case ($\sim_t p \Rightarrow \sim_t p$): The last inference of P is of the form: $\sim_t p \Rightarrow \sim_t p$ for any $p \in \Phi$. In this case, we obtain IC $\vdash f(\sim_t p) \Rightarrow f(\sim_t p)$, i.e., IC $\vdash p' \Rightarrow p'$ ($p' \in \Phi'$), by the definition of f.

Case ($\sim_t\!\rightarrow_t$r): The last inference of P is of the form:

$$\frac{\sim_t\alpha, \Gamma \Rightarrow \quad \Delta \Rightarrow \sim_t\beta}{\Gamma, \Delta \Rightarrow \sim_t(\alpha \rightarrow_t \beta)}\ (\sim_t\!\rightarrow_t\mathrm{r}).$$

By induction hypothesis, we have IC $\vdash f(\sim_t\alpha), f(\Gamma) \Rightarrow$ and IC $\vdash f(\Delta) \Rightarrow f(\sim_t\beta)$. Then, we obtain the required fact:

$$\dfrac{\begin{matrix}\vdots & & \vdots\\ f(\sim_t\alpha), f(\Gamma) \Rightarrow & & f(\Delta) \Rightarrow f(\sim_t\beta)\end{matrix}}{f(\Gamma), f(\Delta) \Rightarrow f(\sim_t\alpha) \leftarrow_t f(\sim_t\beta)}\ (\leftarrow_t\mathrm{r})$$

as $f(\sim_t\alpha) \leftarrow_t f(\sim_t\beta) = f(\sim_t(\alpha \rightarrow_t \beta))$.

Case ($\sim_t\!\rightarrow_t$l1): The last inference of P is of the form:

$$\frac{\sim_t\beta, \Gamma \Rightarrow \gamma}{\sim_t(\alpha \rightarrow_t \beta), \Gamma \Rightarrow \gamma}\ (\sim_t\!\rightarrow_t\mathrm{l1}).$$

By induction hypothesis, we have IC $\vdash f(\sim_t\beta), f(\Gamma) \Rightarrow f(\gamma)$. Then, we obtain the required fact:

$$\dfrac{\begin{matrix}\vdots\\ f(\sim_t\beta), f(\Gamma) \Rightarrow f(\gamma)\end{matrix}}{f(\sim_t\alpha) \leftarrow_t f(\sim_t\beta), f(\Gamma) \Rightarrow f(\gamma)}\ (\leftarrow_t\mathrm{l1})$$

as $f(\sim_t\alpha) \leftarrow_t f(\sim_t\beta) = f(\sim_t(\alpha \rightarrow_t \beta))$.

Case ($\sim_t\!\rightarrow_t$l2): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \sim_t\alpha}{\sim_t(\alpha \rightarrow_t \beta), \Gamma \Rightarrow}\ (\sim_t\!\rightarrow_t\mathrm{l2}).$$

By induction hypothesis, we have IC $\vdash f(\Gamma) \Rightarrow f(\sim_t\alpha)$. Then, we obtain the required fact:

$$\dfrac{\begin{matrix}\vdots\\ f(\Gamma) \Rightarrow f(\sim_t\alpha)\end{matrix}}{f(\sim_t\alpha) \leftarrow_t f(\sim_t\beta), f(\Gamma) \Rightarrow}\ (\leftarrow_t\mathrm{l2})$$

as $f(\sim_t\alpha) \leftarrow_t f(\sim_t\beta) = f(\sim_t(\alpha \rightarrow_t \beta))$.

(2): By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\gamma)$ in IC $-$ (t-cut). We distinguish the cases according to the last inference of Q, and show only the following case.

Case ($\rightarrow_t$l): The last inference of Q is ($\rightarrow_t$l).

