Stability and Performance of Scheduling Policies in a Transportation Node

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Abstract. In this paper we consider the dynamic model of a logistic node of a transportation network and study dispatching feedback policies in terms of stability and optimality. A necessary and sufficient condition for the existence of a stable feedback policy is given and a policy is presented which would be optimal if the transportation resources were continuous.

1 Introduction

An intermodal logistic system can be modeled as a network comprising a set of nodes (hubs and terminals) connected by the links established by the transport operations, which, in general, take place under different modes. The management of logistic nodes in this network is a complex problem where several factors have to be taken into account, from the availability of carriers and their assignment to particular tasks (in terms of products to be shipped, destinations or routes), and fulfillment of various performance criteria such as timely delivery, minimization of transportation and inventory costs (possibly, both at the logistic nodes and at the destinations); see among others, [1–4].

Many instances of decisional problems for these systems are presented and solved in the literature; often transportation problems can be addressed in terms of linear programming problems, see e.g., [5–8], and developing ad-hoc techniques to obtain the solutions, such as dynamic programming with linear approximation of the (unknown) value function. It is worth mentioning that by the approach of [5, 6] the framework of the Logistic Queuing Networks is introduced.

A slightly different paradigm considers shipping policies for simplified models of a logistic network (or a part of it) and addresses the minimization of transportation and inventory costs, see for example [9, 10] and [15] where a stochastic setting is adopted. Still another example of management problems for a logistic node is represented by the optimization of space allocated for containers in ports (e.g., [13]); or the optimization of the operations of discharging containers from a ship, their location in the terminal yard and the upload of new containers [14]. In these last two cases, the performances considered can be also viewed in terms of the necessity to maintain low levels of stocked products in the logistic node (in this case, a port). The stability of the dynamics of the stock at a logistic node is therefore a relevant issue to be taken into account.
In order to study the stability (and the performance of stabilizing policies) from a dynamical point of view, in this paper a simplified model of a logistic node in a transportation network is considered and feedback policies based on the current state of the system are defined to control the node. The scenario is similar to those arising in other applicative domains (like in manufacturing, communications, computer systems or queuing networks in general), so the feedback policies considered in this paper have been inspired by various well established techniques developed in those domains. The stability of these policies will be investigated and a necessary and sufficient condition for the possibility of stabilizing the system will be determined. The sufficiency will be established in a constructive way by determining a class of policies which guarantee the stability of the system. A comparison among the performances of different stabilizing policies will be carried out through simulations, showing that a policy, inspired by a control known as optimal for the fluid version of the problem, will provide the best results among the policies considered in the paper.

2 Problem Formulation

Consider the discrete time model of a logistic node collecting $Q$ different types of wares which have to be shipped to $P$ different locations, and let $x_{ij}(k) \geq 0$ be the quantity of items of type $j = 1, \ldots, Q$, with destination $i = 1, \ldots, P$, stocked at the logistic node at time $t_k$, and collected in the buffer $B_{ij}$. In this model a destination could be more in general considered as a route among different locations, established through some routing algorithm.

The time evolution of each $x_{ij}$ is observed at various decisional time instants $t_k$, and as such characterized by a discrete time dynamics. Denoting $d_{ij}(k)$ the amount of goods of type $j$ to be sent to destination $i$ arriving in the node in the interval $(t_k, t_{k+1})$ and $u_{ij}(k)$ the amount of goods of type $j$ shipped to destination $i$ from the node in the same interval $(t_k, t_{k+1})$, we have:

$$x_{ij}(k+1) = x_{ij}(k) + d_{ij}(k) - u_{ij}(k)$$

(1)

In addition to this dynamics, we consider that of the vehicles executing the shipping task. Let $n_i(k)$ be the number of vehicles assigned to destination $i$ in the interval $(t_k, t_{k+1})$; the total number $N(k)$ of vehicles present in the node at time $t_k$ obeys to the following equation:

$$N(k+1) = N(k) + R(k) - \sum_{i=1}^{P} n_i(k)$$

(2)

where $R(k)$ is the number of vehicles arriving from outside in the interval $(t_k, t_{k+1})$. Notice that, according to the above dynamics, the total number of vehicles available for a shipping task at time $t_k$ is given by

