A UNIFIED APPROACH TO GEOMETRIC MODELING OF CURVES AND SURFACES

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Abstract: A unified approach to geometric modeling of curves and surfaces is given. Both a vector-valued fourth and sixth order partial differential equations (PDEs) of motion are proposed. The fourth order PDE covers all existing PDEs used for surface modeling, and the sixth order PDE considers the curvature effect on curves and surfaces. In order to apply these PDEs to create curves and surfaces in real time, we have presented a composite power series method which guarantees the exact satisfaction of boundary conditions, and represents curves and surfaces with analytical mathematical formulae. We have examined the accuracy and efficiency of the proposed method, and employed it to a number of applications of static and dynamic modeling of curves and surfaces, including free-form surface generation and surface blending. It is found that this method has similar computational accuracy and efficiency to the corresponding closed form solution method, and creates curves and surfaces far more efficiently and accurately than numerical methods. In addition, it can deal with complicated shape modeling problems.

1 INTRODUCTION

Free-form surfaces and curves are conventionally created by surface modelers, such as Bézier, B-spline and NURBS (Farin, 1997). Normally, designers obtain control over the shape of a curve or surface by adjusting control points. This involves a lot of manual manipulations, especially when a large number of control points are involved. To overcome this weakness, active research is being undertaken for many complementary free-form modeling approaches. (Hyodo, 1990) proposed a method generating a free-form surface defined by contours and sectional curves. (Miura, 2000) proposed a unit quaternion integral curve which is used to specify the tangent of a curve in order to manipulate its curvature more directly. (Ochiai and Yasutomi, 2000) demonstrated a method of generating a free-form surface using the boundary integral equation.

With the drive towards realism, especially in computer animation, physically-based modeling represents another on-going research area where forces, dynamics and time-dependent deformation are considered. (Terzopoulos et al., 1987) employed continuous elasticity theory to model the shapes and motions of deformable bodies. Later on, this model was extended to viscoelasticity, plasticity and fracture (Terzopoulos and Fleischer, 1988), and dynamic deformations (Terzopoulos and Qin, 1994), (Qin and Terzopoulos, 1995). (Celniker and Gossard, 1991) developed curve and surface finite elements for interactive sculpting of curves and surfaces. By minimizing the energy functional of a surface, (Vassilev, 1997) proposed an interactive sculpting method for deformable non-uniform B-splines. With introduction of bar network mechanics, a deformation method of surfaces was developed (Guillet and Léon, 1998). A comprehensive survey into physics-based modeling methods was made by (Nealen et al., 2006) which reviews the existing finite element/difference /volume methods, mass-spring systems, meshfree methods, coupled particle systems and reduced deformable models based on modal analysis.

However, all these methods have to solve a large set of linear algebra equations, hence are computationally expensive and require large computer memory.

Surface blending, another important application field of shape modeling, has also been an active research subject and many methods have been developed. (Vida et al., 1994) classified methods of constructing parametric blends as rolling-ball based blends, spine-based blends, trimline-based blends, blends based on polyhedral methods and other

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methods including a cyclicde solution, PDE based-blends and the Fourier-based blends.

The PDE based methods, since their advent two decades ago, have found their applications in a lot of surface modeling tasks, including free-form surface generation (Bloor and Wilson, 1990a), *n*-sided patch modeling (Bloor and Wilson, 1989a), surface blending (Bloor and Wilson, 1989b), and industrial applications (Athanasopoulos et al., 2009). Compared with the conventional surface modeling methods, the PDE-based methods provide the user with a higher level control of the shape of the generated surfaces using the parameters and the boundary conditions of the PDE instead of many hundreds of control points. Therefore they can be easily implemented as an easy to use interactive modeling package. However, before that can be realized, we need to overcome one serious hurdle, that is to solve the corresponding PDE efficiently. Currently, it is done either *ad hoc* or only for simple problems. For complicated problems, expensive numerical methods are still the only available choice, such as the finite element method (Li, 1998, 1999, Li and Chang, 1999), finite difference method (Du and Qin, 2005), and collocation point method (Bloor and Wilson, 1990b). In order to improve the computational efficiency, the Fourier series method was proposed (Bloor and Wilson, 1996) although it is effective only when the high frequency modes are not strongly represented in the boundary conditions. In addition, another issue to be addressed is that the existing PDE based approaches only considered static modeling of surfaces. Dynamic modeling of curves and surfaces with up to curvature continuities using analytical PDEs has not been investigated yet.

