AN INTERPOLATION APPROACH FOR CONSTRAINED OUTPUT FEEDBACK

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Abstract: The purpose of this paper is twofold. In the first part, we provide a solution to the problem of the state construction through measurement and storage of appropriate previous measurements. In the second part we consider the robust control problem of constrained discrete-time linear-time invariant systems with disturbance and bounded input. Based on an interpolation technique, feasibility and a robustly asymptotically stable closed loop behavior are guaranteed.

1 INTRODUCTION

This paper considers the problem of output feedback control design for a class of linear discrete time systems in presence of output and control constraints and subject to bounded disturbance. The boundedness assumptions on the different manipulated signals will be modeled by means of polyhedral constraints which assure a global linear system description (linear difference equation and linear equalities/inequalities).

There are several papers in the literature dealing with the output feedback synthesis problem. Due to the presence of input and state constraints, the robust model predictive control (MPC) design seems to best fit our objectives. Indeed, based on a Luenberger observer, an approach that incorporates the error on the state estimation as an additive bounded disturbance has been proposed in (Mayne et al., 2006). The estimation error is then taken in to account in the classical design of the constrained controller. A different approach is taken in (Goulart and Kerrigan, 2007), where the authors include the observer dynamics in the computation of the domain of attraction of the closed loop system.

The main drawback of the observer-based approaches is that, when the constraints become active, the nonlinearity dominates the properties of the state feedback control system and one cannot expect the separation principle to hold. Moreover there is no guarantee that the constraints will be satisfied along the closed-loop trajectories.

The work of (Wang and Young, 2006) proposed an approach to MPC based on a non-minimal state space model, in which the states are represented by measured past inputs and outputs. This approach eliminates the need of an observer. However the resulting state space model is unobservable and the state dimension may be large.

The main aim of the present paper is twofold. In the first part, we revisit the problem of state construction through measurement and storage of appropriate previous measurements. We recall that, there exists a *minimal* state space model with the structural constraints of having a state variable vector available though measurement and storage of appropriate previous measurements. Even if this model might be *non-minimal* from the classical state space representation point of view, it is directly measurable and will pro-

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vide an appropriate model for the control design with constraints handling guarantees.

In the second part, starting from this state space model, we consider the robust control problem of constrained discrete-time linear invariant systems with disturbance and bounded input. For this purpose, two types of controller will be used in this paper. The first one is the global vertex controller (Gutman and Cwikel, 1986). The second one is the local unconstrained robust optimal control. Based on an interpolation technique and by minimizing an appropriate objective function, feasibility and a robustly asymptotically stable closed loop behavior are achieved.

The following notations will be used throughout the paper. We call a C-set a convex and compact set and containing the origin as an interior point. A polyhedron, or a polyhedral set, is the intersection of a finite number of half spaces. A polytope is a closed and bounded polyhedral set. Given two sets $X_1 \subset R^n$ and $X_2 \subset R^n$, the Minkowski sum of the sets X_1 and X_2 is defined by $X_1 \oplus X_2 \triangleq \{x_1 + x_2 | \; x_1 \in X_1, x_2 \in X_2\}$. The set X_1 is a proper subset of the set X_2 if and only if X_1 lies strictly inside X_2. For the set X, let Fr(X) be the boundary of X, Int(X) be the interior of X.

The paper is organized as follows. Section 2 is concerned with the problem statement. Section 3 is dedicated to the state space realization. Section 4 deals with the problem of computing an invariant set, while Section 5 is concerned with an interpolation technique. The simulation results are evaluated in Section 6 before drawing the conclusions.

2 PROBLEM STATEMENT

Consider the regulation problem for the following discrete linear time-invariant system, described by the input-output relationship

$$\begin{aligned} & y(t+1) + D_1 y(t) + D_2 y(t-1) + \ldots + D_n y(t-n+1) \\ & = N_1 u(t) + N_2 u(t-1) + \ldots + N_m u(t-m+1) + w(t) \end{aligned} \tag{1}$$

where: $y(t) \in R^q$, $u(t) \in R^p$, $w(t) \in R^q$ and $D_i, i = 1, \ldots, n$ and $N_i, i = 1, \ldots, m$ are matrices of suitable dimension.

It is assumed that $m \leq n$.

The output and control are subject to the following hard constraints

$$y(t) \in Y, \; u(t) \in U \tag{2}$$

where $Y = \{y : F_y y \leq g_y\}$ and $U = \{u : F_u u \leq g_u\}$ are polyhedral sets and contain the origin in their interior.

The signal $w(t)$ represents the disturbance input.

In this paper, we assume that the disturbance $w(t)$ is unknown, additive and lie in the polytope W, i.e. $w(t) \in W$, where $W = \{w : F_w w \leq g_w\}$ is a C-set.