3 These buffers could be considered as virtual, in the sense that in some cases we may have items which are physically stocked in different places according to their type (in such a way that the physical content of a buffer is given by $\sum_{i=1}^{P} x_{ij}$) as it happens for the stocked finished products in a factory.
To model the inflow of vehicles $R(k)$, first consider the simple scenario where there is a fixed quantity $N_c$ of vehicles which could serve the logistic node (as if for example, the logistic node is a shipper who owns a certain quantity $N_c$ of trucks). A traveling time $T_i \in \mathbb{N}$ is associated to each route (i.e. destination $i$); in particular $T_i$ denotes the round-trip time, i.e. the interval after which a vehicle is again available at the node after completion of a shipping task to destination $i$. In this case $R(k) = \sum_{i=1}^{P} n_i(k - T_i)$ is the number of vehicles coming back from their expedition, and therefore (2) reads as follows:

$$N(k+1) = N(k) + \sum_{i=1}^{P} n_i(k - T_i) - \sum_{i=1}^{P} n_i(k); \quad N(0) = N_c \quad (4)$$

Notice that the total sum of vehicles (those at the logistic node and those traveling) equals $N_c$ at each time instant.

In other cases we can consider the logistic node and the shippers as separate entities, so that the total number of vehicles which are going to access the logistic node varies with time; in such situations $N_c$ can be obtained through a suitable average of the expedition history in the node, and can be possibly perturbed when new vehicles are assigned to (or removed from) the node. A possible way to model this situation is by perturbing the signal $R(k)$, i.e., $R(k) = \sum_{i=1}^{P} n_i(k - T_i) + \Delta(k)$, where $\Delta$ is a disturbance signal characterized by certain statistical properties (e.g. zero mean). Another interesting extension would be to add to the round trip time some noise (possibly asymmetric, in the sense that positive perturbations, so that $T_\triangleright > T_i$, are more likely to occur than negative ones). As a first approach to the problem, in the following we consider the simplified model (4), i.e., assuming that the $T_i$’s are deterministic quantities and $N_c$ is fixed. In this case the number of available vehicles at time $t_k$ is:

$$N_a(k) := N(k) + R(k) \quad (3)$$

Let’s now consider the interaction between the stock dynamics (1) and the vehicle dynamics (4). To this end, assume that each vehicle has identical volume capacity and that each item of type $j = 1, \ldots, Q$ has a relative volume with respect to vehicle capacity $v_j \leq 1$ (that is, a vehicle has unit capacity). Accordingly, we have the following constraint for any route $i$:

$$\sum_{j=1}^{Q} v_j u_{ij}(k) \in [0, n_i(k)]. \quad (5)$$

Since $n_i(k)$ vehicles are used at time $t_k$ for route $i$, it is reasonable that the above quantity is larger than $n_i(k) - 1$ (actually, by the policies that will be considered in this paper, vehicles travel completely loaded).

The objectives of this work will be essentially two. First, derive conditions on the stability of the system, that is conditions on the inflow process $d(\cdot)$, (relative) part volumes $v_j$, traveling times $T_i$ and number of vehicles $N_c$ such that there exists a policy of selection of $n_i(\cdot)$ and $u_{ij}(\cdot)$ which maintains limited all the buffers $x_{ij}(\cdot)$. Second, analyze the performance of some class of policies, trying to solve the optimization problem.
consisting in the selection of the $n_i(k)$ and of the $u_{ij}(k)$ to minimize:

$$\begin{align*}
J &= \sum_{k=1}^{K} g[x(k)]\gamma^k \\
\end{align*}$$

(6)

where $\gamma \in (0, 1]$ is a discount factor and $K$ a planning horizon, possibly infinite. The function $g(\cdot)$ penalizes waiting freights in the node, e.g., for a linear $g(x)$.

$$g(x) = \sum_{i=1}^{P} \sum_{j=1}^{Q} c_{ij} x_{ij}$$

(7)

We now make a fluid approximation for the variables involved in (1), considering $x_{ij}$, $d_{ij}$ and $u_{ij}$ as continuous quantities. Accordingly, the information about the volume of each type $j = 1, \ldots, Q$ is now carried by the continuous variables (now the relative volumes $v_j$ have no sense per se, hence they will be dropped in the following) and each cost $c_{ij}$, assuming a fixed $i$, now has the meaning of holding cost of part $j = 1, \ldots, Q$ per unit volume⁴. Notice that also the variables $N, N_c, n_i$ will represent volumes (multiples of the unit volume).