In this paper, we propose a PDE approach to tackle both static and dynamic modeling of curves and surfaces. It represents curves and surfaces analytically, solves the PDEs quickly and accurately, and has a capacity to carry out complicated shape modeling. Therefore, it is applicable to interactive geometric modeling applications (Ugail et al., 1999a, 1999b). Unlike the existing PDE-based methods, which can only generate dynamic surfaces of tangent continuity, this approach will be able to generate curves and surfaces of curvature continuity.

2 THEORY AND METHOD

In this section, we introduce two partial differential equations of motion for static and dynamic modeling of curves and surfaces, and determine their composite power series solutions.

2.1 Partial Differential Equations of Motion

Based on the existing research on dynamic simulation of cloth deformation (You et al., 1999, Zhang et al., 1999), deformable moving surfaces (You and Zhang, 2003), and introducing the notations of $w_{,u^i}=\dfrac{\partial^i w}{\partial u^i}$ and $w_{,u^i v^j}=\dfrac{\partial^{i+j} w}{\partial u^i \partial v^j}$, we use the following vector-valued fourth and sixth order partial differential equations of motion for both static and dynamic modeling of curves and surfaces

$$\mathbf{a}_1\mathbf{x}_{,u^4}+\mathbf{a}_2\mathbf{x}_{,u^2v^2}+\mathbf{a}_3\mathbf{x}_{,v^4}+\rho\mathbf{x}_{,t^2}+c\mathbf{x}_{,t}=\mathbf{F} \tag{1}$$

$$\begin{aligned}&\mathbf{a}_1\mathbf{x}_{,u^6}+\mathbf{a}_2\mathbf{x}_{,u^4v^2}+\mathbf{a}_3\mathbf{x}_{,u^2v^4}+\mathbf{a}_4\mathbf{x}_{,v^6}+\rho\mathbf{x}_{,t^2}\\&+c\mathbf{x}_{,t}=\mathbf{F}\end{aligned} \tag{2}$$

where u and v are parametric variables, t is time variable, ρ is the density, c is the damping coefficient, $\mathbf{a}_l=\{a_{kl}\}$ $(k=x,y,z;\ l=1\sim 3$ for the fourth order, $1\sim 4$ for the sixth order$)$ are vector-valued shape control parameters, and the vector-valued position function $\mathbf{x}=\{x,\ y,\ z\}$ and vector-valued force function $\mathbf{F}=\{F_x,\ F_y,\ F_z\}$ involve variables u, v and t, or u and v, or u and t, or u only depending on different modeling tasks of curves and surfaces.

Because of the introduction of time variable t, Eqs. (1) and (2) integrate both static and dynamic modeling. When time variable t in these equations is taken to be a constant, Eqs. (1) and (2) become static PDEs and can be used to solve various static problems of curve and surface modeling.

It is worth pointing out that when time variable t is set to a constant, Eqs. (1) and (2) actually represent the generalization of all forms of existing fourth and sixth order PDEs used for surface modeling.

Equation (2) provides enough degrees of freedom to consider not only tangent but also curvature properties of curves and surfaces at boundary points or boundary curves. This gives an advantage in two applications: firstly, it is able to generate curves and surfaces requiring curvature continuity; and secondly, the specified curvature values are useful for shape control and producing more varieties of different curves and surfaces, since boundary curvature also has a great influence on curves and surfaces.

Equations (1) and (2) can be reduced to suit the modeling of curves, both statically and dynamically.

This is undertaken by setting one parametric variable v constant.

In order to apply the above equations to static and dynamic modeling of curves and surfaces, we must define boundary curves and surface properties at these curves for surface modeling, and boundary points and curve properties at these points for curve modeling. A compound surface may consist of multiple patches separated by a number of boundary curves. Similarly, a compound curve can be divided into a number of segments joined together at the boundary points. Using Eq. (1), the tangential properties of surfaces or curves at the boundary curves or boundary points can be taken into account. Thus the boundary conditions can be given by

$$u = u_i \qquad \mathbf{x} = \mathbf{S}_{i1} \qquad \mathbf{x}_{,u} = \mathbf{S}_{i2} \tag{3}$$

where i denotes the index of the boundary curves or boundary points, $\mathbf{S}_{i1}$ and $\mathbf{S}_{i2}$ are the function of v and t for dynamic modeling of surfaces, of v for static modeling of surfaces, of t for dynamic modeling of curves, and are constants for static modeling of curves.

