Illustrating the Difficulties of Zimmermann Method for Solving the Fuzzy Linear Programming by the Geometric Approach

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Abstract: In this paper we first recall Zimmermann method and the Geometric approach for solving fuzzy linear programming problem. We show, by the geometric approach, Zimmermann method has some difficulties. Numerical examples are provided for illustrating the difficulties. Finally, the IZM algorithm for improving Zimmermann method is recalled.

1 INTRODUCTION

Following "Decision Making in Fuzzy Environment" proposed by (Bellman and Zadeh, 1970) and "On Fuzzy Mathematical Programming" proposed by (Tanaka et al., 1974), Zimmermann, (1976) first introduced FLP as a conventional LP. Since then, FLP has been developed in a number of directions with many wide applications. Among the others, the approach of (Verdegay, 1982) and (Chanas, 1983) which presents a parametric programming method for solving FLP, is the most often used. Guu and Wu (1999) developed a two-phase approach for solving the problem, which concentrates on the fuzzy efficiency of solutions. Safi et al., (2007) showed some difficulties in ZM by algebraic approach. They proposed an algorithm (IZM algorithm), that eliminates these difficulties.

The majority of studies for handling FLP problems focus on developing different algebraic methods. Safi et al., (2007) used the fuzzy geometry proposed by (Rosenfeld, 1994) and presented a geometric approach for solving FLP problems.

In this note we illustrate the difficulties of Zimmermann method (ZM) by the geometric approach.

2 THE ZIMMERMANN METHOD

Consider the following general form of the FLP problem:

\[
\begin{align*}
\max z &= \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad &\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
&x_j \geq 0
\end{align*}
\]

(2.1)

where, \(\max\) and \(\leq\) denote the relaxed or fuzzy versions of the ordinary \(\max\) and \(\leq\) symbols, respectively. For representing the fuzzy goal, let us assume that the objective function must be essentially greater than or equal to an aspiration level \(b_0\) that has been chosen by the decision maker (DM). Then we consider the following problem:

\[
\begin{align*}
\text{find} \quad x &= (x_1, \ldots, x_n) \\
\text{s.t.} \quad &\sum_{j=1}^{n} c_j x_j \geq b_0 \\
&\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
&x_j \geq 0
\end{align*}
\]

(2.2)

The above fuzzy inequalities can be interpreted as the fuzzy subsets \(\tilde{C}^i, i = 0, 1, \ldots, m\) of \(\mathbb{R}^n\) such that \(\tilde{C}^i = \{(x, \tilde{c}^i(x)), x \geq 0\}, i = 0, 1, \ldots, m\), and

\[
\tilde{c}^i(x) = \begin{cases}
1 & \text{if} \quad \sum_{j=1}^{n} x_j \geq b_i \\
\frac{h_i - \sum_{j=1}^{n} x_j}{p_i} & \text{if} \quad h_i - p_i \leq \sum_{j=1}^{n} x_j < b_i, \quad i = 1, 2, \ldots, m \\
0 & \text{if} \quad \sum_{j=1}^{n} x_j < b_i - p_i
\end{cases}
\]

(2.3)
3.2 Feasibility and Optimality

The following definitions and theorems are from Safi et al., (2007).

Definition 3.2.1. Consider FLP problem (2.2). \( S = \bigcap_{i=1}^{m} C_i \) is called the fuzzy feasible space, and \( D = S \cap C^0 \) is called the fuzzy decision space. Here we use the min-operator for intersection.

Definition 3.2.2. A point \( x \in \mathbb{R} \) is called a \( \lambda \)-feasible point of \( S \) if \( \tilde{S}(x) \geq \lambda \).

Theorem 3.2.3. Every convex combination of two \( \lambda \)-feasible points of \( S \) is again a \( \lambda \)-feasible point of \( S \).

Definition 3.2.4. For \( \tilde{A} \neq \phi \), set \( \alpha^* = \text{sup} \{ \alpha | \tilde{A}_\alpha = \phi \} \). Then \( \tilde{A}_{\alpha^*} \) is called the nonempty supremum cut (NSC) of the fuzzy set \( \tilde{A} \) and denoted by NSC(\( \tilde{A} \)).

Definition 3.2.5. Let \( \tilde{D} \) be the fuzzy decision space for the problem (2.1). For \( \tilde{D} \neq \phi \), NSC(\( \tilde{D} \)) = \( \tilde{D}_{\alpha^*} \) is called the set of optimal solutions with the optimal value \( \alpha^* \). If \( \tilde{D} = \phi \), we say that the problem does not have any optimal solution.

Safi et al., (2007) discussed the optimal solution and the optimal objective value in Definition 3.2.5 which are completely consistent with those in ZM.