3 STATE SPACE MODEL

In this section, the measured plant input, output and their past measured values are used to represent the states of the plant.

To simplify the description, it is assumed that $m = n$. Note that this assumption is always true, by supposing $N_{m+1} = N_{m+2} = \ldots = N_n = 0$.

The state of the system along the lines of (Taylor et al., 2000). All the state construction is detailed such that the presentation of the results to be self contained.

$$x(t) = \begin{pmatrix} x_1(t)^T & x_2(t)^T & \ldots & x_n(t)^T \end{pmatrix}^T \tag{3}$$

where $(*)^T$ denotes the transposed of matrix $(*)$ and

$$\begin{cases} x_1(t) = y(t) \\ x_2(t) = -D_n x_1(t-1) + N_n u(t-1) \\ x_3(t) = -D_{n-1} x_1(t-1) + x_2(t-1) + N_{n-1} u(t-1) \\ x_4(t) = -D_{n-2} x_1(t-1) + x_3(t-1) + N_{n-2} u(t-1) \\ \vdots \\ x_n(t) = -D_2 x_1(t-1) + x_{n-1}(t-1) + N_2 u(t-1) \end{cases} \tag{4}$$

It is clear that

$$\begin{aligned} & x_2(t) = -D_n y(t-1) + N_n u(t-1) \\ & x_3(t) = -D_{n-1} y(t-1) - D_n y(t-2) + \\ & \qquad + N_{n-1} u(t-1) + N_n u(t-2) \\ & \vdots \\ & x_n(t) = -D_2 y(t-1) - D_3 y(t-2) - \ldots - D_n y(t-n+1) + \\ & \qquad + N_2 u(t-1) + N_3 u(t-2) + \ldots + N_n u(t-n+1) \end{aligned}$$

One has

$$\begin{aligned} & y(t+1) = -D_1 y(t) - D_2 y(t-1) - \ldots - D_n y(t-n+1) \\ & \quad + N_1 u(t) + N_2 u(t-1) + \ldots + N_n u(t-n+1) + w(t) \end{aligned}$$

or

$$x_1(t+1) = -D_1 x_1(t) + x_n(t) + N_1 u(t) + w(t)$$

The state space model is then defined as follows

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) + Ew(t) \\ y(t) = Cx(t) \end{cases} \tag{5}$$

where

$$A = \begin{pmatrix} -D_1 & 0_q & 0_q & \ldots & 0_q & I_q \\ -D_n & 0_q & 0_q & \ldots & 0_q & 0_q \\ -D_{n-1} & I_q & 0_q & \ldots & 0_q & 0_q \\ -D_{n-2} & 0_q & I_q & \ldots & 0_q & 0_q \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -D_2 & 0_q & 0_q & \ldots & I_q & 0_q \end{pmatrix},$$

$$B = \begin{pmatrix} N_1^T & N_n^T & N_{n-1}^T & N_{n-2}^T & \ldots & N_2^T \end{pmatrix}^T,$$

$$E = \begin{pmatrix} I_q & 0_q & 0_q & 0_q & \ldots & 0_q \end{pmatrix}^T,$$

$$C = \begin{pmatrix} I_q & 0_q & 0_q & 0_q & \ldots & 0_q \end{pmatrix}.$$

Here I_q, 0_q denote the identity and zeros matrices of dimension $q \times q$, respectively.

It should be noted that, the state space realization (5) is minimal in the single input single output case. In the other cases this realization might not be minimal, as showing in the following example.

Consider the SIMO discrete time system:

$$y(t+1)+\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} y(t)+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y(t-1)= \\ =\begin{pmatrix} 0.5 \\ 2 \end{pmatrix} u(t)+\begin{pmatrix} 0.5 \\ 1 \end{pmatrix} u(t-1)+w(t) \tag{6}$$

Using the above construction, the state space model is given as follows:

$$\begin{cases} x(t+1)=Ax(t)+Bu(t)+Ew(t) \\ y(t)=Cx(t) \end{cases}$$

where

$$A=\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, B=\begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ -1.5 \end{pmatrix}$$

$$E=\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, C=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It is obvious that, this realization is not minimal. One of the minimal realizations of the system is given by:

$$A=\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, B=\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, E=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Denote

$$z(t)=\begin{matrix} (y(t)^T & y(t-1)^T & \ldots & y(t-n+1)^T \\ u(t-1)^T & u(t-2)^T & \ldots & u(t-n+1)^T)^T \end{matrix} \tag{7}$$

The state vector $x(t)$ (3) is related to the vector $z(t)$ as follows

$$x(t)=Tz(t) \tag{8}$$

where

$$T=(T_1 \quad T_2)$$

$$T_1=\begin{pmatrix} I_q & 0_q & 0_q & \ldots & 0_q \\ 0_q & -D_n & 0_q & \ldots & 0_q \\ 0_q & -D_{n-1} & -D_n & \ldots & 0_q \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0_q & -D_2 & -D_3 & \ldots & -D_n \end{pmatrix}$$

$$T_2=\begin{pmatrix} 0_{q\times p} & 0_{q\times p} & 0_{q\times p} & \ldots & 0_{q\times p} \\ N_n & 0_{q\times p} & 0_{q\times p} & \ldots & 0_{q\times p} \\ N_{n-1} & N_n & 0_{q\times p} & \ldots & 0_{q\times p} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ N_2 & N_3 & N_4 & \ldots & N_n \end{pmatrix}$$

From the formula (8), it is apparent that at any time instant t, the state variable vector is available though measurement and storage of appropriate previous measurements.