With Eq. (2), higher order derivatives were introduced which provide more degrees of freedom to accommodate the curvature property of surfaces at boundary curves and that of curves at boundary points. Therefore, boundary conditions for Eq. (2) are given by

$$u = u_i \quad \mathbf{x} = \mathbf{S}_{i1} \quad x_{,u} = \mathbf{S}_{i2} \quad \mathbf{x}_{,u^2} = \mathbf{S}_{i3} \tag{4}$$

where the definition of $\mathbf{S}_{i3}$ is the same as those of $\mathbf{S}_{i1}$ and $\mathbf{S}_{i2}$, and $\mathbf{S}_{il} = \{S_{ilx} \quad S_{ily} \quad S_{ilz}\}$ $(l = 1, 2, 3)$.

2.2 Solution to PDEs of Motion

Many modeling applications of curves and surfaces in computer graphics and computer-aided design such as interactive design and computer animation require real-time performance. Numerical solutions of PDEs are too expensive to fulfil this requirement. Closed form solutions of PDEs, which are the fastest, are obtainable only for some simple boundary conditions. In the following, we present a solution method making use of the composite power series.

Like the treatment given by You and Zhang (2004), we first define linearly independent basic functions as constant 1, parametric variable v, time variable t, their various elementary functions excluding polynomials, and their combinations not in a polynomial form. Then we can decompose the boundary conditions (3) and (4) into such basic functions.

To facilitate the description, we also define a new vector product operator whose operands are two vectors of the same dimension and each element of the resultant vector is the product of the corresponding elements of the two vectors, i. e.,

$$\mathbf{pq} = \{p_x q_x \;\; p_y q_y \;\; p_z q_z\} \tag{5}$$

where $\mathbf{p} = \{p_x \;\; p_y \;\; p_z\}$ and $\mathbf{q} = \{q_x \;\; q_y \;\; q_z\}$ are two column vectors.

According to the decomposed linearly independent basic functions, boundary conditions (3) and (4) can be rewritten as follows, respectively

$$u = u_i \quad \mathbf{x} = \sum_{j=0}^{J} \mathbf{b}_{ij}\mathbf{s}_{ij} \quad \mathbf{x}_{,u} = \sum_{j=0}^{J} \mathbf{c}_{ij}\mathbf{s}_{ij} \tag{6}$$

$$u = u_i \quad \mathbf{x} = \sum_{j=0}^{J} \mathbf{b}_{ij}\mathbf{s}_{ij} \quad \mathbf{x}_{,u} = \sum_{j=0}^{J} \mathbf{c}_{ij}\mathbf{s}_{ij} \quad \mathbf{x}_{,u^2} = \sum_{j=0}^{J} \mathbf{d}_{ij}\mathbf{s}_{ij} \tag{7}$$

where $\mathbf{b}_{ij} = \{b_{ijx} \;\; b_{ijy} \;\; b_{ijz}\}$, $\mathbf{c}_{ij} = \{c_{ijx} \;\; c_{ijy} \;\; c_{ijz}\}$ and $\mathbf{d}_{ij} = \{d_{ijx} \;\; d_{ijy} \;\; d_{ijz}\}$ are the known constants, and $\mathbf{s}_{ij} = \{s_{ijx} \;\; s_{ijy} \;\; s_{ijz}\}$ are the linearly independent basic functions which involve the same variables as those of $\mathbf{S}_{il}$ $(l = 1, 2, 3)$ depending on different modeling tasks.

The curve or surface to be generated can now be approximately represented with a composite power series which combines the power series of the parametric variable u with the linearly independent basic functions $\mathbf{s}_{ij}$. Thus the ith curve or surface segment can be given by

$$\mathbf{x}_i = \sum_{j=0}^{J} \sum_{m=0}^{M} \mathbf{r}_{ijm} u^m \mathbf{s}_{ij} \tag{8}$$

where $\mathbf{r}_{ijm}$ $(i = 0,\ 1,\ 2,\ \cdots)$ are the unknown constants to be determined, and M may be set to the same or different integers for different position function components and different terms of the same position function component.

When $\mathbf{s}_{ij} = \mathbf{1}$ or its some component is 1, the corresponding M should be set to 3 for Eq. (1) and 5 for Eq. (2) because these two equations have been satisfied for these cases and only the boundary conditions require to be considered.

