A NETWORK MODEL FOR PRICE STABILIZATION

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Abstract: We consider a simple network model for economic agents where each can buy commodities in the neighborhood. Their prices may be initially distinct in any node. However, by assuming some rules on new prices, we show that the distinct prices will be converged to unique by iterating buy and sell operations. If we consider the price determination process as a kind of consensus problem, we can apply the stabilization proof to it. So we first present a naive protocol in which each agent always offers half of the difference between his own price and the lowest price in the neighborhood, called max price difference. Then, we consider game theoretic price determination in two ways, that is, by using different payoff functions. Finally, we propose a protocol in which each agent makes a bid uniformly distributed over the max price difference.

1 INTRODUCTION

Conventionally, the topics of price determination have been discussed in the context of microeconomics approach (J. E. Stiglitz, 1993). A famous model, supply and demand curves, has been used as an abstract, theoretical method which explains a price equilibrium. However, for example, there is no distance concept in the model. To know a detailed process to the equilibrium, we need more sophisticated model, e.g., multiagent approach, which gives us another insight into the price determination.

We construct a price determination model by applying the idea of stabilization to the multiagent approach. The self-stabilization (S. Dolev, 2000) has been originally studied as the recovery from transient faults in distributed systems. From any initial state, self-stabilizing algorithms eventually lead to a legitimate state without any aid of external actions. In particular, a self-stabilizing consensus algorithm is associated with the price determination because every agent eventually has the same value.

We show a network model consisting of nodes and edges as cities and their links to neighbors, respectively. Each node contains an agent which represents people in the city. Any interaction among agents is governed by micro-rules, that is, the agents who want to buy a commodity make bids to their neighboring nodes. Then, the agents who want to sell the commodity accept the highest bid, like an auction (V. Krishna, 2002). By iterating these rules, the prices will reach an equilibrium.

First, we present a naive protocol in which each agent always offers a fixed price without considering other bidders’ strategies. Then, we analyze the stabilization time of the protocol for a special case. Next, we consider game theoretic price determination in two ways, that is, by using different payoff functions. Finally, we propose a protocol in which each agent offers a random price and show that it stabilizes with high probability.

2 MODEL

Our system can be represented by a connected network \( G = (V, E) \), consisting of a set of \( n \) nodes \( V \) and edges \( E \), where the nodes represent cities and a pair of neighboring nodes (cities) is linked by an edge. We assume that each node \( i \in V \) has a commodity and its initial price may be different. Let \( N_i = \{ i \} \cup \{ j \} \). Let \( P_i(t) \) be the commodity price in the node \( i \) at time \( t \). It is also denoted by \( P_i \) if time \( t \) is not important. We say that the price \( P_i(t) \) is maximal if \( P_i(t) \geq P_j(t) \) for any
\section{Naive Protocol}

In this section, we consider a naive protocol, called \textbf{HalfBid}, in which each agent always offers half of the max price difference. Here, we focus on a star, the part of a network \( G \), with a center node \( c \).

\textbf{HalfBid}

- Each agent \( a_j \) makes a bid with an integer price
  \[ P_c(t) + \left\lfloor \frac{P_j(t) - P_c(t)}{2} \right\rfloor \]
  to node \( c \in N_j^+ \) which has the lowest-priced commodity in \( N_j^+ \). The agent \( a_j \) contracts with the neighboring \( a_j \) who has submitted the highest bid. That is, the commodity price at time \( t + 1 \) is
  \[ P_c(t+1) := P_c(t) + \max_{j \in N_c} \left\lfloor \frac{P_j(t) - P_c(t)}{2} \right\rfloor \]
- If \( P_c(t) \) is maximal and \( a_i \) accepts no bidding from \( N_c \), the price at time \( t + 1 \) will be cut to
  \[ P_c(t+1) := P_c(t) - \max_{j \in N_c} \left\lfloor \frac{P_c(t) - P_j(t)}{2} \right\rfloor \]
- If several agents make bids to node \( c \) with the same highest price, agent \( a_i \) contracts with one of them with equal probability.

Example 1. Figure 1 shows an example of our network system consisting of 4 nodes \( V = \{0, 1, 2, 3\} \). At time 1, the prices of commodities are \( (P_0(t), P_1(t), P_2(t), P_3(t)) = (50, 10, 110, 70) \) as shown in Figure 1(a). Each agent \( a_i \) wants to buy the commodity if its price is lower than \( P_i(t) \), i.e., \( P_i(t) > \min_{j \in N_i} P_j(t) \). Thus, agent \( a_2 \) makes a bid to node \( 0 \) with price \( 50 + (110 - 50)/2 \). Likewise, agents \( a_0 \) and \( a_1 \) make bids to nodes \( 1 \) and \( 0 \), respectively.

Then, \( a_3 \)'s bid and \( a_0 \)'s bid are successful, \( a_2 \) (resp. \( a_0 \)) makes a contract with \( a_3 \) (resp. \( a_1 \)). At time \( t + 1 \), the prices become \( (80, 30, 80, 70) \) as shown in Figure 1(b). Since node 2 was maximal and no bid was submitted, the price is cut to 80 at time \( t + 1 \).

