A TRACTABLE FORMALISM FOR COMBINING RECTANGULAR CARDINAL RELATIONS WITH METRIC CONSTRAINTS

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Abstract: Knowledge representation and reasoning in real-world applications often require to integrate multiple aspects of space. In this paper, we focus our attention on the so-called Rectangular Cardinal Direction calculus for qualitative spatial reasoning on cardinal relations between rectangles whose sides are aligned to the axes of the plane. We first show how to extend a tractable fragment of such a calculus with metric constraints preserving tractability. Then, we illustrate how the resulting formalism makes it possible to represent available knowledge on directional relations between rectangles and to derive additional information about them, as well as to deal with metric constraints on the height/width of a rectangle or on the vertical/horizontal distance between rectangles.

1 INTRODUCTION

Qualitative spatial representation and reasoning play an important role in various areas of computer science such as, for instance, geographic information systems, spatial databases, document analysis, layout design, and image retrieval. Different aspects of space, such as direction, topology, size, and distance, which must be dealt with in a coherent way in many real-world applications, have been modeled by different formal systems (Broxvall, 2002; Condotta, 2000; Gerevini and Renz, 2002; Liu et al., 2009) (see (Cohn and Hazarika, 2001) for a survey). For practical reasons, a bidimensional space is commonly assumed, and spatial entities are represented by points, boxes, or polygons with a variety of shapes, depending on the required level of detail.

Information about spatial configurations is usually specified by constraint networks describing the allowed binary relations between pairs of spatial variables. The central problem in qualitative reasoning is consistency checking, which is the problem of deciding whether or not a network has a solution, that is, the problem of establishing whether or not there exists an assignment of domain values to variables that satisfies all constraints.

Cardinal relations are directional relationships that allow one to specify how spatial objects are placed relative to one another either by making use of a fixed reference system, e.g., to say that an object is to the “north” or “southwest” of another one in a geographic space, or, alternatively, by exploiting directions as “above” or “below and left” in a local space. Cardinal relations are of particular interest for geographic information systems, spatial databases, and image databases (Frank, 1996; Goyal, 2000; Papadias and Theodoridis, 1997; Skiaidopoulos et al., 2005).

The most expressive formalism with cardinal relations between extended spatial objects is the Cardinal Direction calculus, CD-calculus for short (Goyal and Egenhofer, 2000; Liu et al., 2010; Skiaidopoulos and Koubarakis, 2005). The consistency problem for the CD-calculus is NP-complete, and no tractable fragment of it has been identified so far, with the only exception of the fragment obtained by forbidding disjunctive relations (Skiadopoulos and Koubarakis, 2005). Such a restriction is a serious limitation when we have to deal with incomplete or indefinite information in spatial applications.

In (Navarrete and Sciavicco, 2006), the authors introduce a restricted version of the CD-calculus called Rectangular Cardinal Direction calculus (RCD-calculus), where cardinal relations are defined only between rectangles whose sides are parallel to the axes of the Euclidean plane. Rectangles of this type (boxes) can be seen as minimum bound-
ing rectangles (MBRs) that enclose plane regions (the actual spatial objects). MBRs have been widely used in spatial databases (El-Geresy and Abdelmoty, 2001; Papadias and Theodoridis, 1997), in web-document analysis (Gatterbauer and Bohunsky, 2006), and in 2D-layout design, e.g., in architecture (Baykan and Fox, 1997). On the one hand, approximating regions by rectangles implies a loss of accuracy in the representation of the relative direction between regions; on the other hand, reasoning tasks become more efficient.

The RCD-calculus has a strong connection with the Rectangle Algebra (RA) (Balbiani et al., 1998), which can be viewed as a bidimensional extension of Interval Algebra (IA), the well-known temporal formalism for dealing with qualitative binary relations between time intervals (Allen, 1983). A tractable fragment of the RCD-calculus, named convex RCD-calculus, has been identified by Navarrete et al. (Navarrete et al., 2011). It includes all basic relations and a large number of disjunctive relations, making it possible to represent and reason about indefinite information efficiently.

This paper aims at adding metric features to formalisms for qualitative spatial reasoning. Metric constraints between points over a dense linear order have been dealt with by the Temporal Constraint Satisfaction Problem formalism (TCSP) (Dechter et al., 1991). In such a formalism, one can constrain the distance between a pair of points to belong to a given set of intervals. If each constraint consists of one interval only, we get a tractable fragment of TCSP, called Simple Temporal Problem formalism (STP).

In the following, we propose a metric extension to the convex RCD-calculus that allows one to represent available knowledge on directional relations between rectangles and to derive additional information about them, as well as to deal with metric constraints on the height/width of a rectangle or on the vertical/horizontal distance between rectangles. We will show that the resulting formalism is expressive enough to capture various scenarios of practical interest and still computationally affordable.