Eq. (8) represents the approximate analytical solution of PDEs (1) and (2) under boundary conditions (3) and (4). Substituting Eq. (8) into boundary conditions (6), we determine the unknown constant $\mathbf{r}_{ijm}(j = 0,\ 1,\ 2,\ \cdots,\ J;\ \ m = 0,\ 1,\ 2,\ 3)$. Then the vector-valued function $\mathbf{x}_i$ is written in the following form which satisfies boundary conditions (3) exactly

$$\mathbf{x}_i = \sum_{j=0}^{J}\left[\mathbf{g}_i(u,\mathbf{b}_{ij},\mathbf{c}_{ij},\mathbf{b}_{i+1j},\mathbf{c}_{i+1j}) + \sum_{m=4}^{M}\overline{\mathbf{g}}_i(u,m)\mathbf{r}_{ijm}\right]\mathbf{s}_{ij} \quad (9)$$

where $\mathbf{g}_i(u,\mathbf{b}_{ij},\mathbf{c}_{ij},\mathbf{b}_{i+1j},\mathbf{c}_{i+1j})$ is the function of parametric variable u and the known constants $\mathbf{b}_{ij}$, $\mathbf{c}_{ij}$, $\mathbf{b}_{i+1j}$ and $\mathbf{c}_{i+1j}$. That is $\mathbf{g}_i = \{g_{ik}\}$ and $g_{ik} = g_{ik}(u, b_{ijk}, c_{ijk}, b_{i+1jk}, c_{i+1jk})$ $(k = x, y, z)$. $\overline{\mathbf{g}}_i(u,m)$ is the function of the parametric variable u and index m.

Similarly, substituting Eq. (8) into boundary conditions (7), the unknown constants $\mathbf{r}_{ijm}(j = 0, 1, 2, \cdots, J;\ m = 0, 1, 2, \cdots, 5)$ can be determined and the vector-valued function $\mathbf{x}_i$ meeting the boundary conditions (4) accurately is written as

$$\mathbf{x}_i = \sum_{j=0}^{J}\Big[\mathbf{G}_i(u,\mathbf{b}_{ij},\mathbf{c}_{ij},\mathbf{d}_{ij},\mathbf{b}_{i+1j},\mathbf{c}_{i+1j},\mathbf{d}_{i+1j}) + \sum_{m=6}^{M}\overline{\mathbf{G}}_i(u,m)\mathbf{r}_{ijm}\Big]\mathbf{s}_{ij} \quad (10)$$

Respectively substituting Eq. (9) into (1), and (10) into (2), we can obtain the residual value functions of these PDEs. Within the region where the curve or surface is defined, choosing N collocation points and substituting the coordinate values of these collocation points into the residual value functions, the residual values at these collocation points can be written as

$$\mathbf{R} = \mathbf{AC} - \mathbf{B} \quad (11)$$

By minimizing the squared sum of the residual values of Eq. (11) using the least squares technique (You et al. 2000), we obtain the following linear algebra equations

$$\mathbf{A}^T\mathbf{AC} = \mathbf{A}^T\mathbf{B} \quad (12)$$

The solution of Eq. (12) determines the rest unknown constants of Eq. (9) or (10). Then curves or surfaces can be generated from the analytical mathematical equation (8).

3 COMPUTATIONAL ACCURACY AND EFFICIENCY

Equations (9) and (10) are analytical expressions. Although to determine some of the unknown constants, the least squares technique was employed for the solution of a very small number of linear equations, the efficiency is very close to a closed form solution. Also because we ensure the boundary conditions are met exactly and errors in the inner region of the generated curve and surface are minimized, we can expect to have a good accuracy. To verify the speculation that the above method provides both good accuracy and efficiency, in this section we are undertaking a numerical study.

This study is to make comparisons between the proposed composite power series solution, finite difference solution and the corresponding closed form solution for a specified example where a closed form solution exists. Firstly, we investigate the error and efficiency between the proposed solution and closed form solution for both Eq. (1) and Eq. (2). Then we compare the efficiency and accuracy of the three methods only for the static form of Eq. (1). For a surface, since the determination of x, y and z components are the same, we only discuss the x component. In order to obtain its closed form solution, the damping term and force function are set to zero, the density is assumed to be $\rho = 1$, the vector-valued parameters are taken to be $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{1}$ and $\mathbf{a}_3 = \mathbf{a}_4 = -\mathbf{1}$, and the boundary conditions have the form of

$$\begin{aligned} u &= 0 \qquad x = x_{,u} = x_{,u^2} = \cos v \sin t \\ u &= 1 \qquad x = x_{,u} = x_{,u^2} = 2\cos v \sin t \end{aligned} \quad (13)$$

where the derivatives of the second order in the above equation are redundant for Eq. (1).