Subcase (1): The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\alpha) \quad f(\beta), f(\Delta) \Rightarrow f(\gamma)}{f(\alpha \rightarrow_t \beta), f(\Gamma), f(\Delta) \Rightarrow f(\gamma)}\ (\rightarrow_t\mathrm{l})$$

where $f(\alpha \rightarrow_t \beta)$ coincides with $f(\alpha) \rightarrow_t f(\beta)$ by the definition of f. By induction hypothesis, we have DV $-$ (t-cut) $\vdash \Gamma \Rightarrow \alpha$ and DV $-$ (t-cut) $\vdash \beta, \Delta \Rightarrow \gamma$. We thus obtain the required fact:

$$\dfrac{\begin{matrix}\vdots & & \vdots\\ \Gamma \Rightarrow \alpha & & \beta, \Delta \Rightarrow \gamma\end{matrix}}{\alpha \rightarrow_t \beta, \Gamma, \Delta \Rightarrow \gamma}\ (\rightarrow_t\mathrm{l}).$$

Subcase (2): The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\sim_t\alpha) \quad f(\sim_t\beta), f(\Delta) \Rightarrow f(\gamma)}{f(\sim_t(\alpha \leftarrow_t \beta)), f(\Gamma), f(\Delta) \Rightarrow f(\gamma)}\ (\rightarrow_t\mathrm{l})$$

as $f(\sim_t(\alpha \leftarrow_t \beta)) = f(\sim_t\alpha) \rightarrow_t f(\sim_t\beta)$. By induction hypothesis, we have DV $-$ (t-cut) $\vdash \Gamma \Rightarrow \sim_t\alpha$ and DV $-$ (t-cut) $\vdash \sim_t\beta, \Delta \Rightarrow \gamma$. We thus obtain the required fact:

$$\dfrac{\begin{matrix}\vdots & & \vdots\\ \Gamma \Rightarrow \sim_t\alpha & & \sim_t\beta, \Delta \Rightarrow \gamma\end{matrix}}{\sim_t(\alpha \leftarrow_t \beta), \Gamma, \Delta \Rightarrow \gamma}\ (\sim_t\!\leftarrow_t\mathrm{l}).$$

∎

Using Theorem 2.6 and the cut-elimination theorem for IC, we obtain the following cut-elimination theorem for DV.

Theorem 2.7 (Cut-elimination for DV)**.** *The rule* (t-cut) *is admissible in cut-free* DV.

Proof. Suppose DV $\vdash \Gamma \Rightarrow \gamma$. Then, we have IC $\vdash f(\Gamma) \Rightarrow f(\gamma)$ by Theorem 2.6 (1), and hence IC $-$ (t-cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$ by the cut-elimination theorem for IC. By Theorem 2.6 (2), we obtain DV $-$ (t-cut) $\vdash \Gamma \Rightarrow \gamma$. ∎

Using Theorem 2.6 and the cut-elimination theorem for IC, we obtain the following strong theorem for syntactically embedding DV into IC.

Theorem 2.8 (Strong embedding from DV into IC)**.** *Let Γ be a set of formulas in $\mathcal{L}_{\mathrm{DV}}$, γ be a formula in $\mathcal{L}_{\mathrm{DV}}$ or the empty sequence, and f be the mapping defined in Definition 2.5.*

1. DV $\vdash \Gamma \Rightarrow \gamma$ *iff* IC $\vdash f(\Gamma) \Rightarrow f(\gamma)$.
2. DV $-$ (t-cut) $\vdash \Gamma \Rightarrow \gamma$ *iff* IC $-$ (t-cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$.

Proof. (1): ($\Longrightarrow$): By Theorem 2.6 (1). ($\Longleftarrow$): Suppose IC $\vdash f(\Gamma) \Rightarrow f(\gamma)$. Then we have IC $-$ (t-cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$ by the cut-elimination theorem for IC. We thus obtain DV $-$ (t-cut) $\vdash \Gamma \Rightarrow \gamma$ by Theorem 2.6 (2). Therefore we have DV $\vdash \Gamma \Rightarrow \gamma$.

(2): ($\Longrightarrow$): Suppose DV $-$ (t-cut) $\vdash \Gamma \Rightarrow \gamma$. Then we have DV $\vdash \Gamma \Rightarrow \gamma$. We then obtain IC $\vdash f(\Gamma) \Rightarrow f(\gamma)$ by Theorem 2.6 (1). Therefore we obtain IC $-$ (t-cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$ by the cut-elimination theorem for IC. ($\Longleftarrow$): By Theorem 2.6 (2). ∎

Using Theorem 2.7, we show the paraconsistency of DV with respect to $\sim_t$.