The rest of the paper is organized as follows. In Section 2, we provide background knowledge on qualitative calculi and we shortly recall Interval Algebra and Rectangle Algebra. In Section 3, we introduce RCD-calculus and its convex fragment. In Section 4, we extend the convex RCD-calculus with metric constraints, and we devise a sound and complete polynomial algorithm for consistency checking. We conclude the section with a simple application example. Conclusions provide an assessment of the work and outline future research directions.

### 2 PRELIMINARIES

In this section, we introduce basic notions and terminology.

Temporal knowledge, as well as spatial knowledge, is commonly represented in a qualitative calculus by means of a qualitative network consisting of a complete constraint-labeled digraph $N = (V, C)$, where $V = \{v_1, \ldots, v_n\}$ is a finite set of variables, interpreted over an infinite domain $D$, and the labeled edges in $C$ specify the constraints describing qualitative spatial or temporal configurations. An edge from $v_i$ to $v_j$ labeled with $R$ corresponds to the constraint $v_i R v_j$, where $R$ denotes a binary relation over $D$ which restricts the possible values for the pair of variables $(v_i, v_j)$. The full set of relations of the calculus is usually taken as the powerset $2^D$, where $\mathcal{B}$ is a finite set of binary basic relations that forms a partition of $D \times D$. Thus, a relation $R_j \in 2^\mathcal{B}$ is of the form $R = \{r_1, \ldots, r_m\}$, where each $r_i$ is a basic relation, and $R$ represents the union of the basic relations it contains. If $m = 1$, we call $R$ a basic relation; otherwise, we call it a disjunctive relation. A special case of disjunctive relation is the universal relation, denoted by " $\top"$, which contains all the basic relations. A basic constraint $v_i r_j v_j$ expresses definite knowledge about the values that the two variables $v_i, v_j$ can take, while a disjunctive constraint $v_i \{r_1, \ldots, r_m\} v_j$ expresses indefinite or imprecise knowledge about these values. In particular, the universal constraint $v_i ? v_j$ states that the relation between $v_i$ and $v_j$ is totally unknown. From a logical point of view, a disjunctive constraint $v_i \{r_1, \ldots, r_m\} v_j$ can be viewed as the logical disjunction $v_i \{r_1\} v_j \lor \cdots \lor v_i \{r_m\} v_j$.

An instantiation (or interpretation) of the constraints of a qualitative network $N$ is a mapping $\tau$ representing an assignment of domain values to the variables of $N$. A constraint $v_i R v_j$ is said to be satisfied by an instantiation $\tau$ if the pair $(\tau(v_i), \tau(v_j))$ belongs to the binary relation represented by $R$. A consistent
instantiation, or solution, of a network is an assignment of domain values to variables satisfying all the constraints. If such a solution exists, then the network is consistent, otherwise it is inconsistent.

The main reasoning task in qualitative reasoning is consistency checking, which amounts to deciding if a network is consistent. If all relations are considered, consistency checking is usually NP-hard. Hence, finding subsets of $2^n$ for which consistency checking turns out to be polynomial (tractable subsets) is an important issue to address. Another common task is instantiation in a minimal network is feasible. If all relations are considered, some solution of the network. It can be easily shown that each basic relation is entailed by the constraints of the network. If the empty relation is obtained, the network is consistent. In some special cases, the network is inconsistent. Consistency checking techniques are usually exploited. The most prominent method for constraint propagation is the path-consistency algorithm, PC-algorithm for short (Mackworth, 1977). Such an algorithm refines relations by successively applying the operation $R_{ij} \leftarrow R_{ij} \cap (R_{ik} \circ R_{kj})$ for every triple of variables $(v_i, v_k, v_j)$, until a stable network is reached, where $R_{ij}, R_{ik}, R_{kj}$ are the relations constraining the pair of variables $(v_i, v_j), (v_k, v_j), (v_i, v_k)$, respectively ($\circ$ stands for the composition of relations). If the empty relation is obtained during the process, then the network is inconsistent; otherwise, we can conclude that the output network is path consistent, which does not necessarily imply that it is consistent. In some special cases, the PC-algorithm can be used to decide the consistency of a qualitative network and to get the minimal one.

2.1 Interval Algebra and Point Algebra

Allen’s Interval Algebra (IA) allows one to model the relative position of two temporal intervals (Allen, 1983). An interval $I$ is usually interpreted as a closed interval over the rational numbers $[-1, 1]$, whose endpoints $I'$ and $I''$ satisfy the relation $I' < I''$. Let $B_{IA}$ be the set of the thirteen basic interval relations capturing all possible ways to order the four endpoints of two intervals, usually denoted by the symbols $b, o, d, m, s, f, e, b_i, o_i, d_i, m_i, s_i$, and $f_i$. The semantics of basic IA-relations is defined in terms of ordering relations between the endpoints of the intervals, as shown in Figure 1. Notice that, given a basic relation $r$ between two intervals $I$ and $J$, the inverse relation $ri$ is defined by simply exchanging the roles of $I$ and $J$ (see Figure 1). IA can be viewed as a constraint algebra defined by the power set $2^{B_{IA}}$ and the operations of intersection, inverse ($^{-1}$), and composition ($\circ$) of relations.