4 INVARIANT SET CONSTRUCTION

Using (4), it is clear that $x_i(t) \in X_i$ where X_i is given by

$$X_1 = Y \\ X_2 = D_n(-X_1)\oplus N_n(U) \\ X_i = D_{n+2-i}(-X_1)\oplus X_{i-1}\oplus N_{n+2-i}U, \forall i=3,\ldots,n$$

In summary, the constraints on the state are $x \in X$, where $X=\{x:\ F_x x \le g_x\}$.

4.1 Maximal Robustly Admissible Set for $u = Kx$

Using the results in the control theory (LQR, LQG, LMI based, ...), one can find a feedback gain K, that quadratically stabilizes the system (5) with some desired properties. The details of such a synthesis procedure are not reproduced here, but we assume that the feasibility of such an optimization based robust control design is guaranteed, leading to a closed loop transition matrix $A_c = A+BK$.

Definition 1 (Robustly Positively Invariant Set). The set $\Omega \subseteq X$ is a robustly positively invariant (RPI) set with respect to $x(t+1)=A_c x(t)+Ew(t)$ if and only if

$$\forall x \in \Omega \Rightarrow A_c x + Ew \in \Omega \tag{9}$$

for any $w \in W$.

Definition 2 (Minimal RPI). The set $\Omega_\infty \subseteq X$ is a minimal RPI (mRPI) set with respect to $x(t+1)=A_c x(t)+Ew(t)$ if and only if Ω_∞ is a RPI and contained in any RPI set.

It is possible to show that if the mRPI set Ω_∞ exists, then it is unique, bounded and contains the origin in its interior (Kolmanovsky and Gilbert, 1998), (Rakovic et al., 2005). Moreover, all trajectories of the system $x(t+1)=A_c x(t)+Ew(t)$ starting from the origin, are bounded by Ω_∞. It follows from linearity and asymptotic stability of A_c, that Ω_∞ is the limit set of all trajectory of the system $x(t+1)=A_c x(t)+Ew(t)$.

It is clear that, it is impossible to devise a controller $u(t)=Kx(t)$ such that $x(t)\to 0$ as $t\to\infty$. The best that can be hoped for is that the controller steers any initial state to the mRPI set Ω_∞, and maintains the state in this set once it is reached. In other words, the set Ω_∞ can be considered as the origin of the system (5).

In the sequel, it is assumed that the set Ω_∞ is a proper subset Y.

Definition 3 (Maximal RPI). The set $O_\infty \in X$ is a maximal RPI (MRPI) set with respect to $x(t+1) = A_x x(t) + Ew(t)$ if and only if O_∞ is a RPI and contains every RPI set.

If the MRPI set is non-empty, then it is unique. Furthermore if X, U and W are a C-set, then the MRPI set O_∞ is also a C-set.

The mRPI set Ω_∞ and the MRPI set O_∞ are connected be the following theorem:

Theorem 1. The following statements are equivalent:

1. The MRPI set O_∞ is non-empty.
2. $\Omega_\infty \subset X$

Proof. Interested readers are referred to (Kolmanovsky and Gilbert, 1998) for the details of the proof. □

Define the polytope P_{xu} as follows

$$P_{xu} = \{x : \; F_{xu}x \leq g_{xu}\} \tag{10}$$

where

$$F_{xu} = \begin{pmatrix} F_x \\ F_u K \end{pmatrix}, \quad g_{xu} = \begin{pmatrix} g_x \\ g_u \end{pmatrix}$$

Under the assumption that the Ω_∞ is a proper subset of X, a constructive procedure is used to compute the MRPI set, as follows (Blanchini and Miani, 2008).

Procedure 1. Maximal robustly positively invariant set computation.

1. Set $t = 0$, $F_t = F_{xu}$, $g_t = g_{xu}$ and $P_t = P_{xu}$.
2. Set $P_t^1 = P_t$
3. Solve the following linear program
$$d = \max F_t E w, \text{ s.t. } w \in W$$
4. Compute a polytope
$$P_t^2 = \{x : F_t A_c x \leq g_t - d\}$$
5. Set P_t as an intersection
$$P_t = P_t^1 \cap P_t^2$$
6. If $P_t = P_t^1$ then stop and set $O_\infty = P_t$. Else continue.
7. Set $t = t+1$, go to step 2.