Definition 2.9. *Let $\sharp$ be a negation (-like) connective. A sequent calculus L is called* explosive *with respect to $\sharp$ if for any formulas α and β, the sequent $\alpha, \sharp\alpha \Rightarrow \beta$ is provable in L. It is called* paraconsistent *with respect to $\sharp$ if it is not explosive with respect to $\sharp$.*

Theorem 2.10 (Paraconsistency for DV)**.** DV *is paraconsistent with respect to $\sim_t$.*

Proof. Consider a sequent $p, \sim_t p \Rightarrow q$ where p and q are distinct propositional variables. Then, the unprovability of this sequent can be shown using Theorem 2.7. ∎

Using Theorem 2.8 and the decidability of IC, we show the decidability of DV.

Theorem 2.11 (Decidability for DV)**.** DV *is decidable.*

Proof. By decidability of IC, for each α, it is possible to decide if $\Rightarrow f(\alpha)$ is provable in DV. Then, by Theorem 2.8, IC is decidable. ∎

3 INTUITIONISTIC DE MORGAN FALSIFICATION LOGIC

The language of *intuitionistic De Morgan falsification logic* consists of logical connectives $\wedge_f$ (dual-conjunction), $\vee_f$ (dual-disjunction), $\rightarrow_f$ (dual-implication), $\leftarrow_f$ (dual-co-implication) and $\sim_f$ (dual-paraconsistent negation). A *negative sequent* is an expression of the form $\gamma \Rightarrow \Gamma$ where γ denotes a single formula or the empty sequence.

A Gentzen-type sequent calculus DF for intuitionistic De Morgan falsification logic is defined as follows based on negative sequents.

Definition 3.1 (DF)**.** *The initial sequents of* DF *are of the following form, for any propositional variable p:*

$$p \Rightarrow p \qquad\qquad \sim_f p \Rightarrow \sim_f p.$$

The structural rules of DF *are of the form:*

$$\frac{\gamma \Rightarrow \Gamma, \alpha \quad \alpha \Rightarrow \Delta}{\gamma \Rightarrow \Gamma, \Delta}\ (\text{f-cut}) \qquad \frac{\gamma \Rightarrow \Gamma, \alpha, \alpha}{\gamma \Rightarrow \Gamma, \alpha}\ (\text{f-co-r})$$

$$\frac{\gamma \Rightarrow \Gamma}{\gamma \Rightarrow \Gamma, \alpha}\ (\text{f-we-r}) \qquad \frac{\Rightarrow \Gamma}{\alpha \Rightarrow \Gamma}\ (\text{f-we-l}).$$

The positive logical inference rules of DF *are of the form:*

$$\frac{\alpha \Rightarrow \Gamma}{\alpha \wedge_f \beta \Rightarrow \Gamma}\ (\wedge_f \text{l1}) \qquad \frac{\beta \Rightarrow \Gamma}{\alpha \wedge_f \beta \Rightarrow \Gamma}\ (\wedge_f \text{l2})$$

$$\frac{\gamma \Rightarrow \Gamma, \alpha \quad \gamma \Rightarrow \Gamma, \beta}{\gamma \Rightarrow \Gamma, \alpha \wedge_f \beta}\ (\wedge_f \text{r}) \qquad \frac{\alpha \Rightarrow \Gamma \quad \beta \Rightarrow \Gamma}{\alpha \vee_f \beta \Rightarrow \Gamma}\ (\vee_f \text{l})$$

$$\frac{\gamma \Rightarrow \Gamma, \alpha}{\gamma \Rightarrow \Gamma, \alpha \vee_f \beta}\ (\vee_f \text{r1}) \qquad \frac{\gamma \Rightarrow \Gamma, \beta}{\gamma \Rightarrow \Gamma, \alpha \vee_f \beta}\ (\vee_f \text{r2})$$

$$\frac{\Rightarrow \Gamma, \alpha \quad \beta \Rightarrow \Delta}{\alpha \rightarrow_f \beta \Rightarrow \Gamma, \Delta}\ (\rightarrow_f \text{l})$$

$$\frac{\gamma \Rightarrow \Gamma, \beta}{\gamma \Rightarrow \Gamma, \alpha \rightarrow_f \beta}\ (\rightarrow_f \text{r1}) \qquad \frac{\alpha \Rightarrow \Gamma}{\Rightarrow \Gamma, \alpha \rightarrow_f \beta}\ (\rightarrow_f \text{r2})$$

$$\frac{\beta \Rightarrow \Gamma, \alpha}{\alpha \leftarrow_f \beta \Rightarrow \Gamma}\ (\leftarrow_f \text{l}) \qquad \frac{\alpha \Rightarrow \Gamma \quad \gamma \Rightarrow \Delta, \beta}{\gamma \Rightarrow \Gamma, \Delta, \alpha \leftarrow_f \beta}\ (\leftarrow_f \text{r}).$$