IA subsumes Point Algebra, PA for short (Violain and Kautz, 1986), a simpler qualitative calculus whose binary relations specify the relative position of pairs of time points. PA binary relations are $\prec, \succ, =, \leq, \geq, \neq$ (disjunctive), plus the empty relation. The endpoint relations defining an IA-relation (Figure 1) are basic relations of PA.

2.2 Rectangle Algebra

Rectangle Algebra (RA), proposed by Balbiani et al. (1998), is an extension of IA to a bidimensional space. We assume here the domain of RA to consist of the set of rational rectangles whose sides are parallel to the axes of the Euclidean plane. To avoid a notational overload, with an abuse of notation, hereafter we will denote by $a, b$ both rectangles in the domain of RA and constraint (rectangle) variables. A rectangle $a$ is completely characterized by a pair of intervals $(a_x, a_y)$, where $a_x$ and $a_y$ are the projections of $a$ onto the $x$- and $y$-axis, respectively. We call $B_{RA}$ the set of basic relations of RA, which is obtained by considering all possible pairs of basic IA-relations. Hence, a basic RA-relation $r$ is denoted by a pair $r = (r_x, r_y)$ of basic IA-relations, representing the set of pairs of rectangles $(a, b)$ such that $a \succ r_x b$ and $a \succ r_y b$. Given a basic RA-relation $r = (r_x, r_y)$, let $t = \pi_x(r)$ and $t' = \pi_y(r)$ be the $x$- and $y$-projection of $r$, respectively.

Example 1. Figure 2 shows a spatial realization of the basic RA-constraint $a \{o, bi\} b$. We have that $\pi_x(o, bi) = o, \pi_y(o, bi) = bi, a_x$ overlaps $b_x$, and $a_y$ is after $b_y$. The left endpoints of the intervals assigned to $a_x$ and $a_y$ (1 and 5.9, respectively) and their right endpoints (4.6 and 8, respectively) are the coordinates of the lower-left and upper-right vertices of the given instantiation of $a$, respectively. The same for $b$. Thus, the values assigned to the endpoints of the projections of $a$ and $b$ represent an assignment for $a$ and $b$ that satisfies the constraint $a \{o, bi\} b$.

In the case of an arbitrary RA-relation $R \in 2^{B_{RA}}$, the projections of $R$ are defined as follows:

$$\pi_x(R) = \{\pi_x(r) \mid r \in R\}, \quad \pi_y(R) = \{\pi_y(r) \mid r \in R\}.$$  

Note that, in general, $\pi_x(R) \times \pi_y(R)$ may be different from $R$ or, equivalently, we may have $\pi_x(R_1) = \pi_x(R_2)$ and $\pi_y(R_1) = \pi_y(R_2)$ for some $R_1 \neq R_2$.

The mappings $\pi_x$ and $\pi_y$ can be generalized to RA-networks. We define the projections $\pi_x$ and $\pi_y$.  

Footnote 1: An extension of RA to n-dimensional spaces can be found in Balbiani et al. (2002).
of an RA-network \(N = (V, C)\) as the two IA-networks \(\pi_i(N) = (V_i, C_i)\) and \(\pi_j(N) = (V_j, C_j)\), where \(V_i, V_j\) are the sets of interval variables corresponding to the rectangle variables in \(V\) and the set of IA-constraints \(C_i\) (resp., \(C_j\)) is obtained by replacing each relation \(R_{ij}\) in \(C\) by \(\pi_i(R_{ij})\) (resp., by \(\pi_j(R_{ij})\)).

### 2.3 Convex Subalgebras

The consistency problem for both IA and RA is known to be NP-complete. Several tractable fragments of both calculi have been identified in the literature. In this paper, we focus our attention on convex subalgebras of RA (Balbiani et al., 1998), which consist of the set of IA-constraints (basic RA-relations) or between the endpoints of interval variables (convex IA-relations) or between the endpoints of the projections of rectangle variables (convex RA-relations). It is worth to mention that a convex RA-relation is equivalently characterized as a RA-relation which can be obtained as the Cartesian product of two convex IA-relations. A PC-algorithm can be used to solve both the consistency and the minimality problems in the convex fragments of PA, IA, and RA in \(O(n^3)\), where \(n\) is the number of variables of the input network.