We will deal with the two problems above by restricting the control policies to those which make vehicles travel completely full (this is possible under the fluid approximation of the materials): this should represent, as remarked below, a correct choice under heavy traffic conditions. Notice, also, that transportation costs have not been included in the cost index: this depends on the fact that (i) transportation costs are considered constant in time; (ii) we restrict the analysis to policies which make all vehicles travel completely full. The assumptions above imply that the transportation cost is a fixed component that does not influence the optimization problem. The choice of considering the vehicles fully loaded is reasonable under heavy traffic conditions (where allowing the possibility of sending partially full vehicles may even compromise the stability), but may become significantly sub-optimal in the case of reduced inflow rates, large holding costs $c_{ij}$ and small traveling costs.

3 Stability

As an introduction, consider a one part-type system ($Q = 1$) with constant inflow processes $d_i$ and equal transportation times $T_i = T, \forall i$. The equations are then:

$$\begin{align*}
x_i(k+1) &= x_i(k) + d_i - u_i(k), \quad i = 1, \ldots, P \\
u_i(k) &\in [0, n_i(k)] \\
\sum_{i=1}^{P} n_i(k) &\leq N_a(k) \\
N(k+1) &= N(k) + \sum_{i=1}^{P} n_i(k-T) - \sum_{i=1}^{P} n_i(k) \\
\end{align*}$$

(8) (9) (10) (11)

⁴ Formally, as if the system were described by new variables $x_{ij}' = x_{ij} v_j$ (and similarly for $d_{ij}$ and $u_{ij}$) and $c_{ij}' = c_{ij} / v_j$; dropping the “prime” and remaining with the same notation.
Based on the Little’s law, the necessary and sufficient condition of stability for this system should be:

\[ \sum_{i=1}^{P} d_i \leq \frac{N_c}{T} \]  

(12)

In fact, \( N_c/T \) is actually the effective number of vehicles available at each time unit, and hence also the volume of goods the node may handle in each unit of time. This must be equal to the volume arriving from outside, i.e. \( \sum_{i=1}^{P} d_i \).

The stronger condition that there exists a static vehicle allocation such that:

\[ T d_i \leq n_i \]  

(13)

for all \( i \), which implies condition (12), actually is not necessary (but clearly sufficient, since if it holds, allows to apply a policy where vehicles are divided once for ever among the tasks and each task is fulfilled, with no interaction among them), as shown in the following simple example.

**Example.** Consider a system with \( d_1 = d_2 = 0.5 \), \( T = 1 \), \( N_c = 1 \). Clearly it is not possible to distribute vehicles once for ever (in fact for any static selection of \( n_i \), condition (13) does not hold). However (12) holds and, in fact, the periodic allocation \( n_1(k) = \{1, 0, 1, \ldots \} \) and \( n_2(k) = 1 - n_1(k) \), maintains the buffers bounded. \( \square \)

Let us now return to the general case, but considering at first a constant inflow process. Condition (12) should be substituted by:

\[ \sum_{i=1}^{P} \sum_{j=1}^{Q} d_{ij} T_i \leq N_c \]  

(14)

which will be shown to be necessary and sufficient for the stability of the node. In this case, in fact, the quantity \( d_{ij} T_i \) plays the role of a work inflow in the system per unit of time (in the sense that for each item of type \( j \) to be sent to \( i \), the system must allocate a working capacity of \( T_i \), where the total working capacity is \( N_c \)). In the case of time varying inflow rates (but with the inflow rate oscillating in a bounded interval), the same condition should hold with average inflow rates \( \bar{d}_{ij} \).