The negative logical inference rules of DF *are of the form:*

$$\frac{\alpha \Rightarrow \Gamma}{\sim_f \sim_f \alpha \Rightarrow \Gamma}\ (\sim_f \text{l}) \qquad \frac{\gamma \Rightarrow \Gamma, \alpha}{\gamma \Rightarrow \Gamma, \sim_f \sim_f \alpha}\ (\sim_f \text{r})$$

$$\frac{\sim_f \alpha \Rightarrow \Gamma \quad \sim_f \beta \Rightarrow \Gamma}{\sim_f (\alpha \wedge_f \beta) \Rightarrow \Gamma}\ (\sim_f \wedge_f \text{l})$$

$$\frac{\gamma \Rightarrow \Gamma, \sim_f \alpha}{\gamma \Rightarrow \Gamma, \sim_f (\alpha \wedge_f \beta)}\ (\sim_f \wedge_f \text{r1})$$

$$\frac{\gamma \Rightarrow \Gamma, \sim_f \beta}{\gamma \Rightarrow \Gamma, \sim_f (\alpha \wedge_f \beta)}\ (\sim_f \wedge_f \text{r2})$$

$$\frac{\sim_f \alpha \Rightarrow \Gamma}{\sim_f (\alpha \vee_f \beta) \Rightarrow \Gamma}\ (\sim_f \vee_f \text{l1})$$

$$\frac{\sim_f \beta \Rightarrow \Gamma}{\sim_f (\alpha \vee_f \beta) \Rightarrow \Gamma}\ (\sim_f \vee_f \text{l2})$$

$$\frac{\gamma \Rightarrow \Gamma, \sim_f \alpha \quad \gamma \Rightarrow \Gamma, \sim_f \beta}{\gamma \Rightarrow \Gamma, \sim_f (\alpha \vee_f \beta)}\ (\sim_f \vee_f \text{r})$$

$$\frac{\sim_f \beta \Rightarrow \Gamma, \sim_f \alpha}{\sim_f (\alpha \rightarrow_f \beta) \Rightarrow \Gamma}\ (\sim_f \rightarrow_f \text{l})$$

$$\frac{\sim_f\alpha \Rightarrow \Gamma \quad \gamma \Rightarrow \Delta, \sim_f\beta}{\gamma \Rightarrow \Gamma, \Delta, \sim_f(\alpha \rightarrow_f \beta)}\ (\sim_f\rightarrow_f\mathrm{r})$$

$$\frac{\Rightarrow \Gamma, \sim_f\alpha \quad \sim_f\beta \Rightarrow \Delta}{\sim_f(\alpha \leftarrow_f \beta) \Rightarrow \Gamma, \Delta}\ (\sim_f\leftarrow_f\mathrm{l})$$

$$\frac{\gamma \Rightarrow \Gamma, \sim_f\beta}{\gamma \Rightarrow \Gamma, \sim_f(\alpha \leftarrow_f \beta)}\ (\sim_f\leftarrow_f\mathrm{r1})$$

$$\frac{\sim_f\alpha \Rightarrow \Gamma}{\Rightarrow \Gamma, \sim_f(\alpha \leftarrow_f \beta)}\ (\sim_f\leftarrow_f\mathrm{r2}).$$

Some remarks are given as follows.

1. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in DF. This fact can be shown by induction on α.
2. The following sequents are provable in DF: for any formulas α and β,
   (a) $\sim_f\sim_f\alpha \Leftrightarrow \alpha$,
   (b) $\sim_f(\alpha\wedge_f\beta) \Leftrightarrow (\sim_f\alpha\vee_f\sim_f\beta)$,
   (c) $\sim_f(\alpha\vee_f\beta) \Leftrightarrow (\sim_f\alpha\wedge_f\sim_f\beta)$,
   (d) $\sim_f(\alpha\rightarrow_f\beta) \Leftrightarrow (\sim_f\alpha\leftarrow_f\sim_f\beta)$,
   (e) $\sim_f(\alpha\leftarrow_f\beta) \Leftrightarrow (\sim_f\alpha\rightarrow_f\sim_f\beta)$.
3. A sequent calculus for a dual-like version of N4 is obtained from the $\leftarrow_f$-free fragment of DF by replacing $\{(\sim_f\rightarrow_f\mathrm{l}), (\sim_f\rightarrow_f\mathrm{r})\}$ with the negative inference rules of the form:

$$\frac{\alpha \Rightarrow \Gamma}{\sim_f(\alpha \rightarrow_f \beta) \Rightarrow \Gamma}\ (\mathrm{dn}\sim_f\rightarrow_f\mathrm{l1})$$

$$\frac{\sim_f\beta \Rightarrow \Gamma}{\sim_f(\alpha \rightarrow_f \beta) \Rightarrow \Gamma}\ (\mathrm{dn}\sim_f\rightarrow_f\mathrm{l2})$$

$$\frac{\gamma \Rightarrow \Gamma, \alpha \quad \gamma \Rightarrow \Gamma, \sim_f\beta}{\gamma \Rightarrow \Gamma, \sim_f(\alpha \rightarrow_f \beta)}\ (\mathrm{dn}\sim_f\rightarrow_f\mathrm{r})$$

   which correspond to the axiom scheme: $\sim_f(\alpha\rightarrow_f\beta) \leftrightarrow (\alpha\wedge_f\sim_f\beta)$.

Definition 3.2 (DC)**.** *A sequent calculus* DC *for positive dual-intuitionistic logic with co-implication is defined as the $\sim_f$-free fragment of* DF, *i.e., it is obtained from* DF *by deleting the negative initial sequents $\sim_f p \Rightarrow \sim_f p$ and the negative logical inference rules concerning $\sim_f$.*

The following result is known (Urbas, 1996).

Proposition 3.3 (Cut-elimination and decidability for DC)**.** *We have:*

1. The rule (f-cut) *is admissible in cut-free* DC.
2. DC *is decidable.*

The following definition is similar to Definition 2.5.

Definition 3.4. *We fix a set Φ of propositional variables and define the set $\Phi' := \{p' \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}_{\mathrm{DF}}$ of* DF *is defined using Φ, $\wedge_f, \vee_f, \rightarrow_f, \leftarrow_f$ and $\sim_f$. The language $\mathcal{L}_{\mathrm{DC}}$ of* DC *is obtained from $\mathcal{L}_{\mathrm{DF}}$ by adding Φ' and deleting $\sim_f$.*

A mapping g from $\mathcal{L}_{\mathrm{DF}}$ to $\mathcal{L}_{\mathrm{DC}}$ is defined inductively by:

1. for any $p \in \Phi$, $g(p) := p$ and $g(\sim_f p) := p' \in \Phi'$,
2. $g(\alpha \circ \beta) := g(\alpha) \circ g(\beta)$ where $\circ \in \{\wedge_f, \vee_f, \rightarrow_f, \leftarrow_f\}$,
3. $g(\sim_f\sim_f\alpha) := g(\alpha)$,
4. $g(\sim_f(\alpha\wedge_f\beta)) := g(\sim_f\alpha)\vee_f g(\sim_f\beta)$,
5. $g(\sim_f(\alpha\vee_f\beta)) := g(\sim_f\alpha)\wedge_f g(\sim_f\beta)$,
6. $g(\sim_f(\alpha\rightarrow_f\beta)) := g(\sim_f\alpha)\leftarrow_f g(\sim_f\beta)$,
7. $g(\sim_f(\alpha\leftarrow_f\beta)) := g(\sim_f\alpha)\rightarrow_f g(\sim_f\beta)$.

Theorem 3.5 (Weak embedding from DF into DC)**.** *Let Γ be a set of formulas in $\mathcal{L}_{\mathrm{DF}}$, γ be a formula in $\mathcal{L}_{\mathrm{DF}}$ or the empty sequence, and f be the mapping defined in Definition 3.4.*

1. If DF $\vdash \gamma \Rightarrow \Gamma$, *then* DC $\vdash g(\gamma) \Rightarrow g(\Gamma)$.
2. If DC $-$ (f-cut) $\vdash g(\gamma) \Rightarrow g(\Gamma)$, *then* DF $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma$.