### 3 RECTANGULAR CARDINAL DIRECTION CALCULUS

The Rectangular Cardinal Direction calculus (RCD-calculus, for short) (Navarrete and Sciavicco, 2006; Navarrete et al., 2011) deals with cardinal direction relations between rectangles. Hence, its domain is the same as that of RA. Let \(b\) be a reference rectangle. We denote by \(b^+\) (resp., \(b^*\) and \(b^-\)) the left and the right endpoint of the projection of \(b\) onto the \(x\)-axis (resp., \(y\)-axis), respectively. The

Figure 2: An instantiation of the RA-constraint \(a((a, bi)b).\) The corresponding RCD-relation is \(a\{NW:N\}b\)

Figure 3: (a) Cardinal tiles with respect to rectangle \(b\). (b) A possible instantiation of the RCD-constraint \(aB:N:NE:E\).

straight lines \(x = b^-\), \(x = b^+\), \(y = b^-\), \(y = b^+\) divide the plane into nine tiles \(\tau_i(b)\), with \(1 \leq i \leq 9\), as shown in Figure 3-(a), where \(\tau_i\) is a tile symbol from the set \(TS = \{B,S,SW,W,NW,N,NE,E,SE\}\), denoting the cardinal directions in the Bounds of, to the South of, to the Southwest of, to the West of, to the SouthEast of, to the East of, and to the Southeast of, respectively.

**Definition 1.** A basic rectangular cardinal relation (basic RCD-relation) is denoted by a tile string \(\tau_1\tau_2\ldots\tau_k\), where \(\tau_i \in TS\), for \(1 \leq i \leq k\), such that \(a\tau_1\tau_2\ldots\tau_k b\) holds iff for all \(\tau_i \in \{\tau_1, \tau_2, \ldots, \tau_k\}\), \(a^* \cap \tau_i(b) \neq \emptyset\), and for all \(\tau_i \in TS \setminus \{\tau_1, \tau_2, \ldots, \tau_k\}\), \(a^* \cap \tau_i(b) = \emptyset\), where \(a^*\) is the interior of \(a\). A rectangular cardinal relation (RCD-relation) is represented by a set \(R = \{r_1, \ldots, r_m\}\), where each \(r_i\) is a basic RCD-relation.

As usual, if \(R\) is a singleton, then it is a basic RCD-relation; otherwise, it is a disjunctive one.

The set \(\beta_{RCD}\) of basic RCD-relations consists of 36 elements (see Figure 4). Qualitative networks with labels in \(2^{\beta_{RCD}}\), as well as the consistency problem for such networks, are defined in the standard way.

The RCD-calculus can be viewed as a restricted version of the CD-calculus over the domain of regular regions (Goyal and Egenhofer, 2000; Liu et al., 2010; Skiadopoulos and Koubarakis, 2005), which includes all rectangles aligned to the axes. Let \(a, b\) denote regions. A cardinal relation is defined by considering the exact shape of a primary region \(a\) and the minimum bounding rectangle (MBR) of the reference region \(b\), where MBR(b) is the smallest rectangle aligned to the axes of the plane that encloses \(b\). There are 218 CD-relations over connected regions, that become 512 if we allow disconnected regions. Cardinal relationships between regions may be approximated by RCD-relations between their MBRs, with a possible loss of accuracy when the regions are non-convex or diagonal. The advantage of the RCD-calculus over the CD-calculus is its simplicity (only 36 basic rela-
tions), which leads to a better computational behavior, also when disjunctive relations are considered.

**Example 2.** Figure 3-(b) shows a possible instantiation of the CD-constraint \( a:B:N:E:b \). We indeed have that a lies partly in the bounds, partly to the north, and partly to the east of \( MBR(b) \). Alternatively, the pair \((MBR(a), MBR(b))\) in Figure 3-(b) can be viewed as an instantiation of the RCD-constraint \( a:B:N:NE:E:b \), as it holds that \( MBR(a):B:N:NE:E MBR(a) \). Notice that while the CD-constraint exactly specifies the direction of region \( a \) with respect to the minimum bounding rectangle of region \( b \), the direction expressed by the RCD-constraint is just approximated, since \( a \) does not intersect the tile \( NE(b) \) \((= NE(MBR(b)))\), that is, \( a \) does not lie partly to the northeast of \( NE(b) \). Notice also that, in general, a basic CD-constraint \( aRb \) alone does not provide definite information about the relative direction of pairs of regions. For that purpose, both \( aRb \) and \( bRa \) must be specified.

### 3.1 RCD and RA

The relationships between RCD and RA have been systematically investigated in (Navarrete et al., 2011). For instance, consider the RCD-constraint \( a:\{NW:N\} b \). A possible instantiation of such a constraint is depicted in Figure 2. The very same pair of rectangles can be viewed as an instantiation of the RA-constraint \( a:b \). A possible instantiation of such a constraint is given in Figure 2. In general, for a given RCD-constraint there exist more than one corresponding RA-constraints, while for a given RA-constraint there exist exactly one corresponding RCD-constraint. This is due to the coarseness of RCD-relations with respect to RA-relations. As an example, RCD does not allow one to precisely state that two given rectangles are externally connected or strictly disconnected, or to constrain their sides to be (or to be not) vertically (resp., horizontally) aligned. As a general rule, given an RCD-relation, we can always determine the strongest RA-relation it implies. As an example, the strongest RA-relation implied by \( NW:N \) is \( \{fi,o \} \times \{mi,bi\} \). Notice that such an RA-relation, which is entailed by a basic RCD-relation, is not a basic RA-relation.