The closed form solutions of Eqs. (1) and (2) subject to the above boundary conditions are denoted with x, and the composite power series solutions and finite difference solutions are represented with $\widetilde{x}$. In order to quantify the difference between these methods, we choose $I_u \times J_v$ points within the solving region and introduce the following error equation

$$\overline{E} = \frac{1}{I_u \times J_v}\sum_{i=1}^{I_u}\sum_{j=1}^{J_v}\left|\frac{x(u_i,v_j,t) - \widetilde{x}(u_i,v_j,t)}{x(u_i,v_j,t)}\right| \quad (14)$$

According to the proposed method, we can obtain the composite power series solutions $\widetilde{x}$ of this problem for both Eqs. (1) and (2). In this case study, the collocation points are taken to be 9 for both equations, and the number of terms of the composite power series is 5 for Eq. (1) and 7 for Eq. (2).

Taking the resolution region to be $\{0 \le u \le 1;\ 0 \le v \le \pi\}$, uniformly choosing 101×101 points within the resolution region, substituting the values of the two solutions at these points into the above equation, we find that the relative error between the closed form solution and the proposed solution is $\overline{E} = 7.37 \times 10^{-3}$ for Eq. (1) and

$\overline{E} = 1.05 \times 10^{-4}$ for Eq. (2). Clearly, these errors are very small.

The computational efficiency of the proposed method is also very high. We have timed the process determining the unknown constants in the closed form solutions and composite power series solutions. It was found that both methods took less than 10^{-6} second on an ordinary PC to solve Eqs. (1) and (2).

In order to further demonstrate good accuracy and efficiency of proposed composite power series method, in the following, we carry out the finite difference calculation of Eq. (1) subject to boundary conditions. For simplicity, we only study the static problem of the above example, i. e., set $\sin t = 1$ in the boundary conditions (13), neglect the force function and all other terms containing the partial derivatives with respect to the time variable t in Eq. (1), and take the resolution region to be $\{0 \le u \le 1;\ 0 \le v \le 1\}$. The collocation points and terms of the proposed composite power series are the same as above. The boundary conditions for the finite difference calculation are taken from the closed from solution of the same problem. Within the resolution region, uniformly set $N_u \times N_v$ nodes, and determine the values of x component at these nodes using the finite difference formulae of Eq. (1) and all the boundary conditions of this problem. Then calculate the values of the proposed power series solution and closed form solution at these nodes, and use Eq. (14) to find the errors among them. In Table 1, PS means the errors between the proposed power series solution and the closed form solution, FD stands for the errors between the finite difference solution and the closed form solution, and the last row of the table gives the time of the finite difference solution. The time of the proposed solution and closed form solution is less than 10^{-6} second once again.

Table 1: Comparison of accuracy and efficiency.

$N_u \times N_v$	15×15	25×25	35×35
PS	0.00437	0.0045	0.00455
FD	0.738	0.622	0.579
Time(seconds)	3.08	71.5 4	573.78

It is very clear that the proposed method has much better computational accuracy and efficiency than the finite difference method. Although the total number of the nodes was greatly increased leading to very expensive computational cost, the computational accuracy of the finite difference method was not improved obviously. From the tendency of the computational errors given by the finite difference method, it appears difficult to reach as high accuracy as that of the composite power series solution although we increase the nodes for the finite difference calculation. Low computational efficiency of numerical methods indicates they are less ideal for the computer graphics applications requiring real-time performance.

In summary, the proposed composite power series solution is both accurate and efficient. It can generate surfaces with the similar efficiency and accuracy to the closed form solution method, far more quickly and accurately than numerical methods.

The proposed method can be employed to a wide range of shape modeling applications. In the following, we will apply this method to solve a number of dynamic and static modeling problems of curves and surfaces.

4 DYNAMIC MODELLING

Dynamic modeling of curves and surfaces is a very interesting subject of computer animation. With the developed composite power series method based on Eqs. (1) and (2) together with boundary conditions (3) and (4), we can perform dynamic modeling of curves and surfaces analytically.