Proof. Similar to Theorem 2.6. ∎

Theorem 3.6 (Cut-elimination for DF)**.** *The rule* (f-cut) *is admissible in cut-free* DF.

Proof. Similar to Theorem 2.7. By Theorem 3.5 and the cut-elimination theorem for DC. ∎

Theorem 3.7 (Strong embedding from DF into CD)**.** *Let Γ be a set of formulas in $\mathcal{L}_{\mathrm{DF}}$, γ be a formula in $\mathcal{L}_{\mathrm{DF}}$ or the empty sequence, and g be the mapping defined in Definition 3.4.*

1. DF $\vdash \gamma \Rightarrow \Gamma$ *iff* DC $\vdash g(\gamma) \Rightarrow g(G)$.
2. DF $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma$ *iff* DC $-$ (f-cut) $\vdash g(\gamma) \Rightarrow g(\Gamma)$.

Proof. Similar to Theorem 2.8. By Theorem 3.5 and the cut-elimination theorem for CD. ∎

Theorem 3.8 (Paraconsistency for DF)**.** DF *is paraconsistent with respect to $\sim_f$.*

Proof. By Theorem 3.6. ∎

Theorem 3.9 (Decidability for DF)**.** DF *is decidable.*

Proof. By Theorem 2.8 and the decidability of CD. ∎

4 DUALITY

Next, we introduce a translation from DF into DV. The idea of this translation comes from (Czermak, 1977; Urbas, 1996).

Definition 4.1. *We fix a common set* Φ *of propositional variables. The language* $\mathcal{L}_{\mathrm{DF}}$ *of* DF *is defined using* Φ, $\wedge_f, \vee_f, \rightarrow_f, \leftarrow_f$ *and* $\sim_f$. *The language* $\mathcal{L}_{\mathrm{DV}}$ *of* DV *is defined using* Φ, $\wedge_t, \vee_t, \rightarrow_t, \leftarrow_t$ *and* $\sim_t$.

A mapping h *from* $\mathcal{L}_{\mathrm{DF}}$ *to* $\mathcal{L}_{\mathrm{DV}}$ *is defined inductively by:*

1. $h(p) := p$ *for any* $p \in \Phi$,
2. $h(\alpha \wedge_f \beta) := h(\alpha) \vee_t h(\beta)$,
3. $h(\alpha \vee_f \beta) := h(\alpha) \wedge_t h(\beta)$,
4. $h(\alpha \rightarrow_f \beta) := h(\alpha) \leftarrow_t h(\beta)$,
5. $h(\alpha \leftarrow_f \beta) := h(\alpha) \rightarrow_t h(\beta)$,
6. $h(\sim_f \alpha) := \sim_t h(\alpha)$.

Theorem 4.2 (Strong embedding from DF into DV). *Let* Γ *be a set of formulas in* $\mathcal{L}_{\mathrm{DF}}$, γ *be a formula in* $\mathcal{L}_{\mathrm{DF}}$ *or the empty sequence, and* h *be the mapping defined in Definition 4.1.*

1. $\mathrm{DF} \vdash \gamma \Rightarrow \Gamma$ *iff* $\mathrm{DV} \vdash h(\Gamma) \Rightarrow h(\gamma)$.
2. DF − (f-cut) $\vdash \gamma \Rightarrow \Gamma$ *iff* DV − (t-cut) $\vdash h(\Gamma) \Rightarrow h(\gamma)$.

Proof. We show only (1) since (2) can be obtained as a subproof of (1). We show only the direction ($\Longrightarrow$) of (1) by induction on the proof P of $\gamma \Rightarrow \Gamma$ in DF. We distinguish the cases according to the last inference of P, and show some cases.

Case ($\sim_f\rightarrow_f$l): The last inference of P is of the form:

$$\frac{\sim_f\beta \Rightarrow \Gamma, \sim_f\alpha}{\sim_f(\alpha \rightarrow_f \beta) \Rightarrow \Gamma}\ (\sim_f\rightarrow_f\mathrm{l}).$$

By induction hypothesis, we have $\mathrm{DV} \vdash h(\Gamma), h(\sim_f\alpha) \Rightarrow h(\sim_f\beta)$ where $h(\sim_f\alpha)$ and $h(\sim_f\beta)$ respectively coincide with $\sim_t h(\alpha)$ and $\sim_t h(\beta)$ by the definition of h. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ h(\Gamma), \sim_t h(\alpha) \Rightarrow \sim_t h(\beta)\end{array}}{h(\Gamma) \Rightarrow \sim_t(h(\alpha) \leftarrow_t h(\beta))}\ (\sim_t\leftarrow_t\mathrm{r})$$

as $\sim_t(h(\alpha) \leftarrow_t h(\beta)) = h(\sim_f(\alpha \rightarrow_f \beta))$.