The weaker expressive power of RCD with respect RA is not necessarily a problem. As an example, if an application is interested in pure cardinal information only, the expressiveness of RCD-relations suffices. Moreover, the constraint language of the RCD-calculus is closer to the natural language than the one of the RA. For example, stating that "rectangle \( a \) lies partly to the northwest and partly to the north of \( b \)" \((a\{NW:N\}b)\) is much more natural than stating that "the \( x \)-projection of \( a \) is overlapping or finished by the \( x \)-projection of \( b \), and the \( y \)-projection of \( b \) is ... " \((a\{fi,o\} \times \{mi,bi\} \times \{x,bi\})\).

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![Figure 4: Translation from basic RCD-relations to RA-relations via toRA mapping.](Image)

Figure 4 describes a translation function, called \( toRA \), to map a basic RCD-relation into the strongest entailed RA-relation. This mapping can be extended to translate arbitrary relations, constraints, and networks of RCD-calculus to their counterparts in RA, preserving consistency. More precisely, given a disjunctive relation \( R \), \( toRA(R) \) is obtained as the union of the translation of the basic relations in \( R \), while, given an RCD-network \( N = (V,C) \), the corresponding RA-network \( toRA(N) \) is obtained by replacing each relation \( R_{ij} \) in \( C \) by \( toRA(R_{ij}) \). As the following theorem states, to decide the consistency of an RCD-network \( N \), one can compute the corresponding RA-network \( toRA(N) \) and then apply any algorithm for deciding the consistency of RA-networks (Navarrete et al., 2011).

**Theorem 1.** An RCD-network \( N \) is consistent if and only if the RA-network \( toRA(N) \) is consistent.

### 3.2 The Convex Fragment of RCD

In (Navarrete and Sciavicco, 2006), the authors prove that the consistency problem for the RCD-calculus is NP-complete, and they identify a tractable subset of RCD-relations. A larger tractable fragment of RCD-calculus, called convex RCD-calculus, has been identified in (Navarrete et al., 2011). Such a fragment consists of all and only the RCD-relations \( R \) whose translation \( toRA(R) \) is a convex RA-relation (convex RCD-relations). It is possible to show that there exist 400 such relations.

As we already pointed out, the convex subclasses
Algorithm 3.1: the algorithm con-cRCD.

Require: a convex RCD-network $N$

1: $N_c \leftarrow \text{toRA}(N)$;
2: $N_i \leftarrow \pi_{\overline{c}}(N_c)$; $N_r \leftarrow \pi_{\overline{r}}(N_c)$;
3: $N^P_c \leftarrow \text{toPA}(N_i)$; $N^P_r \leftarrow \text{toPA}(N_r)$;
4: If $\text{CSPAN}(N^P_c)$ or $\text{CSPAN}(N^P_r)$ returns an empty network, then return ‘inconsistent’; otherwise, return ‘consistent’.

of IA, PA, and RA are tractable and PC-algorithms can be used to decide their consistency. In particular, the following result holds for RA (Balbiani et al., 1998):

Theorem 2. Let $N$ be a convex RA-network. $N$ is path-consistent (resp., consistent) iff its projections $\pi_{\overline{c}}(N)$ and $\pi_{\overline{r}}(N)$ are path-consistent (resp., consistent). Moreover, if $N$ is path consistent, then it is consistent.

Making use of the above results, polynomial-time algorithms to solve the consistency and the minimality problems for convex RCD-networks have been proposed in (Navarrete et al., 2011). In the following, we will exploit one of these algorithms, called con-cRCD, that solves the two PA-networks corresponding to a convex RCD-network. Such an algorithm can be summarized as follows. Let $N$ be a convex RCD-network. First, it applies the mapping toRA to get the convex RA-network $N_c$ corresponding to $N$. Then, it computes the projections $N_i$ and $N_r$ of $N_c$. Next, it applies the mapping toPA to translate the convex IA-networks $N_i$ and $N_r$ into two equivalent PA-networks $N^P_c$ and $N^P_r$ with convex relations between intervals endpoints. Such a mapping is based on the list of the convex IA-relations and of their translations to PA given in (van Beek and Cohen, 1990). Finally, the algorithm CSPAN (van Beek, 1992) is applied to decide the consistency of the two convex PA-networks in $O(n^2)$ (we assume that this algorithm returns an empty network in case the input network is inconsistent). It can be easily shown that such an algorithm runs in $O(n^2)$. Algorithm 3.1 provides a pseudocode encoding of con-cRCD.

4 CONVEX-METRIC RCD

In this section, we propose a tractable metric extension of the convex RCD-calculus, called convex-metric RCD, to represent and to reason with both qualitative cardinal constraints between rectangles and metric constraints on the distance between the endpoints of their projections.