4.1 Dynamic Surface Modeling

For dynamic surface modeling, we here give an example to show how an original surface is consecutively changed to a series of different surfaces using Eq. (1). The boundary conditions for this dynamic modeling are

$$\begin{aligned}
&u = 0\\
&\{x \quad x_{,u}\} = \{r_1 \quad r_1'\}(3-2t)\cos 2\pi v + \{r_2 \quad r_2'\} t \sin 12\pi v\\
&\{y \quad y_{,u}\} = \{r_1 \quad r_1'\}(2-t)\sin 2\pi v + \{r_2 \quad r_2'\} t^2 \cos 12\pi v\\
&\{z \quad z_{,u}\} = \{h_0 \quad h_0'\} + \{h_1 \quad h_1'\} t \sin 10\pi v\\
&u = 1\\
&\{x \quad x_{,u}\} = \{r_3 \quad r_3'\}\cos 2\pi v + \{r_4 \quad r_4'\}(1-t)\sin 12\pi v\\
&\{y \quad y_{,u}\} = \{r_3 \quad r_3'\}\sin 2\pi v + \{r_4 \quad r_4'\}(1-t^2)\cos 12\pi v\\
&\{z \quad z_{,u}\} = \{0 \quad 0\}
\end{aligned} \tag{15}$$

From the above boundary conditions, we can obtain the linearly independent basic functions $\cos 2\pi v$, $t\cos 2\pi v$, $\sin 12\pi v$, $t \sin 12\pi v$ for x component, $\sin 2\pi v$, $t \sin 2\pi v$, $\cos 12\pi v$, $t^2 \cos 12\pi v$

for y component, and 1 and $t\sin 10\pi v$ for z components. According to these linearly independent basic functions, we can construct the following composite power series functions

$$
\begin{aligned}
x &= \sum_{m=0}^{M} r_{00mx}u^m \cos 2\pi v + \sum_{m=0}^{M} r_{01mx}u^m t\cos 2\pi v \\
&\quad + \sum_{m=0}^{M} r_{02mx}u^m \sin 12\pi v + \sum_{m=0}^{M} r_{03mx}u^m t\sin 12\pi v \\
y &= \sum_{m=0}^{M} r_{00my}u^m \sin 2\pi v + \sum_{m=0}^{M} r_{01my}u^m t\sin 2\pi v \\
&\quad + \sum_{m=0}^{M} r_{02my}u^m \cos 12\pi v + \sum_{m=0}^{M} r_{03my}u^m t^2 \\
&\quad \cos 12\pi v \\
z &= \sum_{m=0}^{3} r_{00mz}u^m + \sum_{m=0}^{M} r_{01mz}u^m t\sin 10\pi v
\end{aligned}
\tag{16}
$$

The unknown constants r_{0jmk} ($j = 0 \sim 3$ for x and y components and $j=0,1$ for z component; $m = 0 \sim 3$; $k = x, y, z$) in the above equation can be determined from boundary conditions (15). Then substituting Eq. (16) into (1), uniformly choosing 9 collocation points within the solving region and taking 6 terms for each composite power series which means only 2 unknown constants in Eq. (12) are to be determined, we can obtain the rest unknown constants and analytical mathematical equations of the surface to be created. Specifying the values of the geometric parameters in Eq. (15) and vector-valued parameters in Eq. (1), we can generate the surface at any time t.

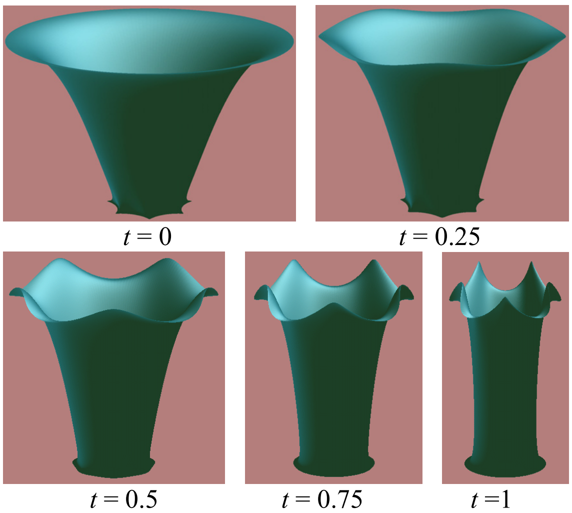

Figure 1: Dynamic modeling of a surface.