Case ($\sim_f\rightarrow_f$r): The last inference of P is of the form:

$$\frac{\sim_f\alpha \Rightarrow \Gamma \quad \gamma \Rightarrow \Delta, \sim_f\beta}{\gamma \Rightarrow \Gamma, \Delta, \sim_f(\alpha \rightarrow_f \beta)}\ (\sim_f\rightarrow_f\mathrm{r}).$$

By induction hypothesis, we have $\mathrm{DV} \vdash h(\Gamma) \Rightarrow h(\sim_f\alpha)$ and $\mathrm{DV} \vdash h(\Delta), h(\sim_f\beta) \Rightarrow h(\gamma)$ where $h(\sim_f\alpha)$ and $h(\sim_f\beta)$ respectively coincide with $\sim_t h(\alpha)$ and $\sim_t h(\beta)$ by the definition of h. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ h(\Gamma) \Rightarrow \sim_t h(\alpha)\end{array} \quad \begin{array}{c}\vdots\\ h(\Delta), \sim_t h(\beta) \Rightarrow h(\gamma)\end{array}}{\sim_t(h(\alpha) \leftarrow_t h(\beta)), h(\Gamma), h(\Delta) \Rightarrow h(\gamma)}\ (\sim_t\leftarrow_t\mathrm{l})$$

as $\sim_t(h(\alpha) \leftarrow_t h(\beta)) = h(\sim_f(\alpha \rightarrow_f \beta))$.

Case ($\sim_f\leftarrow_f$l): The last inference of P is of the form:

$$\frac{\Rightarrow \Gamma, \sim_f\alpha \quad \sim_f\beta \Rightarrow \Delta}{\sim_f(\alpha \leftarrow_f \beta) \Rightarrow \Gamma, \Delta}\ (\sim_f\leftarrow_f\mathrm{l}).$$

By induction hypothesis, we have $\mathrm{DV} \vdash h(\Gamma), h(\sim_f\alpha) \Rightarrow$ and $\mathrm{DV} \vdash h(\Delta) \Rightarrow h(\sim_f\beta)$ where $h(\sim_f\alpha)$ and $h(\sim_f\beta)$ respectively coincide with $\sim_t h(\alpha)$ and $\sim_t h(\beta)$ by the definition of h. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ h(\Gamma), \sim_t h(\alpha) \Rightarrow\end{array} \quad \begin{array}{c}\vdots\\ h(\Delta) \Rightarrow \sim_t h(\beta)\end{array}}{h(\Gamma), h(\Delta) \Rightarrow \sim_t(h(\alpha) \rightarrow_t h(\beta))}\ (\sim_t\rightarrow_t\mathrm{r})$$

as $\sim_t(h(\alpha) \rightarrow_t h(\beta)) = h(\sim_f(\alpha \leftarrow_f \beta))$.

Case ($\sim_f\rightarrow_f$r1): The last inference of P is of the form:

$$\frac{\gamma \Rightarrow \Gamma, \sim_f\beta}{\gamma \Rightarrow \Gamma, \sim_f(\alpha \leftarrow_f \beta)}\ (\sim_f\leftarrow_f\mathrm{r1}).$$

By induction hypothesis, we have $\mathrm{DV} \vdash h(\Gamma), h(\sim_f\beta) \Rightarrow h(\gamma)$ where $h(\sim_f\beta)$ coincides with $\sim_t h(\beta)$ by the definition of h. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ h(\Gamma), \sim_t h(\beta) \Rightarrow h(\gamma)\end{array}}{\sim_t(h(\alpha) \rightarrow_t h(\beta)), h(\Gamma) \Rightarrow h(\gamma)}\ (\sim_t\rightarrow_t\mathrm{l1})$$

as $\sim_t(h(\alpha) \rightarrow_t h(\beta)) = h(\sim_f(\alpha \leftarrow_f \beta))$.