Non-emptiness property of the MRPI set O_∞ assures that the above procedure terminates in finite time and lead to the MRPI in form of a polytope

$$O_\infty = \{x : F_o x \leq g_o\} \tag{11}$$

4.2 Robustly Positively Controlled Invariant Set for $u \in U$

Recall the following definitions (Blanchini and Miani, 2008)

Definition 4: Robustly Positively Controlled Invariant Set. Given the system (5), the set $\Psi \subseteq X$ is invariant if and only if for any $x(t) \in \Psi$, there exists a control action $u(t) \in U$ such that for any $w(t) \in W$, one has $x(t+1) = Ax(t) + Bu(t) + Ew(t) \in X$.

Definition 5: Pre-image Set. Given the polytopic system (1), the one-step pre-image set of the set $P_0 = \{x : F_0 x \leq g_0\}$ is given by all states that be steered in one step in P_0 when a suitable control is applied. The pre-image set, called $P_1 = Pre(P_0)$ can be shown to be:

$$P_1 = \{x \in R^n : \exists u \in U : F_0(Ax + Bu) \leq g_0 - \max F_0 E w\} \tag{12}$$

where $w \in W$.

Remark 1: It is clear that if the set Ψ is contained in its pre-image set the Ψ is invariant.

Recall that the set O_∞ is the MRPI. Define P_N as the set of states, that can be steered to the O_∞ in no more that N steps along an admissible trajectory, i.e. a trajectory satisfying control, state and disturbance constraints. This set can be generated recursively by the following procedure:

Procedure 2. Invariant set computation

1. Set $k = 0$ and $P_0 = O_\infty$.
2. Define
$$P_{k+1} = \text{Pre}(P_k) \bigcap X$$
3. If $P_{k+1} = P_k$, then stop and set $P_N = P_k$. Else continue.
4. If $k = N$, then stop else continue.
5. Set $k = k+1$ and go to the step 2.

A consequence of the fact that O_∞ is an invariant set, it follows that for each k, $P_{k-1} \subset P_k$ and therefore P_k is an invariant set and a sequence of nested polytopes.

Note that the complexity of the set P_N does not have an analytic dependence on N and may increase without bound, thus placing a practical limitation on the choice of N.

For further use, the controlled invariant set resulting from the Procedure 2 is denoted

$$P_N = \{x : F_N x \leq g_N\} \tag{13}$$

5 INTERPOLATION BASED CONTROLLER WITH LINEAR PROGRAMMING

The purpose of this section is to show how an interpolation technique can be used together with linear programming.

5.1 Vertex Control Law

Given a positve invariant polytope $P_N \in R^n$, this polytope can be decomposed in a sequence of simplices P_N^k each formed by n vertices $x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}$ and the origin. These simplices have following properties:

- $\mathrm{Int}(P_N^k) \neq \emptyset$,
- $\mathrm{Int}(P_N^k \cap P_N^l) = \emptyset$ if $k \neq l$,
- $\bigcup_k P_N^k = P_N$,

Denote by $X^{(k)} = (x_1^{(k)}\ x_2^{(k)}\ \ldots\ x_n^{(k)})$ the square matrix defined by the vertices generating P_N^k. Since P_N^k has nonempty interior, $X^{(k)}$ is invertible. Let $U^{(k)} = (u_1^{(k)}\ u_2^{(k)}\ \ldots\ u_n^{(k)})$ be the matrix defined by the admissible control values at these vertices. For $x \in P_N^k$ consider the following linear gain K^k:

$$K^k = U^{(k)}(X^{(k)})^{-1} \tag{14}$$

Remark 2: By the admissible control value we understand any control action, that keeps the state inside the invariant set. Generally, one would like to maximize the control action at the vertices of the feasible invariant set. This can be done by using the following program.

$$J = \max \|u\|_p \text{ s.t. } \begin{cases} F_N(Ax+Bu) \leq g_N - \max F_N E w \\ F_u u \leq g_u. \end{cases} \tag{15}$$

where $\|u\|_p$ is a $p-$ norm of u and $w \in W$.

Due to the properties of the positive invariant set, the above program is always feasible.

Theorem 2. The piecewise linear control $u = K^k x$ is feasible and asymptotically stable for all $x \in P_N$.

Proof. The proof of this theorem is not reported here, with (Gutman and Cwikel, 1986) and (Blanchini, 1992) providing the necessary details. □

5.2 Interpolation via Linear Programming

Any state $x(t)$ in P_N can be decomposed as follows:

$$x(t) = c x_v(t) + (1-c) x_o(t) \tag{16}$$

where $x_v(t) \in P_N$, $x_o(t) \in \Omega$ and $0 \leq c \leq 1$.