4.1 STP

The main tool we use to deal with metric information in convex-metric RCD is the STP formalism, which was introduced in (Dechter et al., 1991) to process metric information about time points. More precisely, we use STP to elaborate information on the endpoints of MBR projections onto the Cartesian axes.

Formally, an STP is specified by a constraint network $S = (P,M)$, where $P$ is a set of point variables, whose values range over a dense domain (we assume it to be $\mathbb{Q}$), and $M$ is a set of binary metric constraints over $P$. A metric constraint $M_{ij} = [q,q']$ (open and semi-open intervals can be used), with $q,q' \in \mathbb{Q}$, on the distance between (the values of) $p_i,p_j \in P$ states that $p_j - p_i \in [q,q']$, or, equivalently, that $q \leq p_j - p_i \leq q'$. Hence, the constraint $M_{ij}$ defines the set of possible values for the distance $p_j - p_i$.

In the constraint graph associated to $S$, $M_{ij} = [q,q']$, is represented by an edge from $p_i$ to $p_j$, labeled by the rational interval $[q,q']$. Unary metric constraints restricting the domain of a point variable $p_i$ can be encoded as binary constraints between $p_i$ and a special starting-point variable with a fixed value, e.g., 0. The universal constraint is $]-\infty, +\infty[$. The operations of composition ($\circ$) and inverse ($\cdot$) of metric constraints are computed by means of interval arithmetic, that is, $[q_1,q_2] \circ [q_3,q_4] = [q_1 + q_3, q_2 + q_4]$ and $[q_1,q_2] \cdot [q_3,q_4] = [q_1q_3, q_2q_4]$. Intersection of constraints (intervals) is defined as usual.

Assuming such an interpretation of the operations of composition, inverse, and intersection, Dechter et al. (1991) showed that any PC-algorithm can be exploited to compute the minimal STP equivalent to a given one, if any (if an inconsistency is detected, the algorithm returns an empty network). In the following, we will denote such an algorithm by $\text{PC}_{\text{STP}}$. Making use of such a result, Meiri (1996) proposed a formalism to combine qualitative constraints between points and intervals with (possibly disjunctive) metric constraints between points (as in TCSP).

An easy special case arises when only convex PA-constraints and STP-constraints are considered. Convex PA-constraints can be encoded as STP-constraints by means of the toSTP translation function described in Table 1. The following result can be found in Meiri (1996):

Theorem 3. Let $N$ be a network with convex PA-constraints and STP-constraints. If $N$ is path-consistent, then $N$ is also consistent and its metric constraints are minimal.

$\text{PC}_{\text{STP}}$ can thus be used to decide the consistency of a network $N$ satisfying the conditions of the above
theorem. To this end, it suffices to encode PA-constraints into equivalent STP-constraints.

<table>
<thead>
<tr>
<th>Convex PA relation</th>
<th>STP constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_i &lt; p_j )</td>
<td>( p_j - p_i \in [0, +\infty) )</td>
</tr>
<tr>
<td>( p_i \leq p_j )</td>
<td>( p_j - p_i \in [0, +\infty) )</td>
</tr>
<tr>
<td>( p_i = p_j )</td>
<td>( p_j - p_i \in [0, 0] )</td>
</tr>
<tr>
<td>( p_i &gt; p_j )</td>
<td>( p_j - p_i \in (-\infty, 0] )</td>
</tr>
<tr>
<td>( p_i \geq p_j )</td>
<td>( p_j - p_i \in [-\infty, 0] )</td>
</tr>
</tbody>
</table>

4.2 Integrating Convex RCD with STP

Combining RCD with STP makes it possible to express both directional constraints and metric constraints in a uniform framework. As an example, the resulting formalism allows one to constrain the position of a rectangle in the plane and to impose minimum and/or maximum values to the width/height of a given rectangle, or on the vertical/horizontal distances between the sides of two rectangles. Obviously, RCD-constraints and STP-constraints are not totally independent, that is, RCD-constraints entail some metric constraints and vice versa.

Example 3. Let \( a \) and \( b \) be two rectangle. We can use the metric constraint \( 0 < a_{x}^* - a_{z}^* \leq 7 \) to state that the maximum width of \( a \) is 7 and, similarly, we can exploit the metric constraint \( 2 \leq a_{x}^* - a_{z}^* \) to state that the minimum height of \( a \) is 2 (leaving the maximum height unbounded). We can also express distance constraints between the boundaries of \( a \) and \( b \). We can constrain the horizontal distance between the right side of \( a \) and the left side of \( b \) to be at least 3 by means of the constraint \( 3 \leq b_{c}^* - a_{z}^* \) and the vertical distance between the upper side of \( a \) and the bottom side of \( b \) to be greater than or equal to 0 by means of the constraint \( 0 \leq b_{c}^* - a_{z}^* \). The two constraints together entail the basic RCD constraint \( a \{SW\} b \). Finally, some metric constraints can be entailed by RCD ones. For instance, the convex relation \( a \{NW, N, NE, NW':N, NW':NE, N:NE\} b \) implies that \( 0 \leq a_{y}^* - b_{y}^* \).