**Remark 1.** Actually, while (14) is necessary for stability, the proof reported below only holds if the inequality in (14) is strict. We believe however that also the equality ensures the stability. Notice, in any case, that a strict inequality should be considered in practical settings to guarantee a certain degree of robustness of the stability property.

The previous discussion can be formalized in the following theorem.

**Theorem 1.** Condition (14) is necessary and sufficient (if taken with strict inequality) to maintain all the buffers in the node bounded at all times.

**Proof.** Necessity. The necessity of (14) can be shown by relaxing the integer constraint on the \( n_i(k) \). If the vehicle resource is not discrete, it is possible to maintain all the buffers bounded only if there exists a static assignment of the vehicles (notice in fact that in our model the inflow process is constant) which balances the freight inflow into
the system for all the routes $i$. The freight inflow into the system of parts to be sent on the route $i$ is given by $D_i := \sum_{j=1}^{Q_i} d_{ij}$. If $n_i$ vehicles are assigned to this route, since each transport requires $T_i$ time units, $n_i$ vehicles are available only every $T_i$ time units. The amount of wares accumulated in such a period is given by $D_i T_i$. So it must be $D_i T_i \leq n_i$. Summing over $i$, we get the condition (14). Since this condition is necessary for the relaxed problem, it is necessary also for the original problem.

**Sufficiency.** The proof of sufficiency is constructive: we exhibit a class of policies which, if (14) holds with strict inequality, ensures that all the buffers remain bounded. The proof is very similar to the proof of Theorem 1 in [11]. The class of policies ensuring stability is like the CAF policies in [11] where, however, the buffer $x_{ij}$ is processed not until it is cleared (level zero) but until its level becomes lower than $N_c$. That is: all the vehicles are assigned to a single route by filling them with the products of a certain buffer $B_{ij}$ (selected according to the CAF rule (15) reported below) only if this buffer has sufficient stock to use all vehicles, and the buffer is changed when this is no more possible. If no buffer can fill all the vehicles, the system remains idle until this becomes possible. Let $\tau_n$ denote the time a buffer has been finished to be processed. At each time $\tau_n$ the next buffer will be the one (denoted with a $*$) satisfying:

$$x^*(\tau_n) \geq \epsilon \sum_{i,j} x_{ij}(\tau_n)$$

for some $\epsilon > 0$ (e.g. the policy which selects the buffer with the largest content will belong to this class, satisfying (15) with any $\epsilon \in (0, 1/P)$, see [11]). Let $\bar{T}_i := T_i / N_c$.

Performing a derivation similar to the one reported in [11], it is possible to show that:

$$\tau_{n+1} - \tau_n \leq \bar{T}_i x^*(\tau_n) + \frac{N_c}{d_*}$$

where the $*$ denotes the quantities corresponding to the buffer selected at time $\tau_n$ and $d_* := \bar{T}_i d_*$. The terms in (16) have been obtained as follows: the first term $\bar{T}_i x^*(\tau_n)$ corresponds to the time to bring the buffer $x^*$ from its initial level $x^*(\tau_n)$ to a value below $N_c$ and is derived from [11] setting the setup time $\delta$ to 0 and considering that we only need to reach a value below $N_c$ and not 0; the second term $\frac{N_c}{d_*}$ takes into account that when a buffer is selected, perhaps its content is less than $N_c$. We define, as in [11]:

$$w(k) = \sum_{i,j} \bar{T}_i x_{ij}(k)$$

Then we have:

$$w(\tau_{n+1}) = \sum_{i,j} \bar{T}_i x_{ij}(\tau_{n+1}) =$$

$$= \sum_{i,j \neq *} \bar{T}_i [x_{ij}(\tau_n) + d_{ij}(\tau_{n+1} - \tau_n)] + \bar{T}_i x^*(\tau_{n+1})$$

$$= w(\tau_n) + \sum_{i,j \neq *} \bar{T}_i [x_{ij}(\tau_{n+1}) - x_{ij}(\tau_n)] + \bar{T}_i [x^*(\tau_{n+1}) - x^*(\tau_n)]$$
\[ w(\tau_n) + \sum_{i,j \neq *} \bar{T}_i d_{ij}(\tau_{n+1} - \tau_n) + \bar{T}^* N_c - \bar{T}^* x^*(\tau_n) \]

where the last inequality is implied by the fact that \( x^*(\tau_{n+1}) \leq N_c \) (we stop processing \( x^* \) at time \( \tau_{n+1} \), when its content is below \( N_c \)). Exploiting (16),

\[ w(\tau_{n+1}) \leq w(\tau_n) + \sum_{i,j \neq *} \bar{T}_i d_{ij}(\tau_n + 1 - \tau_n) + \bar{T}^* N_c - \bar{T}^* x^*(\tau_n) \]