Consider the following control law:

$$u(t) = c u_v(t) + (1-c) u_o(t) \tag{17}$$

where $u_v(t)$ is obtained by applying the vertex control law and $u_o(t) = K x_o(t)$ is the control law, that is feasible in O_∞.

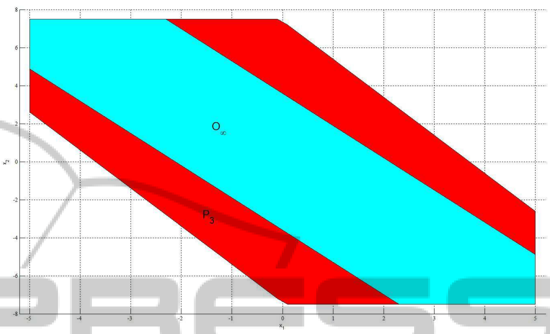

Figure 1: Feasible regions for example 1. The blue one is the MRPI O_∞, when applying the control law $u = Kx$. The red one is the positive invariant set P_N.

Theorem 3. The above linear control is feasible for all $x \in P_N$.

Proof.

Corresponding to the decomposition, the control law is given by (17).

One has to prove that $F_u u(t) \leq g_u$ and $x(t+1) = Ax(t) + Bu(t) + Ew(t) \in P_N$ for all $x(t) \in P_N$ and for any $w(t) \in W$.

One has

$$\begin{aligned} F_u u(t) &= F_u(c u_v(t) + (1-c) u_o(t)) \\ &= c F_u u_v(t) + (1-c) F_u u_o(t) \\ &\leq c g_u + (1-c) g_u = g_u \end{aligned}$$

and

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Ew(t) \\ &= A(c x_v(t) + (1-c) x_o(t)) + \\ &\quad + B(c u_v(t) + (1-c) u_o(t)) + Ew(t) \\ &= c(A x_v(t) + B u_v(t) + Ew(t)) + \\ &\quad + (1-c)(A x_o(t) + B u_o(t) + Ew(t)) \end{aligned}$$

We have $A x_v(t) + B u_v(t) + Ew(t) \in P_N$ and $A x_o(t) + B u_o(t) + Ew(t) \in O_\infty \subset P_N$, it follows that $x(t+1) \in P_N$. □

In order to give a maximal control action, one would like to minimize c, so the following program is given:

$$c^*(x) = \min_{c, x_v, x_o} c, \text{ s.t. } \begin{cases} F_N x_v \leq g_N, \\ F_o x_o \leq g_o, \\ c x_v + (1-c) x_o = x, \\ 0 \leq c \leq 1 \end{cases} \tag{18}$$

Denote $r_v = cx_v$, $r_o = (1-c)x_o$. It is clear that $r_v \in cP_N$ and $r_o \in (1-c)\Omega$ or equivalently $F_N r_v \leq cg_N$ and $F_w r_o \leq (1-c)g_w$. The above non-linear program is translated into a linear program as follows.

Interpolation based on Linear Programming.

$$c^*(x) = \min_{c, r_v} c, \text{ s.t. } \begin{cases} F_N r_v \leq cg_N \\ F_o(x - r_v) \leq (1-c)g_o \\ 0 \leq c \leq 1 \end{cases} \quad (19)$$

Remark 3. If one would like to maximize c, it is obvious that $c = 1$ for all $x \in P_N$. In this case the controller turns out to be the vertex controller.

Theorem 4. The control law using interpolation based on linear programming (16), (17), (19) guarantees robustly asymptotic stability for all initial state $x(0) \in P_N$.

Proof. The complete proof of this theorem is given in (Nguyen et al., 2011).

6 EXAMPLES

To show the effectiveness of the proposed approach, two examples will be presented in this section. For both of these examples, to solve linear programs and to implement polyhedral operations, we used the Multi-parametric toolbox, (Kvasnica et al., 2004).

6.1 Example 1

Consider the following discrete-time system

$$\begin{array}{c} y(t+1) - 2y(t) + y(t-1) = \\ = 0.5u(t) + 0.5u(t-1) + w(t) \end{array} \quad (20)$$

The constraints are

$$\begin{array}{c} -5 \leq y(t) \leq 5 \\ -5 \leq u(t) \leq 5 \end{array}$$

and

$$-0.1 \leq w(t) \leq 0.1$$

The state space model is given by

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) + Ew(t) \\ y(t) = Cx(t) \end{cases}$$

where

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, E = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

The state $x(t)$ is available though the measured plant input, output and their past measured values as follows

$$x(t) = Tz(t)$$

where

$$z(t) = \begin{pmatrix} y(t) & y(t-1) & u(t-1) \end{pmatrix}^T,$$
$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0.5 \end{pmatrix}$$

The constraints on the state are

$$\begin{array}{c} -5 \leq x_1 \leq 5 \\ -7.5 \leq x_2 \leq 7.5 \end{array}$$

Using the linear quadratic regulator with weighting matrices $Q = C'C$ and $R = 0.1$ the feedback gain is obtained

$$K = \begin{pmatrix} -2.3548 & -1.3895 \end{pmatrix}$$

Using procedures 1 and 2 one obtains the set O_∞ and P_N as shown in Figure 1. Note that $P_3 = P_4$, in this case P_3 is a maximal invariant set for system (20).