If we allow one to combine arbitrary RCD-constraints with metric constrains, then checking the consistency of the resulting set of constraints turns out to be an NP-complete problem (the consistency problem for RCD-networks is already NP-complete). To preserve tractability, we restrict our attention to the combination of convex RCD-constraints with STP-constraints to establish the convex_metric RCD formalism.

Given a convex RCD-network \( N = (V, C) \), we denote the sets of interval variables belonging to the projections \( \pi_{i}(toRA(N_{i})) \) and \( \pi_{i}(toRA(N_{j})) \) by \( V_{i} \) and \( V_{j} \), respectively. Moreover, we denote by \( P(V_{i}) \) and \( P(V_{j}) \) the sets of point variables representing the endpoints of the interval variables in \( V_{i} \) and \( V_{j} \), respectively. A convex-metric RCD-network is formally defined as follows.

Definition 2. A convex-metric RCD-network (cmRCD-network) is an integrated qualitative and metric constraint network \( N \) consisting of three sub-networks \((N_{c}, S_{x}, S_{y})\), where \( N_{c} = (V, C) \) is a convex RCD-network, and \( S_{x} = (P(V_{i}), M_{x}) \) and \( S_{y} = (P(V_{j}), M_{y}) \) are two STPs.

The convex-metric RCD formalism we propose subsumes the STP formalism and the convex RCD-calculus. Moreover, it also generalizes the convex fragment of the RA, since convex RA-relations are expressible as convex PA-relations and these relations can be, in turn, encoded into an STP.

Now, we provide an algorithm to solve the consistency problem for cmRCD that runs in \( O(n^2)^2 \). First, we extend the translation mapping \( toSTP \) of Table 1 to encode a convex PA-network \( N^{p} \) into an STP \( S \) by replacing each relation \( R_{ij} \) in the network \( N^{p} \) by \( toSTP(R_{ij}) \). By exploiting such a function, we can generalize the algorithm \texttt{con-cRCD} of Section 3.2 to deal with both RCD- and STP-constraints (Algorithm \texttt{con-cmRCD}). First, \texttt{con-cmRCD} computes the PA-networks \( N^{p}_{c} \) and \( N^{p}_{y} \), and then, making use of information about convex RCD-relations encoded as PA-relations, it looks for possible inconsistencies between these constraints and the STP-constraints on the same variables given in \( S_{x} \) and \( S_{y} \), that can be detected at this stage. To this end, it translates the PA-network \( N^{p}_{c} \) (resp., \( N^{p}_{y} \)) into an STP-network by applying the function \( toSTP \), and then it uses the function \texttt{intersect} to compute the “intersection” between \( toSTP(N^{p}_{c}) \) and \( S_{x} \) (resp., \( toSTP(N^{p}_{y}) \) and \( S_{y} \)). This function simply intersects the intervals / metric constraints associated with the same pairs of variables in the two STPs. If an interval intersection produces an empty interval, then \texttt{intersect} returns an empty network, and we can conclude that \( N \) is inconsistent. Otherwise, we apply the path-consistency algorithm to the two STPs computed at lines 4 and 5 independ-
Algorithm 4.1: The algorithm con-cmRCD.

Require: a cmRCD-network \( N = (N_c,S_x,S_y) \)
1: \( N_c \leftarrow \text{toRA}(N_c) \);
2: \( N_x \leftarrow \pi_x(N_c) \), \( N_y \leftarrow \pi_y(N_c) \);
3: \( N^p_x \leftarrow \text{toPA}(N_x) \), \( N^p_y \leftarrow \text{toPA}(N_y) \);
4: \( x\text{STP} \leftarrow \text{intersect}(\text{toSTP}(N^p_x),S_x) \);
5: \( y\text{STP} \leftarrow \text{intersect}(\text{toSTP}(N^p_y),S_y) \);
6: if \( x\text{STP} \) or \( y\text{STP} \) is empty, then return ‘inconsistent’;
7: \( x\text{STP}^{\text{min}} \leftarrow \text{PC}_{\text{app}}(x\text{STP}) \);
8: \( y\text{STP}^{\text{min}} \leftarrow \text{PC}_{\text{app}}(y\text{STP}) \);
9: If \( x\text{STP}^{\text{min}} \) or \( y\text{STP}^{\text{min}} \) is empty, then return ‘inconsistent’;

entirely. The following theorem proves that con-cmRCD is sound and complete.

Theorem 4. Given a cmRCD-network \( N = (N_c,S_x,S_y) \), the algorithm con-cmRCD returns ‘consistent’ if and only if \( N \) is consistent.