Now, introducing the notation \( \rho := \sum_{i,j} \bar{T}_i d_{ij} \), we have that \( \sum_{i,j \neq *} \bar{T}_i d_{ij} = \rho - \rho^* \). Introducing this in the equation above and simplifying, we get:

\[ w(\tau_{n+1}) \leq w(\tau_n) - \bar{T}^* x^*(\tau_n) \left( 1 - \frac{\rho}{1 - \rho^*} \right) + \rho N_c \]

Using (15), the previous becomes:

\[ w(\tau_{n+1}) \leq w(\tau_n) - \bar{T}^* \epsilon \sum_{i,j} x_{ij}(\tau_n) \left( 1 - \frac{\rho}{1 - \rho^*} \right) + \rho N_c \]

where \( \bar{T}_M = \max_{i} \bar{T}_i \). So,

\[ w(\tau_{n+1}) \leq w(\tau_n) \left[ 1 - \epsilon \frac{T^*}{T_M} \left( 1 - \frac{\rho}{1 - \rho^*} \right) + \rho N_c \right] \]

Notice that condition (14) under strict inequality can be written as \( \rho < 1 \), which is exactly the condition considered in [11]. The proof can be continued exactly as in [11] where, however, for us \( \alpha_{ij} = \bar{T}_i \frac{1 - \rho}{1 - \rho^*} \) (the same as in [11]) and \( \beta_{ij} = \frac{1}{d_{ij}} N_c \). So, as in [11], it is possible to obtain:

\[ \sup_n w(\tau_n) \leq \frac{T_M}{\epsilon} \max_{ij} \frac{\beta_{ij}}{\alpha_{ij}} \]

hence

\[ w(t_k) \leq \frac{T_M}{\epsilon} \max_{ij} \frac{\beta_{ij}}{\alpha_{ij}} + \rho N_c \]

where \( d_m = \min_{ij} d_{ij} \). This allows to obtain that

\[ \sum_{i,j} x_{ij}(t_k) \leq \frac{1}{\bar{T}_m} w(t_k) \leq \frac{T_M}{\bar{T}_m \epsilon} \max_{ij} \frac{\beta_{ij}}{\alpha_{ij}} + \rho N_c \]

is bounded for all \( t_k \) (where \( \bar{T}_m := \min_i \bar{T}_i \)). \( \square \)
4 Optimization

Consider for now $Q = 1$ (one part type system). Now, under condition (14), if everything is approximated through continuous variables, the optimal policy is myopic [12], that is, it is the $c\mu$ rule if dealing with a linear cost function $g(x)$ as the one considered in (7). The $c\mu$ rule consists in processing the buffers $B_{ij}$ according to a priority established by the product $c$ times $\mu$, where in the present problem, the cost coefficient $c$ associated to the buffer $B_{ij}$ is given by the coefficient $c_{ij}$ in (7) and the maximum processing rate $\mu$ for this buffer is given by $\mu_i = \frac{N_i}{t_i}$: this is actually the maximum processing capacity for goods with destination (route) $i$. There are however two major differences:

- vehicles are not continuous resources;
- the capacity allocation has an influence also on the future (if we allocate all vehicles to destination $i$ we have to wait $T_i$ time units before we can change allocation) while in the scheduling machine case, where the $c\mu$ policy has been proved optimal, capacity allocations can change instantaneously at each step.