The set of vertices of P_N is given by the matrix $V(P_N)$ below, together with the control matrix U_v

$$V(P_N) = \begin{pmatrix} -5 & -0.1 & 5 & 0.1 & -0.1 & -5 & 0.1 & 5 \\ 7.5 & 7.5 & -2.6 & 7.2 & -7.2 & 2.6 & -7.5 & -7.5 \end{pmatrix}$$

and

$$U_v = \begin{pmatrix} -5 & -5 & -5 & -4.9 & 5 & 5 & 5 & 4.9 \end{pmatrix}$$

Figure 2 shows the state space partition and 6 different trajectories of the closed loop system.

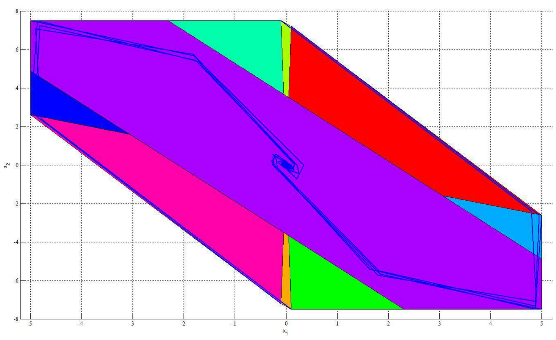

Figure 2: State space partition and trajectories of the closed loop system for example 1.

Corresponding to the initial condition $x_0 = (5.0000 \quad -2.6000)^T$, Figure 3 shows the output and input trajectory.

Figure 4 shows the disturbance input and the interpolating coefficient $c^*(t)$ as a function of t. As expected this function is positive and non-increasing.

In a comparison with the approach, that based on the so called Kalman filter, Figure 5 shows the output

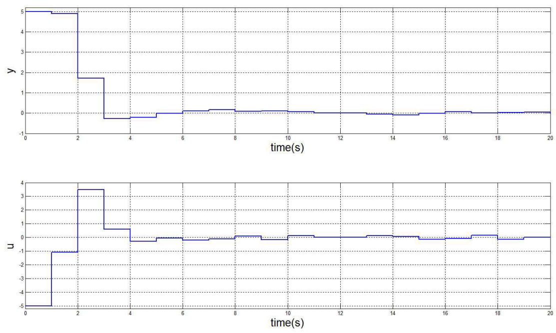

Figure 3: Output and input trajectory for example 1.

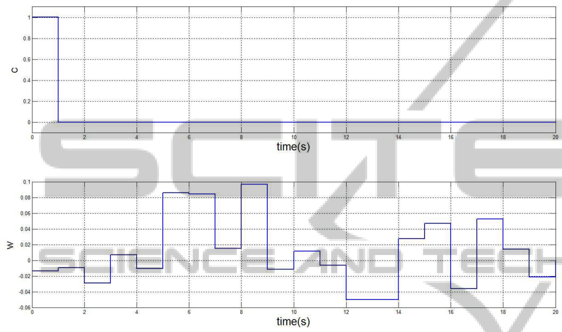

Figure 4: The interpolating coefficient and the disturbance input for example 1.

trajectories using our approach and the Kalman filter based approach. It is obvious that, the mRPI set of the Kalman filter based approach is bigger than the mRPI set of our approach.

The Matlab routine with the command *'kalman'* is used for designing the Kalman filter. The process noise is a white noise with an uniform distribution and there is no measurement noise.

w is a random number with an uniform distribution, $w_l \leq w \leq w_u$. The variance of w is given as follows:

$$C_w = \frac{(w_u - w_l + 1)^2 - 1}{12} = 0.0367$$

The estimator gain of the Kalman filter is obtained:

$$L = (2 \;\; -1)^T$$

The initial condition is $x_0 = (-4 \;\; 6)^T$.

The Kalman filter is used to estimate the state of the system and then this estimation is used to close the loop with the interpolated control law .

In contrast to our approach, where the state is exact, in the Kalman filter approach, the state is not exact and moreover, there is no guarantee that the constraints are satisfied.

Figure 6 shows the output trajectories of our approach and the Kalman filter based approach.