Proof. We basically follow the steps of the algorithm. By Theorem 1, \( N_c \) is consistent if and only if \( N_x \) is consistent; and, by Theorem 2: \( N_x \) is consistent if and only if \( N_y \) are consistent (they can be checked independently). Next, \( N_x \), \( N_y \) and \( S_z \) are consistent if and only if \( N^p_y \) is consistent, since there is no loss in information in the translations (van Beek and Cohen, 1990). The consistency of \( N^p_x \) and \( N^p_y \) could be checked by computing the corresponding STPs and by applying \( \text{PC}_{\text{app}} \). However, we cannot apply \( \text{PC}_{\text{app}} \) directly to the STPs \( \text{toSTP}(N^p_x) \) and \( \text{toSTP}(N^p_y) \) since the metric constraints of \( S_z \) and \( S_x \) must be taken into account. Hence, we compute \( \text{intersect}(\text{toSTP}(N^p_y),S_x) \) and \( \text{intersect}(\text{toSTP}(N^p_x),S_y) \). If one of them returns an empty network, then \( N \) is inconsistent. Otherwise, we independently apply \( \text{PC}_{\text{app}} \) to \( x\text{STP} \) and \( y\text{STP} \). By Theorem 3, if one of the two applications of \( \text{PC}_{\text{app}} \) returns an empty network, then \( N \) is inconsistent; otherwise, the path-consistent STPs \( x\text{STP}^{\text{min}} \) and \( y\text{STP}^{\text{min}} \) are consistent (and minimal), and thus \( N \) is consistent.

Theorem 5. The complexity of the algorithm con-cmRCD is \( O(Rn^3) \), where \( n \) is the number of variables and \( R \) is the maximum range of the network.

Proof. The translation via \( \text{toRA} \), the generation of a projection of a network, the transformation of a JA-network into a RA-network via \( \text{toPA} \) and the last two encodings via \( \text{toSTP} \) require \( O(n^2) \) steps, since there are \( O(n^2) \) constraints and each constraint can be translated in constant time. The function \( \text{toPA} \) introduces two variables for each interval variable, so \( x\text{STP} \) and \( y\text{STP} \) have \( O(n) \) variables each. Finally, \( \text{PC}_{\text{app}} \) runs in \( O(Rn^3) \) time, so the overall complexity is \( O(Rn^3) \) time, where \( R \) is the maximum range of the network (for more details about the complexity of achieving path-consistency for combined networks see (Meiri, 1996)).
Implicit: “buildings must be inside the plot”:

\[ o B p, \quad h B p, \quad s B p; \]

\[ m \{ N W, N, N W \} h; \]

\[ m \{ N W, N, N W \} o; \]

\[ o \{ N E, N E, E, E \} h; \]

\[ s \{ S W, S, S W \}; \]

\[ s \{ S W, S, S W \}; \]

The quantitative part of the network contains the following metric constraints forming two STPs:

\[ b_x^+ - p_x^- \geq 100, \quad p_y^+ - b_y^- \geq 100, \]

\[ m_x^+ - m_x^- = 70, \quad m_y^+ - m_y^- = 70; \]

\[ m_x^+ - h_y^- \geq 100, \quad m_y^- - o_y^+ \geq 70; \]

\[ h_x^- - h_x^- = 100, \quad h_y^+ - h_y^- = 50; \]

\[ a_x^+ - a_x^- = 30, \quad a_y^+ - a_y^- = 70; \]

\[ 60 \leq a_y^- - h_y^- \leq 80; \]

\[ 50 \leq s_x^+ - s_x^- \leq 100, \quad 25 \leq s_y^+ - s_y^- \leq 50 \]

\[ h_x^- - s_x^- \geq 50, \quad o_x^- - s_x^- \geq 50. \]

By applying our consistency algorithm we can verify that it is possible to realize the building project of the example (the corresponding cmRCD-network is consistent). We can also determine the minimum area that the plot should have by using the minimal networks \( x_{STP}^{min} \) and \( y_{STP}^{min} \): in our example the minimum area of \( p \) is \( 390m \times 515m \) while the maximum area is unbounded. The STPs \( x_{STP} \) and \( y_{STP} \), computed by steps 4 and 5 of our algorithm, are sketched in Figure 5, while a solution of the problem is illustrated by Figure 6, showing the minimum feasible values for the point variables. To simplify, we suppose that the origin of the reference system is the lower-left vertex of the plot, since the plot encloses all the build-

5 CONCLUSIONS

In this paper, we have proposed a quite expressive, but tractable, metric extension of RCD (cmRCD), that integrates STP-constraints with convex RCD-constraints. cmRCD allows one to constrain the position of a rectangle in the plane, its width/height, and the vertical/horizontal distance between the sides of two rectangles, as well as to represent cardinal relations between rectangles. We have devised an \( O(n^4) \) consistency-checking algorithm, and we have showed how a spatial realization of a network can be built.

As for future work, we plan to extend cmRCD with topological relations to improve its expressive-
ness (similar results can be found in (Gerevini and Renz, 2002; Liu et al., 2009)). The problem of identifying maximal tractable subsets of RCD is still open. It would be interesting to search for tractable classes (strictly) including the convex fragment.

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