Case ($\sim_f\rightarrow_f$r2): The last inference of P is of the form:

$$\frac{\sim_f\alpha \Rightarrow \Gamma}{\Rightarrow \Gamma, \sim_f(\alpha \leftarrow_f \beta)}\ (\sim_f\leftarrow_f\mathrm{r2}).$$

By induction hypothesis, we have $\mathrm{DV} \vdash h(\Gamma) \Rightarrow h(\sim_f\alpha)$ where $h(\sim_f\alpha)$ coincides with $\sim_t h(\alpha)$ by the definition of h. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ h(\Gamma) \Rightarrow \sim_t h(\alpha)\end{array}}{\sim_t(h(\alpha) \rightarrow_t h(\beta)), h(\Gamma) \Rightarrow}\ (\sim_t\rightarrow_t\mathrm{l2})$$

as $\sim_t(h(\alpha) \rightarrow_t h(\beta)) = h(\sim_f(\alpha \leftarrow_f \beta))$. ∎

We can introduce a translation from DV into DF in a similar way.

Definition 4.3. *Φ, $\mathcal{L}_{\rm DF}$ and $\mathcal{L}_{\rm DV}$ are the same as in Definition 4.1.*

A mapping k from $\mathcal{L}_{\rm DV}$ to $\mathcal{L}_{\rm DF}$ is defined inductively by:

1. $k(p) := p$ *for any* $p \in \Phi$,
2. $k(\alpha \wedge_t \beta) := k(\alpha) \vee_f k(\beta)$,
3. $k(\alpha \vee_t \beta) := k(\alpha) \wedge_f k(\beta)$,
4. $k(\alpha \rightarrow_t \beta) := k(\alpha) \leftarrow_f k(\beta)$,
5. $k(\alpha \leftarrow_t \beta) := k(\alpha) \rightarrow_f k(\beta)$,
6. $k(\sim_t \alpha) := \sim_f k(\alpha)$.

Theorem 4.4 (Strong embedding from DV into DF)**.** *Let Γ be a set of formulas in $\mathcal{L}_{\rm DV}$, γ be a formula in $\mathcal{L}_{\rm DV}$ or the empty sequence, and k be the mapping defined in Definition 4.3.*

1. $\mathrm{DV} \vdash \Gamma \Rightarrow \gamma$ *iff* $\mathrm{DF} \vdash k(\gamma) \Rightarrow k(\Gamma)$.
2. $\mathrm{DV} - (\text{t-cut}) \vdash \Gamma \Rightarrow \gamma$ *iff* $\mathrm{DF} - (\text{f-cut}) \vdash k(\gamma) \Rightarrow k(\Gamma)$.

Proof. Similar to Theorem 4.2. ∎

Some remarks are given as follows.

1. The cut-elimination theorems for DV and DF can be obtained using Theorems 4.2 and 4.4.
2. The following hold for DV and DF:
   (a) $\mathrm{DV} \vdash hk(\Gamma) \Rightarrow hk(\gamma)$ iff $\mathrm{DV} \vdash \Gamma \Rightarrow \gamma$,
   (b) $\mathrm{DF} \vdash kh(\gamma) \Rightarrow kh(\Gamma)$ iff $\mathrm{DF} \vdash \gamma \Rightarrow \Gamma$.
3. A similar theorem for embedding N4 into its dual-like version displayed in Section 3 cannot be shown. Thus, the duality principle for these logics do not hold.

5 CONCLUSIONS

In this paper, the new logics DV and DF which have the De Morgan-like laws with respect to the implication and co-implication connectives were introduced as a Gentzen-type sequent calculus. DV and DF are natural extensions of the positive intuitionistic logic and the positive dual-intuitionistic logic, respectively. Some theorems for embedding DV and DF into their negation-free fragments were proved, and some theorems for embedding DV into DF and vice versa were shown. By using these embedding theorems, the cut-elimination theorems for DV and DF were obtained, and DV and DF were also shown to be paraconsistent and decidable. Also, as remarked in Section 1, the logics DV and DF are suitable for representing verification and falsification of incomplete information. We thus believe that DV and DF are a promising basis for many computer science applications, as for the intuitionistic and dual-intuitionistic logics.

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