4 ILLUSTRATING THE DIFFICULTIES

Safi et al., (2007) has investigated some difficulties in ZM from the algebraic viewpoint. In this section we study the difficulties by means of the geometric approach.

Example 4.1. Consider the following problem:

\[
\begin{align*}
\max z &= x_1 + x_2 \\
\text{s.t.} \\
-2x_1 + 2x_2 &\leq 3 \\
2x_1 + 2x_2 &\leq 12 \\
x_1, x_2 &\geq 0.
\end{align*}
\]

Let \( b_0 = 3 \), \( p_1 = 1 \), \( p_2 = 2 \), \( p_3 = 3 \), \( p_4 = 3 \). The Zimmermann algebraic method solves the associated problem (2.5) and obtains the alternative optimal solutions: \( x^*_1 = (3, 0) \), \( x^*_2 = (0, 3) \), \( x^*_3 = (0, 8.4667), (4.6667, 2.6667) \) with \( \lambda^* = 1 \). The geometric approach provides Figures 4-1...
and 4-2 as $\bar{D}$ and its contour plot, respectively. The set of optimal solutions, $NSC(\bar{D})$, is the innermost (white) 5-gone in Figure 4-2. In this figure the alternative basic optimal solutions are the extreme points: $x_A^*, x_B^*, x_C^*, x_D^*$ and $x_E^*$. Also $NSC(\bar{D}) = \bar{D}_1$, therefore $\lambda^* = 1$.

The line segment between $x_A^*$ and $x_B^*$ is the objective function with the value 3, i.e., $x_1 + x_2 = 3$. Clearly $z = x_1 + x_2$ attains the maximum value in $x_E^*$.

Since the purpose of ZM is to obtain the best value for $\lambda^*$, it does not prefer one of the AOS to the others. Therefore, unless we check the value of $z$ for all AOS, it is possible to introduce (for example) $x_A^*$ to the DM as the optimal solution, ignoring the fact that the best value for $z$ occurs at $x_E^*$.

When we solve this problem by WinQSB, a cycling happens between $x_A^*$ and $x_B^*$, hence only these two solutions is shown, as the alternative basic optimal solutions. Thus, the other three alternative basic optimal solutions, those give better values for $z$, have been lost.

In the final example, ZM obtains an optimal solution with a finite value for $z$, whereas the optimal value of $z$ is unbounded.

**Example 4.2.** Consider the following problem:

$$\begin{align*}
\text{max} & \quad z = x_1 + x_2 \\
\text{s.t.} & \quad 2x_1 - 5x_2 \leq 10 \\
& \quad 5x_1 - 2x_2 \geq 30 \\
& \quad x_1, x_2 \geq 0
\end{align*}$$

Let $b_0 = 6$, $p_0 = 1$, $p_1 = 2$, $p_2 = 3$. The geometric approach gives Figures 4-3 and 4-4 as $\bar{D}$ and its contour plot, respectively. The set of optimal solutions, $NSC(\bar{D})$, is the above region in figure 4-3. Clearly the objective function $z = x_1 + x_2$ can be increased in the white region to infinity. ZM does not distinguish this case and presents a solution with finite value. That is because the alternative optimal basic solutions, are $x_A^* = (5.7143, 0.2857)$, $x_B^* = (6.1905, 0.4762)$ and $x_C^* = (0, 6.0)$, which none of them gives the best value for $z$.

5 THE IZM ALGORITHM

Safi et al., (2007) proposed the following algorithm for improving ZM and called it "Improved
Zimmermann Method (IZM):

**Step 1.** for solving (2.1), take values $b_i$ and $p_i$; $i=0,1,...,m$, from the DM.

**Step 2.** Solve (2.5) for obtaining the optimal $(x^*, \lambda^*)$.

**Step 3.** If problem (2.5) does not have any feasible solution; Stop. If it has AOS, then go to step 4. Else, $z^* = cx^*$ is the best value for $z$. Stop.

**Step 4.** Solve the following LP problem:

$$\begin{align*}
    \text{max} & \quad z = cx \\
    \text{s.t.} & \quad c^T x \geq b_i - (1 - \lambda^*) p_i \\
    & \quad (A^T x) \geq b_i + (1 - \lambda^*) p_i \\
    & \quad x \geq 0.
\end{align*}$$

If problem (4.2) is unbounded, stop. Let $x^*$ be the optimal solution of (4.2). If the set of all AOS is not singleton go to Step 5. Else, Stop.

**Step 5.** (Efficiency, Guu and Wu, 1999) Solve:

$$\begin{align*}
    \text{max} & \quad \sum_{i=0}^{m} \lambda_i \\
    \text{s.t.} & \quad A^T x \geq \lambda_i \geq A^T x^* \\
    & \quad c^T x = c^T x^* \\
    & \quad x \geq 0.
\end{align*}$$

**REFERENCES**