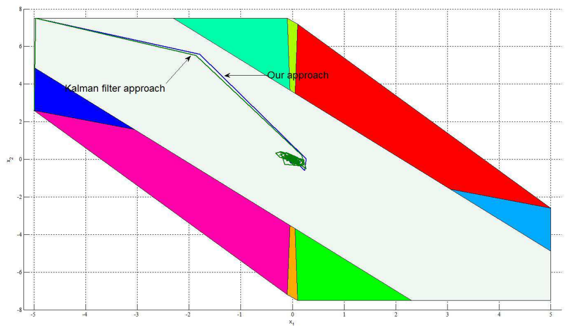

Figure 5: The state trajectory of our approach and the Kalman filter based approach for example 1. The mRPI set of the Kalman filter based approach is bigger than the mRPI set of our approach.

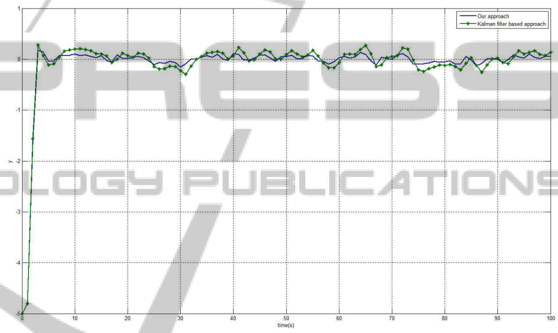

Figure 6: The output trajectories of our approach and the Kalman filter based approach for example 1.

In Figure 7 it is showed that, the constraints might be violated where the Kalman filter is used to estimate the state of the system.

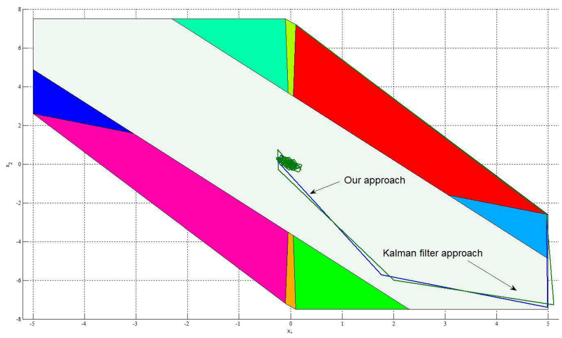

Figure 7: Constraints violation for example 1.

6.2 Example 2

Consider the following discrete-time system

$$y(t+1) + \begin{pmatrix} -1.8787 & 0 \\ 0 & -1.8964 \end{pmatrix} y(t) +$$
$$+ \begin{pmatrix} 0.8787 & 0 \\ 0 & 0.8964 \end{pmatrix} y(t-1) =$$

$$= \begin{pmatrix} -0.3800 & -0.5679 \\ -0.2176 & 0.4700 \end{pmatrix} u(t) + \\ + \begin{pmatrix} 0.3339 & 0.5679 \\ 0.2176 & -0.4213 \end{pmatrix} u(t-1) + w(t) \quad (21)$$

The constraints are

$$-2 \leq y_1 \leq 2, \quad -2 \leq y_2 \leq 2$$
$$-10 \leq u_1 \leq 10, \quad -10 \leq u_2 \leq 10$$

and

$$-0.1 \leq w_1 \leq 0.1, \quad -0.1 \leq w_2 \leq 0.1$$

The state space model is

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) + Ew(t) \\ y(t) = Cx(t) \end{cases}$$

where

$$A = \begin{pmatrix} 1.8787 & 0 & 1.000 & 0 \\ 0 & 1.8964 & 0 & 1 \\ -0.8787 & 0 & 0 & 0 \\ 0 & -0.8964 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.3800 & -0.5679 \\ -0.2176 & 0.4700 \\ 0.3339 & 0.5679 \\ 0.2176 & -0.4213 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It is worth noticing that, the above state space realization is minimal. The state $x(t)$ is available though the measured plant input, output and their past measured values as follows

$$x(t) = Tz(t)$$

where

$$z(t) = \begin{pmatrix} y(t)^T & y(t-1)^T & u(t-1)^T \end{pmatrix}^T,$$

$$T = \begin{pmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.8787 & 0 & 0.3339 & 0.5679 \\ 0 & 0 & 0 & -0.8964 & 0.2176 & -0.4213 \end{pmatrix}$$

The constraints on the state are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0.5460 & -0.8378 \\ 0 & 0 & -0.5958 & -0.8031 \\ 0 & 0 & -1.0000 & 0 \\ 0 & 0 & -0.5460 & 0.8378 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 0.0000 & -1.0000 \\ 0 & 0 & 0.5958 & 0.8031 \\ 0 & 0 & 1.0000 & 0.0000 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 9.0918 \\ 6.2239 \\ 10.7754 \\ 9.0918 \\ 8.1818 \\ 8.1818 \\ 6.2239 \\ 10.7754 \end{pmatrix}$$

Using the linear quadratic regulator with weighting matrices $Q = C'C$ and $R = I$, the feedback gain is obtained

$$K = \begin{pmatrix} 1.9459 & 1.7552 & 1.4968 & 1.3775 \\ 0.8935 & -1.7212 & 0.5524 & -1.2704 \end{pmatrix}$$

Using procedures 1 and 2, one obtains the set O_∞ and P_3 as illustrated in Figure 8. The number of vertices of the set P_3 is 1030 and these are not reported here. The control values at the vertices of the set P_3 are found by applying the program (15). Corresponding to the initial condition $x_0 = \begin{pmatrix} -1.6722 & 0.2088 & 10.7754 & -3.8296 \end{pmatrix}^T$, Figure 9 presents the output and input trajectories.