In Figure 1, we give the images of the surface at t=0, 0.25, 0.5, 0.75 and 1. They were created with one surface patch determined by the analytical mathematical equations. This example indicates that the proposed composite power series method can be used to animate objects directly such as skin deformation of the arms and legs during human motion, or produce a series of key frames of the object to be animated.

4.2 Dynamic Curve Modeling

When the parametric variable v in Eqs. (1) and (2) together with boundary conditions (3) and (4) is taken to be a constant, we can carry out dynamic modeling of curves. In Figure 2, we give an example to show how a straight line is consecutively changed to a human face profile which consists of four curve segments with each segment being determined by a vector-valued position function.

Figure 2: Dynamic modeling of a curve.

5 STATIC MODELING

When the time variable t in Eqs. (1)-(4) is set to a constant, these equations can be applied to perform static modeling of curves and surfaces. In the following, we will give some examples to indicate the applications of these equations in free-form surface generation and surface blending.

5.1 Free-form Surface Generation

The proposed method is also an effective means for free-form surface generation. Rather than moving the control points, the surfaces to be generated can be controlled and deformed simply by changing some parameters, such as the vector-valued shape control parameters, tangential and curvature boundary conditions as well as the force function. Since any complicated boundary curves, planar or spatial, can always be represented by mathematical functions, a surface so defined can be created with the proposed method.

By dividing the object to be created into some surface patches connected by the boundary curves, we can generate complicated free-form surfaces. In Figure 3, the surfaces of a fish were created with the proposed composite power series method.

Figure 3: Surface generation of a fish.

5.2 Surface Blending

Surface blending is another important area of surface modeling. Here we give an example to illustrate the application of the proposed method.

This example is to blend two intersecting cylinders using the solution of the static form of Eq. (1) under boundary conditions (3). We only employ 18 collocation points in the blending region and 6 terms for each composite power series. The obtained blending surface is given in Figure 4. The whole resolution process took less than 10^{-6} second.

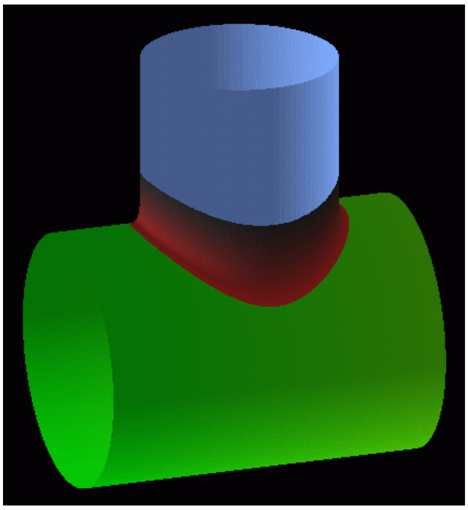

Figure 4: Blending between two intersecting cylinders.

6 CONCLUSIONS

In this paper, we have proposed a unified curve and surface modeling approach so that both static and dynamic problems can be represented in a uniform manner. This approach is based on the use of two partial differential equations of motion, a vector-valued fourth order PDE and a vector-valued sixth order PDE. The former is able to satisfy tangent boundary conditions, while the latter is able to meet curvature conditions.

A key element of making the proposed approach applicable to interactive graphics applications is to solve these PDEs efficiently and effectively. To this point, we have used a composite power series method, which is able to give analytical mathematical equations of curves and surfaces to be created. The positional, tangential and curvature functions in the boundary conditions were firstly decomposed into a number of linearly independent basic functions which combine with the power series of another parametric variable to formulate approximate solution functions. By determining some unknown constants in these solution functions, the boundary conditions are always exactly satisfied. The residual values in the proposed PDEs are minimized using the least squares technique which further reduces the discrepancy between the approximate and the accurate shapes.

The computational accuracy and efficiency of the proposed composite power series method have been investigated. The comparisons between this research, the finite difference approach and the closed form solution indicate that the proposed method can generate surfaces with similar efficiency and accuracy to the closed form solution method, and far more quickly and accurately than numerical methods.

The proposed partial differential equations can also be degenerated for the purpose of curve modeling in a unified format. Since curve modeling is more flexible, it is useful in complex surface modeling as well.

To demonstrate its applications, we have applied this approach to a number of examples of static and dynamic modeling of curves and surfaces.

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