Let \( G \subseteq V \) be the set of nodes that have updated their prices from time \( t \) to \( t + 1 \). Let the highest price be \( P_{\text{max}}(t) = \max_{i \in G} P_i(t) \), and the lowest price be \( P_{\text{min}}(t) = \min_{i \in G} P_i(t) \). The following lemma states that prices continue to move until every node has the same price.

\textbf{Lemma 1.} The protocol \textbf{HalfBid} is deadlock-free. That is, there exist some nodes in \( C_t \) as long as the unique price is not determined.

\textbf{Proof sketch.} The lemma is proved by contradiction. \( \square \)

\textbf{Lemma 2.} Let \( \text{diff}(t) = \max_{i \in G} P_i(t) - \min_{i \in G} P_i(t) \). As long as \( C_t \neq \emptyset \), we have
\[ \text{diff}(t) > \text{diff}(t+1). \]

\textbf{Proof.} First, we consider a node \( i \notin C_t \). Since agent \( a_i \) does not make any bid to other nodes, there is no change in price. Thus, \( \max_{j \in N_i} \left( P_j(t) - P_i(t) \right) = 0 \) holds.

Next, suppose that a node has the maximum price in \( C_t \). Since no neighboring nodes make bids to such a node, the price will be down at time \( t + 1 \). If a node has the minimum price in \( C_t \), there is a neighboring agent who makes a bid to the node. Thus, the price will be up at time \( t + 1 \). Let \( P_{\text{max}}(t) \) be the second maximum price among the nodes in \( C_t \). Then, the price will not exceed \( P_{\text{max}}(t) \) at time \( t + 1 \) because \( P_{\text{max}}(t) \) goes maximally up only when it accepts an offer from \( P_{\text{max}}(t) \). Even if it occurs, the increase is at most the half of the difference between them. Thus,
we have
\[ P_{\text{max}}(t) > P_{\text{max}}(t+1), \quad P_{\text{max}}(t) > P_{\text{max}}(t+1). \]

On the other hand, the node with \( P_{\text{min}}(t) \) accepts a new bid and the price \( P_{\text{min}}(t) \) goes up at time \( t+1 \). Let \( P_{\text{min}}(t) \) be the second minimum price among the nodes in \( C_i \). Then, it maximally decreases without any offers only when it is linked with the node with \( P_{\text{min}}(t) \). Then, we have
\[ P_{\text{min}}(t) \leq P_{\text{min}}(t+1), \quad P_{\text{min}}(t) < P_{\text{min}}(t+1). \]

Thus, \( \text{diff}(t) > \text{diff}(t+1) \) holds.

**Theorem 1.** Our protocol will eventually stabilize the price.

In the sequel, we analyze the stabilization time of a path \((1, \ldots, n)\).

**Theorem 2.** If network \( G \) is a path, the stabilization time of our HalfBid is \( 2\gamma \) rounds, where \( \gamma \) satisfies
\[ \frac{4}{9} \left( \frac{1}{2} \right)^{2\gamma} \left( \frac{1}{2} \right)^{2\gamma} = 1. \]

**Proof.** We call the price difference between neighboring nodes a gap, and call the gaps as 1st gap, 2nd gap ... in the ascending order of the nodes. Let \( d_i(t) \) be the difference of the \( i \)-th gap at time \( t \), where \( i \) means every other time here. Then, we have the following recurrences.
\[
\begin{align*}
d_1(t+1) &= \frac{1}{4} d_{i-1}(t) + \frac{1}{2} d_i(t) + \frac{1}{4} d_{i+1}(t) \quad (1) \\
d_2(t+1) &= \frac{1}{2} d_1(t) + \frac{1}{4} d_2(t) \quad (2) \\
d_h(t+1) &= \frac{1}{2} d_{h-1}(t) + \frac{1}{4} d_h(t) \quad (3)
\end{align*}
\]

Let \( S_{h-j}(t) = \sum_{i=j+1}^{h} d_i(t) \) and \( S_0(0) = D \). Summing (1) from \( i = 1 \) to \( h \) by using (2) and (3) gives
\[ S_h(t+1) = S_h(t) - \frac{1}{4} \left( d_1(t) + d_h(t) \right). \]

Since \( d_1(t) + d_h(t) = S_h(t) - S_{h-1}(t) \), we have
\[ S_h(t+1) = \frac{3}{4} S_h(t) + \frac{1}{4} S_{h-1}(t). \]

By using a generating function, \( S_h(t) \) is given by
\[
D(3/4)^t \sum_{k=0}^{h} \left( \frac{1}{3} \right)^k = \frac{D(3/4)^t}{\left( \frac{1}{3} \right)^{h+1}} \left( t \right) - O(1/3^{h+2})
\]
\[
\leq D \left\{ 1 - \left( \frac{1}{3} \right)^t \left( \frac{1}{3} \right)^{n/2+1} \left( t \right) \right\}
\]
because \( h \approx n/2 \). Hence, \( S_h(t) = 0 \) gives \( \left( \frac{1}{3} \right)^t \left( \frac{1}{3} \right)^{n/2+1} \left( t \right) = 1 \). Since it takes \( 2\gamma \) rounds until convergence, the lemma follows. \( \square \)

## 4 GAME THEORETIC PRICE DETERMINATION

In this section, we consider how to determine a bidding price in two ways by using Bertrand model (J. E. Stiglitz, 1993). Here, as customary, we use “Player” instead of “agent”.