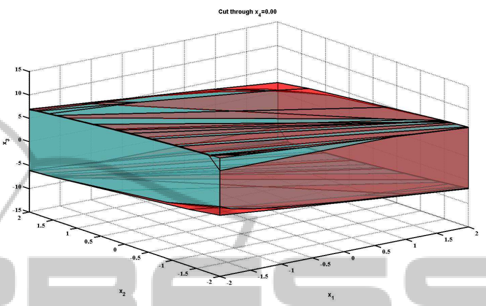

Figure 8: Feasible regions for example 2, cut through $x_4 = 0$. The blue one is the MRPI set O_∞, when applying the control law $u = Kx$. The red one is the positive controlled invariant set P_3.

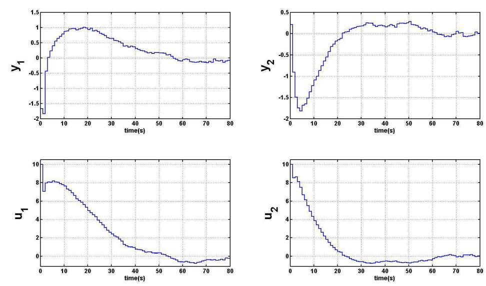

Figure 9: Output and input trajectory for example 2.

Figure 10 shows the disturbance inputs $w_1(t), w_2(t)$ and the interpolating coefficient $c^*(t)$ as a function of t. As expected, this function is positive and non-increasing.

In a comparison with the Kalman filter based approach, Figure 11 shows the output trajectories using our approach and the Kalman filter based approach.

The initial condition is $x_0 = \begin{pmatrix} -1.3378 & 0.1670 & 8.6203 & -3.0637 \end{pmatrix}^T$.

The Matlab routine with the command *'kalman'* is used for designing the Kalman filter. The process

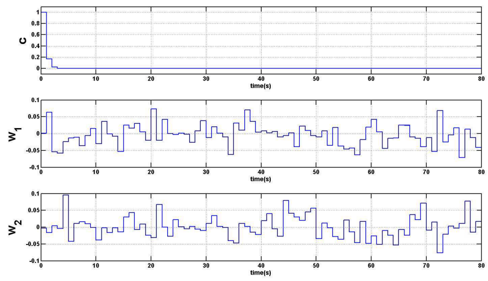

Figure 10: The interpolating coefficient and the disturbance input for example 2.

noise is a white noise with an uniform distribution and there is no measurement noise.

w is a random vector with an uniform distribution, $w_l \leq w \leq w_u$. The covariance matrix of w is given as follows:

$$C_w = \frac{(w_u - w_l + 1)^2 - 1}{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.0367 & 0 \\ 0 & 0.0367 \end{pmatrix}$$

The estimator gain of the Kalman filter is obtained:

$$L = \begin{pmatrix} 1.8787 & 0 \\ 0 & 1.8964 \\ -0.8787 & 0 \\ 0 & -0.8964 \end{pmatrix}$$

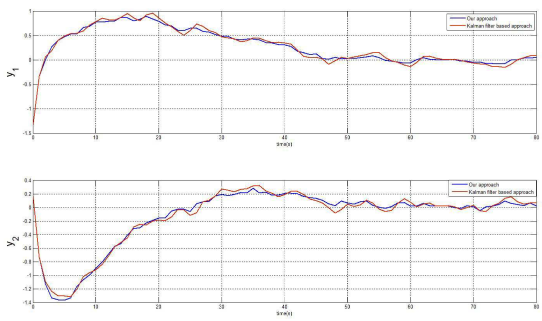

Figure 11: The output trajectories of our approach and the Kalman filter based approach for example 2.

7 CONCLUSIONS

In this paper, a state space realization is detailed for discrete-time linear time invariant systems, with the particularity that the state variable vector is available through measurement and storage of appropriate previous measurements.

A robust control problem is solved based on the interpolation technique and using linear programming. Practically, the interpolation is done between a global vertex controller and a local unconstrained robust optimal control law.

Several simulation examples are presented including a comparison with an earlier solution from the literature and a multi-input multi-output system.

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