4.1 A Possible Heuristic

According to the above observations, we propose here a policy that we believe represents a promising and simple real time rule. We do not give here a proof of optimality for this policy and neither give a proof of stability: the performance of this policy will be explored from a computational point of view. According to the simulations, the stability appears to hold whenever condition (14) holds: this is not surprising since the policy reported below reduces the idle periods with respect to the one considered in the proof of the sufficiency of Theorem 1. This depends on the fact that, even if also this policy (as the one considered in the proof of Theorem 1) does not allow vehicles to travel partially loaded, it is no more required here that all the vehicles travel together to the same destination.

In particular, at each time step, the policy considered in this section allocates the vehicles available at that moment to the buffer $B_{ij}$ which, among the ones with $\sum_j x_{ij} \geq 1$ (that is, among the ones which allow to complete the load of a vehicle) has the largest $c\mu$ index (where, as mentioned above, for the buffer $B_{ij}$, the index $c\mu$ is given by $c_{ij}N_i/T_i$). To illustrate the policy more in detail, assume for simplicity $Q = 1$ and let $i_1, \ldots, i_P$ be the priority established according to the $c\mu$ rule (that is $c_{i_k}/T_{i_k} \geq c_{i_{k+1}}/T_{i_{k+1}}$ for all $k$). Then, the policy is given by:

\[
\begin{align*}
n_{i_1}(k) &= \min \{ N_{a}(k), \lfloor x_{i_1}(k) \rfloor \} \\
n_{i_2}(k) &= \min \{ N_{a}(k) - n_{i_1}(k), \lfloor x_{i_2}(k) \rfloor \} \\
& \quad \vdots \\
\end{align*}
\]

and so on, where $N_{a}(k)$, defined in (3), is the number of available vehicles in the interval $(t_k, t_{k+1})$. Then, to fill all the vehicles assigned to route $i$, we set:

\[
u_i(k) = n_i(k).
\]
4.2 Simulative Results

We tested the policies discussed above in a system with $Q = 1$ (a single product), $P = 3$, characterized by the following parameters: delays $T_1 = 4$, $T_2 = 3$, $T_3 = 5$; arrivals, constant in time, $d_1 = d_2 = 7$, $d_3 = 5$; with this choice the minimum $N_c$ guaranteeing stability is 74, according to condition (14). In figure are shown the performances (6), (7), with unit costs $c_1 = c_3 = 1$, $c_2 = 2$, and $\gamma = 1$, of three policies derived by simulating the system, for a finite time horizon, for various values of the parameter $N_c$. The dash dotted line shows the performances of the stabilizing policy described in Theorem 1; the dashed line the performances of the policy which allocates at each time instant all the available vehicles prioritizing the buffers with higher content, and the continuous line the performances of the “$c\mu$ policy” (those coefficients, by the parameters chosen, make buffer 2 the one with higher priority followed by buffer 1 and 3).

It is possible to observe that for values of $N_c$ lower than the stabilizing value (74), none of the policy described can achieve stability, consistently with Theorem 1 (for $N_c < 74$ the costs reported in Figure 1 result finite as a consequence of the finite time horizon considered). For $N_c > 74$ the $c\mu$ policy performs better than the policy prioritizing the higher buffers.

Fig. 1. Performances of the $c\mu$ policy (continuous line), the serve-largest-buffer policy (dashed), and the basic stabilizing policy (dash-dotted), as a function of $N_c$.

5 Conclusions

In this paper, a simplified model of a logistic node has been considered, where items arrive from outside to the node and must be routed to different destinations. Waiting items are stored in different buffers, according to their class and destination. At first, a necessary and sufficient condition is given in the paper for the possibility of finding dispatching dynamic policies that maintain all the buffers bounded. Subsequently an
optimization problem is considered and a simulative comparison of the performance of different feedback policies is presented in the paper. The problem has been studied under a fluid approximation of the items traveling in the node; this allows to completely fill the vehicles (and to neglect complex combinatorial loading problems). This possibility is actually used by the policies studied in this paper that do not allow the vehicles to travel partially filled. This is actually a reasonable choice under heavy traffic conditions where allowing the possibility of sending partially full vehicles may even compromise the stability, but may become significantly sub-optimal in the case of reduced inflow rates, large holding costs and small traveling costs.

References