### 4.1 Payoff with Second Price

For simplicity, we consider a three-node path \((1,c)\) and \((c,2)\). Suppose that both Player 1 (i.e., agent \( a_1 \)) at node 1 and Player 2 (i.e., agent \( a_2 \)) at node 2 buy a commodity in the center node \( c \) at time \( t + 1 \). Let \( q_1 \) and \( q_2 \) be the bidding prices of Player 1 and Player 2, respectively. Then, the payoff of Player 1 is \( y = P_1(t) - q_1 \). On the other hand, the payoff of Player 2 is \( y = P_2(t) - q_2 \) in the same situation.

As the price goes up, the payoffs of the players go down. In addition, Player 1 (resp. Player 2) regrets if the difference \( q_1 - q_2 \) (resp. \( q_2 - q_1 \)) is large even if he gets the commodity. Next, if \( q_1 = q_2 \), both Player 1 and Player 2 get the commodity with probability 1/2. Thus, the payoff function of Player 1, \( u_1(q_1,q_2) \), is defined as
\[
\begin{align*}
u_1(q_1,q_2) = \begin{cases} \frac{(P_1 - q_1)}{(q_1 - q_2)} & q_1 > q_2 \\
\frac{(P_1 - q_1)}{2} & q_1 = q_2 \\
0 & q_1 < q_2 \end{cases}
\end{align*}
\]
The payoff function of Player 2, \( u_2(q_1,q_2) \), is similarly defined.

Now we temporarily assume that \( q_1 \) and \( q_2 \) are continuous variables. Then,
\[
\frac{\partial u_1}{\partial q_1} = (q_2 - P_1)/(q_1 - q_2)^2 = 0
\]
That is, we obtain \( q_2 = P_1 \). Similarly,
\[ \frac{\partial u_2}{\partial q_2} = (q_1 - P_2)/(q_1 - q_2)^2 = 0 \]
That is, we obtain \( q_1 = P_2 \). Thus, \( (q_1,q_2) = (P_2,P_1) \) is the Nash equilibrium. This is also true when \( q_1 \) and \( q_2 \) are integers.

### 4.2 Payoff without Second Price

In this section, we define the payoff function of Player 1 and that of Player 2 as follows.
\[
u_1(q_1,q_2) = \begin{cases} \frac{P_1 - q_1}{(P_1 - q_1)/2} & q_1 > q_2 \\
0 & q_1 < q_2 \end{cases}
\]
\[
u_2(q_1,q_2) = \begin{cases} \frac{P_2 - q_2}{(P_2 - q_2)/2} & q_1 < q_2 \\
0 & q_1 > q_2 \end{cases}
\]
Based on the consideration in Section 4, we propose a protocol, called RandomBid, using a price with the Bertrand model. Again, we focus on a star, the part of a network $G$, with a center node $c$.

**RandomBid**

- At time $t + 1$, agent $a_i$ offers a random integer price over the range $[P_i, P_i - 1]$ to the neighboring node $c$ with the minimum price $P_c(t) < P_i(t)$.
- If agent $a_c$ with maximal $P_c(t)$ does not accept any bid from $N_c$ and its own offer is accepted by node $i$, the price $P_i(t)$ is decreased to the offered price.
- If several agents make bids to node $c$ with the same highest price at time $t$, agent $a_c$ contracts with one of them with equal probability.

The following lemma states that the agent with a maximal price will win the contract.

**Lemma 3.** The agent $a_i$ with a maximal price will win the contract with probability at least $\frac{P_i - P_c}{P_i - P_c}$, where agent $a_c$ has the second maximal price.

**Proof.** Since only agent $a_i$ can submit a bid with range $[P_i, P_i]$, it can win the contract with agent $a_c$ at the rate. Thus, the probability is at least $\frac{P_i - P_c}{P_i - P_c}$. □

The above lemma means that the agent $a_i$ with a maximal price will win the contract with probability at least $1 - (\frac{P_i - P_c}{P_i - P_c})^m$ after $m$ rounds.

**Theorem 3.** Our RandomBid will eventually stabilize the price with high probability. □

### 6 CONCLUSIONS

In this paper we considered a new network model for the price stabilization. The model shows that the self-stabilization has a wide application to various areas. Our goal is to construct a good multiagent protocol which enables us to simulate a realistic social system. Then, we could analyze and estimate several economic phenomena.

### REFERENCES

